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Solutions

(10/29/2024)

ENG EC/ME/SE 501:

First Quiz

1. (a) Find the Jordan Normal Form $\mathbf{J_A}$ of **15 pts**

$$\mathbf{A} = \begin{pmatrix} 5 & 5 & -4 \\ 8 & 5 & -4 \\ 12 & 6 & -5 \end{pmatrix}.$$

(b) Find a matrix $\mathbf{U}$ such that $\mathbf{U}^{-1}\mathbf{A}\mathbf{U} = \mathbf{J_A}$. **15 pts**

(c) For the $\mathbf{J_A}$ found in part (a), compute $e^{\mathbf{J_A}t}$ in closed form. **10 pts**

2. The characteristic polynomial of the matrix

$$\begin{pmatrix} -1 & 2 & 4 & -2 & 8 \\ 18 & 2 & 24 & -12 & -7 \\ -12 & -8 & -3 & 10 & 4 \\ -18 & -12 & -30 & 4 & 3 \\ 3 & 2 & -3 & 6 & 5 \end{pmatrix}$$

is of the form $p(s) = s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$. Find the value of a_4. **15 pts**

3. Compute e^{At} where $A = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$. **15 pts**

4. Let $g(s) = \frac{6s+8}{s^3+9s^2+23s+15}$. Do any two out of the following four:

(a) Write down the standard controllable realization. Is this system observable?
(b) Write down the standard observable realization. Is this system controllable?
(c) Write down a realization in "tandem form".
(d) Write down a realization in "Jordan form".

15 pts each

501 EXAM SOLUTIONS

FIRST COMPUTE THE CHARACTERISTIC POLYNOMIAL

$$\det\begin{pmatrix} 5-\lambda & 5 & -4 \\ 8 & 5-\lambda & -4 \\ 12 & 6 & -5-\lambda \end{pmatrix}$$

$$= -(\lambda^3 - 5\lambda^2 + 7\lambda - 3)$$

$$= -(\lambda-3)(\lambda-1)^2.$$ $\Rightarrow$ The eigenvalues are

1 (mult. 2) and 3 (mult. 1).

Compute the eigenvectors associated with 1.

Step 1: Solve $(A-\lambda I)\vec{x}=0$.

$$\begin{pmatrix} 4 & 5 & -4 \\ 8 & 4 & -4 \\ 12 & 6 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

EQUIVALENT REDUCED ROW ECHELON FORM IS

$$\begin{pmatrix} 1 & 0 & -1/6 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{pmatrix}$$

IF WE LET $X_3 = S$, THE WE READ OFF THE REST OF THE SOLUTION $X_2 = 2/3\, S$, $X_1 = 1/6\, S$.

THE EIGENVECTOR MAY BE TAKEN TO BE

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = S \begin{pmatrix} 1/6 \\ 2/3 \\ 1 \end{pmatrix}.$$

<u>STEP 2</u>: Solve $(A - \lambda I)^2 \vec{x} = 0$.

$$\begin{pmatrix} 4 & 5 & -4 \\ 8 & 4 & -4 \\ 12 & 6 & -6 \end{pmatrix} \begin{pmatrix} 4 & 5 & -4 \\ 8 & 4 & -4 \\ 12 & 6 & -6 \end{pmatrix} = \begin{pmatrix} 8 & 16 & -12 \\ 16 & 32 & -24 \\ 24 & 48 & -36 \end{pmatrix}$$

R.R.E.F.:

$$\begin{pmatrix} 1 & 2 & -3/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

SOLUTION IS OBTAINED BY SETTING $x_3 = s$, $x_2 = t$. THEN $x_1 = \frac{3}{2}s - 2t$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2}s - 2t \\ t \\ s \end{pmatrix} = s\begin{pmatrix} 3/2 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

Note that for $s=1$, $t = 2/3$, we obtain the previously determined eigenvector.

Note that from this calculation it follows that $\ker(A-\lambda I)^2$ is a <u>two</u> dimensional subspace, and thus there is considerable latitude in choosing a element of the desired JNF basis.

