Lecture 15: Controllability and Observability *

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1 Controllability

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Let

$$G(s) = \left(\frac{1}{s^2+4}, \frac{s+1}{s^2+4} \right) = \frac{E_0 + E_1 s}{s^2+4}$$

where $E_0 = (1,1), E_1 = (0,1)$.
<u>The standard controllable realization</u>:

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \end{array} \right) t + \frac{1}{2} \begin{pmatrix} -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix} t^2 +$$
$$\frac{1}{3!} \left(\begin{array}{cc|cc} 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \\ \hline 16 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \end{array} \right) t^3 + \frac{1}{4!} \begin{pmatrix} 16 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 16 \end{pmatrix} t^4 + \cdots$$
$$= \begin{pmatrix} \cos 2t & 0 & \frac{1}{2} \sin t & 0 \\ 0 & \cos 2t & 0 & \frac{1}{2} \sin 2t \\ -2 \sin 2t & 0 & \cos 2t & 0 \\ 0 & -2 \sin 2t & 0 & \cos 2t \end{pmatrix}$$

*This work is being done by various members of the class of 2012

Consider the 'free response' with initial condition:

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = e^{At} \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\cos 2t - \frac{1}{2}\sin 2t \\ \frac{1}{2}\sin 2t \\ 2\sin 2t - \cos 2t \\ \cos 2t \end{pmatrix},$$

$$y(t) = \begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\cos 2t - \frac{1}{2}\sin 2t \\ \frac{1}{2}\sin 2t \\ 2\sin 2t - \cos 2t \\ \cos 2t \end{pmatrix} = 0$$

Theorem. *There exists a control $u(\cdot)$ which steers the state $x(t)$ of the system*

$$\dot{x} = A(t)x(t) + B(t)u(t)$$

from the value x_0 at time $t = t_0$ to x_1 at time $t = t_1 > t_0$ if and only if $x_0 - \Phi(t_0, t_1)x_1$ belongs to the range space of

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, t)B(t)B(t)^T \Phi(t_0, t)^T dt$$

Proof. The set of points that can be reached along trajectories of the system are:

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \sigma)B(\sigma)u(\sigma)d\sigma \qquad (*)$$

Hence, we must be able to define a $u(\cdot)$ on $[t_0, t_1]$, s.t.

$$x_1 - \Phi(t_1, t_0)x_0 = \int_{t_0}^{t_1} \Phi(t_1, \sigma)B(\sigma)u(\sigma)d\sigma$$

Suppose $x_0 - \Phi(t_0, t_1)x_1$ lies in the range space of $W(t_0, t_1)$. Then there is an $\eta \in \mathbb{R}^n$, s.t.

$$x_0 - \Phi(t_0, t_1)x_1 = W(t_0, t_1)\eta$$

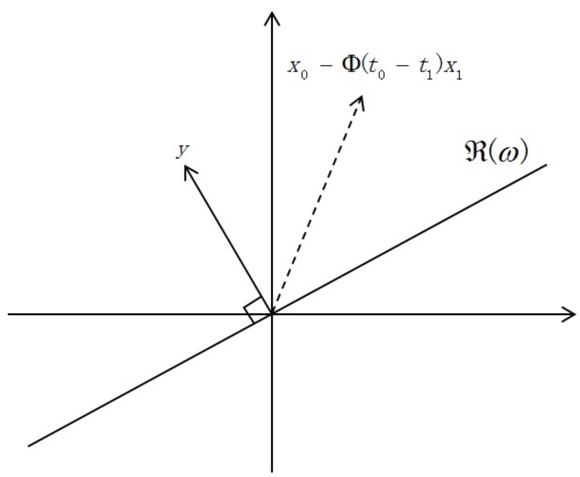
Define

$$\boxed{u(t) = -B(t)^T \Phi(t_0, t)^T \eta}$$

Use this in (*):

