Lecture 11 *

October 11, 2012

1 Last Time

$\dot{x} = Ax + Bu$ Time Domain

$y = Cx$

$\hat{y} = C(Is - A)^{-1}B\hat{(u)}$ Frequency Domain

2 Why is it Called Frequency Domain?

$$m\ddot{x} + Kx = u$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{\frac{k}{m}} \\ -\sqrt{\frac{k}{m}} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = x_1(t)$$

$$= (1, 0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

When the input is $u = 0$, the output is easily found from the matrix exponential (Peano-Baker Series)

$y(t) = Asin(\omega t) + Bcose(\omega t)$

where $\omega = \sqrt{\frac{k}{m}}$

$$sI - A = \begin{pmatrix} s & -\sqrt{\frac{k}{m}} \\ \sqrt{\frac{k}{m}} & s \end{pmatrix} = \begin{pmatrix} s & -\omega \\ \omega & s \end{pmatrix}$$

*This work is being done by various members of the class of 2012

$$c(sI-A)^{-1} = \frac{1}{s^2+\omega^2}\begin{pmatrix} s & -\omega \\ \omega & s \end{pmatrix}$$

Using from above $(A)(B)(C)$

$$A = \begin{pmatrix} 0 & \sqrt{\frac{k}{m}} \\ -\sqrt{\frac{k}{m}} & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1, 0 \end{pmatrix}$$

$$c(sI-A)^{-1} = \begin{pmatrix} 1, 0 \end{pmatrix}\begin{pmatrix} \frac{s}{s^2+\omega^2} & \frac{\omega}{s^2+\omega^2} \\ -\frac{\omega}{s^2+\omega^2} & \frac{s}{s^2+\omega^2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\frac{\omega}{s^2+\omega^2}$$

Let $u(t) = sin(w_o t)$. Consider the system forced by u.

$$\hat{y} = g(s)\hat{u}(s)$$

$$= \frac{\omega}{s^2+\omega^2}\frac{\omega_o}{s^2+\omega_o^2}$$

$$= \lambda\frac{\omega}{s^2+\omega^2}\beta\frac{\omega_o}{s^2+\omega_o^2}$$

where

$$= \lambda = \frac{\omega_o}{\omega_o^2-\omega^2}\beta = \frac{\omega}{\omega^2-\omega_o^2}$$

3 Changing Basis

Changing Basis: The effect on time and frequency doman representations,

Given $$\dot{x} = Ax \tag{1}$$

We have seen that solutions are conveniently represented by

$$e^{At} = P\begin{pmatrix} e^{\lambda_1 t} & 0 & 0 & . & . & . & 0 \\ 0 & e^{\lambda_2 t} & 0 & . & . & . & 0 \\ 0 & 0 & e^{\lambda_3 t} & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & e^{\lambda_n t} \end{pmatrix}P^{-1}$$

In the case that A is <u>normal</u>. How do the representations of

$$\dot{x} = Ax + Bu$$

$$y = Cx \qquad (t)$$

and

$$\hat{y} = c(sI - A)^{-1}B\hat{u}$$

look under a change of basis

Let $z = Tx$, where T is any invertable matrix, Then

$$\dot{z} = T\dot{x}$$

$$= T(Ax + Bu)$$

$$= TAT^{-1}z + TBu$$

$$y = Cx = CT^{-1}z$$

In terms of the state variable z, the system is written as

$$\dot{z} = \bar{A}z + \bar{B}u$$

$$y = \bar{C}z$$

$$\bar{A} = TAT^{-1}$$

$$\bar{B} = TB$$

$$\bar{C} = CT^{-1}$$

The frequency domain representation of the transform system is

$$\hat{y} = \bar{C}(Is - \bar{A})^{-1}\bar{B}\hat{u}$$

$$= CT^{-1}(Is - TAT^{-1})TB\hat{u}$$

$$= CT^{-1}(TT^{-1}s + TAT^{-1})TB\hat{u}$$

$$= CT^{-1}(T[Is - A]T^{-1})TB\hat{u}$$

$$= CT^{-1}(T[Is - A]^{-1}T^{-1})TB\hat{u}$$

$$C(Is - A)^{-1}B\hat{u}$$

4 Realization Theory

Realization Theory: How do we get a state space representation from a transfer function?