$$
\begin{aligned}
x(t_1) &= \Phi(t_1,t_0)x_0 - \int_{t_0}^{t_1} \Phi(t_1,\sigma)B(\sigma)\underbrace{B(\sigma)^T\Phi(t_0,\sigma)^T}_{-u} d\sigma\cdot\eta \\
&= \Phi(t_1,t_0)x_0 - \Phi(t_1,t_0)\int_{t_0}^{t_1} \Phi(t_0,\sigma)B(\sigma)B(\sigma)^T\Phi(t_0,\sigma)^T d\sigma\cdot\eta \\
&= \Phi(t_1,t_0)x_0 - \Phi(t_1,t_0)W(t_0,t_1)\eta \\
&= \Phi(t_1,t_0)x_0 - \Phi(t_1,t_0)[x_0 - \Phi(t_0,t_1)x_1] \\
&= x_1 \quad \rightarrow \textit{ proving that this works}
\end{aligned}
$$

Suppose on the other hand, that $x_0 - \Phi(t_0,t_1)x_1$ does not lie in the range space of $W(t_0,t_1)$. Then there exists a vector, y, such that $W(t_0,t_1)\cdot y = 0$, but $y\cdot(x_0 - \Phi(t_0,t_1)x_1) \neq 0$, as shown in the figure below:



Assume, contrary to what we wish to prove, that there is a $u(\cdot)$ such that

$$x_1 = \Phi(t_1,t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1,\sigma)B(\sigma)u(\sigma)d\sigma$$

i.e.

$$x_0 - \Phi(t_0,t_1)x_1 = -\int_{t_0}^{t} \Phi(t_0,\sigma)B(\sigma)u(\sigma)d\sigma$$

Then,

$$0 \neq y\cdot(x_0 - \Phi(t_0,t_1)x_1) = -y\int_{t_0}^{t_1} \Phi(t_0,\sigma)B(\sigma)u(\sigma)d\sigma \qquad (1)$$

But,

$$
\begin{aligned}
0 &= y^T W(t_0, t_1) y \\
&= \int_{t_0}^{t} y^T \Phi(t_0, \sigma) B(\sigma) B(\sigma)^T \Phi(t_0, \sigma)^T y d\sigma \\
&= \int_{t_0}^{t_1} \|B(\sigma)^T \Phi(t_0, \sigma)^T y\|^2 d\sigma \Rightarrow B(\sigma)^T \Phi(t_0, \sigma)^T y = 0
\end{aligned}
$$

This and (1) cannot both be true, and these mutually contradictory statements imply that if $u(\cdot)$ exists, $x_0 - \Phi(t_0, t_1)x_1$ is in the range space of $W(t_0, t_1)$

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, t) B(t) B(t)^T \Phi(t_0, t)^T dt$$

□

Remark 1. *If $W(t_0, t_1)$ is non-singular (i.e. has rank n), the system*

$$\dot{x} = A(t)x(t) + B(t)u(t)$$

is said to be controllable.

Remark 2. *The proof gives a formula that can be used to steer the system from x_0 to x_1.*

Remark 3. *This will turn out to be the minimum energy control.*

Remark 4. *$W(t_0, t_1)$ is called the controllability grammian.*

Simplification in the case of constant coefficients

$$\dot{x} = Ax + Bu$$

Theorem. *For A,B = constant matrices, the range space and null space of $W(t_0, t_1)$ coincide with the range space and null space of*

$$W_T = [B, AB, \cdots, A^{n-1}B][B, AB, \cdots, A^{n-1}B]^T$$

Proof. Let $x \in N(W(t_0, t_1))$

$$
\begin{aligned}
0 = x^T W(t_0, t_1) x &= \int_{t_0}^{t} x^T e^{A(t_0 - \sigma)} B B^T e^{A^T(t_0 - \sigma)} x d\sigma \\
&= \int_{t_0}^{t} \|B^T e^{A^T(t_0 - \sigma)} x\|^2 d\sigma \Rightarrow B^T e^{A^T(t_0 - \sigma)} x \equiv 0
\end{aligned}
$$

This means that all derivatives are zero, so that

$$\begin{aligned} B^T x &= 0 \\ B^T A^T x &= 0 \\ B^T A^{T(n-1)} x &= 0 \\ &\vdots \end{aligned}$$

Hence $W_T x = 0$. This means $x \in N(W_T)$.

Suppose, on the other hand, that $x \in N(W^T)$. Then

$$W_T x = 0 \Rightarrow x^T W_T x = 0$$

so that

$$\begin{aligned} \|[B, AB, \cdots, A^{n-1}B]^T x\|^2 = 0 \\ \Rightarrow x^T B = 0, x^T AB = 0, \cdots, x^T A^{n-1} B = 0 \end{aligned}$$

By the Cayley-Hamilton theorem,

$$\begin{aligned} e^{A(t_0-\sigma)} &= \sum_{i=0}^{n-1} \alpha_i(t_0-\sigma)A^i \\ x^T W(t_0,t_1) &= \int_{t_0}^{t} \sum_{i=0}^{n-1} \alpha_i(t_0-\sigma) x^T A^i B B^T e^{A^T(t_0-\sigma)} d\sigma \\ &= 0 \end{aligned}$$

$W(t_0, t_1)$ is symmetric $\Rightarrow W(t_0,t_1)x = 0$

Hence, $N(W(t_0,t_1)) = N(W_T)$.

Since $W(t_0,t_1)$ and W_T are symmetric, their range spaces are the orthogonal complements of the null spaces. Hence, these are also equal □

System is controllable $\Leftrightarrow W(t_0,t_1)$ has rank n $\Leftrightarrow [B, AB, \cdots, A^{n-1}B]$ has full rank n.

<u>Example</u>: $m\ddot{x} + kx = u(t)$

The standard first orderization is:

$$\left\{ \begin{array}{l} x_1 = x \\ x_2 = \dot{x} \end{array} \right. \Rightarrow \begin{pmatrix} x1 \\ x2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}}_{b} u(t)$$

$$\begin{pmatrix} b & Ab \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} \\ \frac{1}{m} & 0 \end{pmatrix}$$

rank $= 2 \Rightarrow$ controllable.

2 Observability

$$\dot{x} = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t)$$

We observe $y(\cdot)$, but we would like to know $x(\cdot)$.

$$y(t) = C(t)\Phi(t, t_0)x_0 + \int_{t_0}^{t} C(t)\Phi(t, \sigma)B(\sigma)u(\sigma)d\sigma$$

To reconstruct $x(t)$, we need only to determine x_0. There is thus no loss of generality in considering just

$$\dot{x} = A(t)x(t),\ y(t) = C(t)x(t)$$

Define: $L : \mathbb{R}^n \to C^m[t_0, t_1]$ (=continuous functions on the internal $t_0 \leqslant t \leqslant t_1$ taking values in $\mathbb{R}^m$) by $L_{x_0}(t) = C(t)\Phi(t, t_0)x_0$.

Proposition. *The null space of* L *coincides with the null space of*

$$M(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t, t_0)^T C(t)^T C(t)\Phi(t, t_0)dt$$

Proof. If $M(t_0, t_1)x_0 = 0$, then $x_0^T M(t_0, t_1)x_0 = 0$ and hence:

$$\begin{aligned} 0 &= \int_{t_0}^{t_1} x_0^T \Phi(t, t_0)^T C(t)^T C(t)\Phi(t, t_0)x_0 dt \\ &= \int_{t_0}^{t_1} \|C(t)\Phi(t, t_0)x_0\|^2 dt \\ &\Rightarrow C(t)\Phi(t, t_0)x_0 \equiv 0 \end{aligned}$$

On the other hand:

$$L_{x_0} = 0 \Rightarrow \int_{t_0}^{t} \Phi(t, t_0)^T C(t)^T C(t)\Phi(t, t_0)x_0 dt = 0$$

$\Rightarrow M(t_0, t_1)x_0 = 0$. *This proves the proposition*

□