Case 1: SISO (single input single output) system of the form

$$g(s) = \frac{1}{s^n + a_{n-1}s^{n-1} + ... + a_o}$$

this relates inputs to outputs by $\hat{y} = g(s)\hat{u}$

$$(s^n + a_{n-1}s^{n-1} + ... + a_o)\hat{y}(s) = \hat{u}(s)$$

Taking the inverse laplace transforms

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + ... + a_o y(t) = u(t)$$

First orderizing this in the obvious way.

$$x_1 = y$$

$$x_2 = \dot{y}$$

$$.$$

$$.$$

$$x_n = y^{n-1}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & . & 0 \\ 0 & 0 & 1 & . & 0 \\ . & . & . & . & 0 \\ 0 & 0 & 0 & . & 1 \\ -a_o & -a_1 & -a_2 & . & -a_{n-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ . \\ 0 \\ 1 \end{pmatrix} u(t)$$

$$y = x_1 = \begin{pmatrix} 1, ..., 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix}$$

Case 2: SISO system of the form

$$g(s) = \frac{b_{n-1}S^{n-1} + ... + B_o}{s^n + a_{n-1}s^{n-1} + ... + a_o}$$

Again we wish to find a system such that

$$\hat{y} = g(s)\hat{u}$$

or in other words

$$g(s) = \frac{\hat{y}(s)}{u(s)}$$

To find the state space representation, introduce an intermediate variable $\hat{x}$, and write

$$g(s) = \frac{\hat{y}}{\hat{x}}\frac{\hat{x}}{\hat{u}}$$

we write these factors as

$$\frac{\hat{x}}{\hat{u}} = \frac{1}{s^n + a_{n-1}s^{n-1} + ... + a_o}$$

$$\frac{\hat{y}}{\hat{x}} = b_{n-1}S^{n-1} + ... + B_o$$

The state space representation of x in terms of u has been seen to be

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & . & 0 \\ 0 & 0 & 1 & . & 0 \\ . & . & . & . & 0 \\ 0 & 0 & 0 & . & 1 \\ -a_o & -a_1 & -a_2 & . & -a_{n-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ . \\ 0 \\ 1 \end{pmatrix} u(t)$$

$$x = x_1$$

The time domain rendering of y is then

$$y(t) = b_{n-1}\frac{d^{n-1}}{dt^{n-1}}x + b_o x(t)$$

$$= b_{n-1}x_n + ... + b_o x_1$$

The overall system is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & . & 0 \\ 0 & 0 & 1 & . & 0 \\ . & . & . & . & 0 \\ 0 & 0 & 0 & . & 1 \\ -a_o & -a_1 & -a_2 & . & -a_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ . \\ 0 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} b_o, b_1, ...b_{n-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ x_n \end{pmatrix}$$

Starting from the same transfer function we perform long division

$$\begin{array}{r|l} & \frac{b_{n-1}}{s} + \frac{b_{n-2}-b_{n-1}a_{n-1}}{s^2} + ... \\ \hline (s^n + a_{n-1}s^{n-1} + ... + a_o) & b_{n-1}S^{n-1} + ... + B_o \\ & bn-1s^{n-1} + b_{n-1}a_{n-1}s^{n-2} + ... \\ & b_{n-2} - b_{n-1}a_{n-1})s^{n-2} + ... \end{array}$$

Proper rational functions, in this way, admit Taylor series expansions about infinity

$$g(s) = \frac{\lambda_1}{s} + \frac{\lambda_2}{s} + \frac{\lambda_3}{s} + ...$$

what about MIMO (multiple input multiple output) systems?

$$G(s) = C(sI - A)^{-1}B$$

$$= \frac{1}{s}C\left(I - \frac{A}{s}\right)^{-1} B$$

For s sufficiently large, every entry in the matrix $\frac{A}{s}$ is small hence we may appeal to writing

$$\left(I - \frac{A}{s}\right)^{-1} = f(\frac{A}{s})$$

where

$$f(r) = \frac{1}{1-r}$$

$$\frac{1}{1-A} = 1 + r + r^2 + ...$$

$$\left(I - \frac{A}{s}\right)^{-1} = I + \frac{A}{s} + \frac{A^2}{s^2} + ...$$

The

$$G(s) = \frac{1}{s}C\left(I - \frac{A}{s}\right)^{-1} B = \frac{CB}{s} + \frac{CAB}{s^2} + \frac{CA^2B}{s^3} + ...$$

Note that

$$Ł^{-1}\left(\frac{1}{s^{k+1}}\right) = \frac{t^k}{k!}$$

Hence the inverse Laplace transform of G(s) is

$$CB + C(At)B + C\left(\frac{A^2t^2}{2}\right)B + C\left(\frac{A^3t^3}{3!}\right)B$$

$$= Ce^{At}B$$

Theorem: Let $G(s)$ be a $q \times m$ matrixof rational functions such that the degrees of the numerators exceed the degrees of the denominators for each entry. Then there exist constant matrices A, B, C such that

$$G(s) = C(Is - A)^{-1}B.$$

Proof let $P(s)$ be the monic polynomial that is the least common multiple (l.c.m) of the denominators (monic $\rightarrow$ leading coefficient $= 1$, l.c.m. : if $f(s) = (s-\lambda)(s-\beta)(s-\gamma), g(s) = (s-\beta)(s-\delta)$, then the l.c.m is $(s-\lambda)(s-\beta)(s-\gamma)(s-\delta)$; Then

$$P(s)G(s) = E - 0 + E_1 s + E_2 s^2 + E_{r-1}s^{n-1}$$

Where $r =$ degree of P

Let O_m be the m x m zero matrix, let I x m be the m x m identity matrix. Define A to be the rm x rm matrix

$$A = \begin{pmatrix} O_m & I_m & O_m & . & O_m \\ O_m & O_m & I_m & . & O_m \\ . & . & . & . & . \\ O_m & O_m & O_m & . & I_m \\ -p_o I_m & -p_1 I_m & -p_2 I_m & . & -p_{n-1} I_m \end{pmatrix}$$

where

$$P(s) = s^n + P_{r-1}s^{n-1} + ... + P_1 s + p_o$$

let

$$B = \begin{pmatrix} O_m \\ O_m \\ . \\ O_m \\ I_m \end{pmatrix}, C = \begin{pmatrix} E_o, E_1, ..., Em \end{pmatrix}$$

Now we will show that

$$G(s) = C(Is - A)^{-1}B$$

Note: that $(Is-A)^{-1}B$ is the solution $\hat{x}$ to the matrix equation

$$(Is-A)X = B \qquad (*)$$

Partition

$$\hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ . \\ . \\ \hat{x}_n \end{pmatrix}$$

Compatible with A (i.e. $\hat{x}_k$ is m-dimensional)

then (*) may be written

$$s\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ . \\ . \\ \hat{x}_n \end{pmatrix} \begin{pmatrix} O_m & I_m & O_m & . & O_m \\ O_m & O_m & I_m & . & O_m \\ . & . & . & . & . \\ O_m & O_m & O_m & . & I_m \\ -p_o I_m & -p_1 I_m & -p_2 I_m & . & -p_{n-1} I_m \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ . \\ . \\ \hat{x}_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ . \\ 0 \\ Im \end{pmatrix}$$

component wise this is

$$\left\{ \frac{s\hat{x}_i = \hat{x}_{i+1} \qquad i = 1, ..., r-1}{s\hat{x}_r = -P_o\hat{x}_1 - P_1\hat{x}_2 ..... - P_{r-1}\hat{x}_r + I_m} \right\}$$

$$= (-P_o - P1s - ..... - P_{r-1}S^{n-1})\hat{x}_1 + I_m$$

the last equation may be rendered as

$$S^n\hat{x}_1 = (-P_o - P1s - ..... - P_{r-1}S^{n-1})\hat{x}_1 + I_m$$

$$\hat{x} = \frac{1}{P(s)} I - m$$

Now note that

$$C(Is-A)^{-1}B = C\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ . \\ . \\ \hat{x}_n \end{pmatrix}$$

$$= E_o\hat{x}_1 + E_1\hat{x}_2 + ... + E_{n-1}\hat{x}_r$$

$$= E_o\hat{x}_1 + E_1 s\hat{x}_2 + ... + E_{n-1}s^{n-1}\hat{x}_r$$

$$= (E_o + E_1 s + ... + E_{n-1} s^{n-1})\hat{x}_1$$

$$= \frac{1}{P(s)}(E_o + E_1 s + ... + E_{n-1} s^{n-1}) = G(s)$$

$$A = \begin{pmatrix} O_m & I_m & O_m & . & O_m \\ O_m & O_m & I_m & . & O_m \\ . & . & . & . & . \\ O_m & O_m & O_m & . & I_m \\ -p_o I_m & -p_1 I_m & -p_2 I_m & . & -p_{n-1} I_m \end{pmatrix} B = \begin{pmatrix} O_m \\ O_m \\ . \\ O_m \\ I_m \end{pmatrix}, C = \begin{pmatrix} E_o, E_1, ..., Em \end{pmatrix}$$

Standard controllable realization of G(s)