A somewhat arbitrary choice is

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 0 \\ 1 \end{pmatrix}.$$

With this choice,

$$(A-\lambda I)\vec{x} = \begin{pmatrix} 2 \\ 8 \\ 12 \end{pmatrix},$$

which is in $\ker(A-\lambda I)$ [for $\lambda = 1$], as desired. We have shown that

$$A\begin{pmatrix} 3/2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 12 \end{pmatrix} + \begin{pmatrix} 3/2 \\ 0 \\ 1 \end{pmatrix}$$

$$A\begin{pmatrix} 2 \\ 8 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 12 \end{pmatrix}$$

<u>STEP 3</u>: Solve $(A-\lambda I)\vec{x} = 0$ for $\vec{x}$ with $\lambda = 3$. One choice:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

With respect to the basis vectors determined in Steps 2 and 3, the JNF is seen to be

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

1(b) Lettering $A = \begin{pmatrix} 5 & 5 & -4 \\ 8 & 5 & -4 \\ 12 & 6 & -5 \end{pmatrix}$ we see from problem (a) that

$$A \begin{pmatrix} 2 & 3/2 & 1 \\ 8 & 0 & 2 \\ 12 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & \vdots & 2+3/2 & \vdots & 3 \\ 8 & \vdots & 8+0 & \vdots & 6 \\ 12 & \vdots & 12+1 & \vdots & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3/2 & 1 \\ 8 & 0 & 2 \\ 12 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Hence, this is the desired U.

1(c) Since $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

commute,

$$e^{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}t} = e^{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}t} e^{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}t}$$

$$= \begin{pmatrix} e^t & & \\ & e^t & \\ & & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}$$

2. Given that

$$A = \begin{pmatrix} -1 & 2 & 4 & -2 & 8 \\ 18 & 2 & 24 & -12 & -7 \\ -12 & -8 & -3 & 10 & 4 \\ -18 & -12 & -30 & 4 & 3 \\ 3 & 2 & -3 & 6 & 5 \end{pmatrix}$$

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CharacteristicPolynomial[A, s]
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$$315108 - 18024\, s + 4837\, s^2 + 305\, s^3 - 7\, s^4 + s^5$$

Hence a_4=-7.

3.

$$\begin{pmatrix} e^{t\sigma} \mathrm{Cos}[t\,\omega] & e^{t\sigma} \mathrm{Sin}[t\,\omega] \\ -e^{t\sigma} \mathrm{Sin}[t\,\omega] & e^{t\sigma} \mathrm{Cos}[t\,\omega] \end{pmatrix}$$

4(a)

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -15 & -23 & -9 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$C = (8,\ 6,\ 0)$$

$$\det \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \det \begin{pmatrix} 8 & 6 & 0 \\ 0 & 8 & 6 \\ -90 & -138 & -46 \end{pmatrix}$$

$= 440$ HENCE OBSERVABLE.

4(b)

$$\begin{array}{r|l} & 6/s^2 - 46/s^3 + \cdots \\ s^3 + 9s^2 + 23s + 15 & 6s + 8 \\ & \underline{6s + 54} \\ & -46 + \cdots \\ & -46 + \cdots \end{array}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -15 & -23 & -9 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 6 \\ -46 \end{pmatrix} \qquad C = (1, 0, 0)$$

$$\begin{pmatrix} B & AB & A^2B \end{pmatrix} = \begin{pmatrix} 0 & 6 & -46 \\ 6 & -46 & 276 \\ -46 & 276 & -1516 \end{pmatrix}$$

$$\det\begin{pmatrix} 0 & 6 & -46 \\ 6 & -46 & 276 \\ -46 & 276 & -1516 \end{pmatrix} = -440$$

Hence controllable.

4(c). TANDEM FORM HAS

$$A = \begin{pmatrix} s_1 & 1 & 0 \\ 0 & s_2 & 1 \\ 0 & 0 & s_3 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$C = (C_1, C_2, C_3)$$

where the s_i's are the eigenvalues (= poles of the transfer function) and the c_i's are to be determined. In this case $s_1 = -1$, $s_2 = -3$, $s_3 = -5$.

$$C \cdot (Is - A)^{-1} B$$

$$= \frac{C_1}{15 + 23s + 9s^2 + s^3} + \frac{C_2(1+s)}{15 + 23s + 9s^2 + s^3} + \frac{C_3(3 + 4s + s^2)}{15 + 23s + 9s^2 + s^3}$$

SOLVING FOR THE c_i's TO GET THE DESIRED TRANSFER FUNCTION YIELDS

$$c_1 = 2 \qquad c_2 = 6 \qquad c_3 = 0$$

4(d) The JORDAN FORM IS

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$C = (c_1, c_2, c_3)$$

WHERE, AS IN 4(c), THE r VALUES OF THE c_i's MUST BE DETERMINED TO GIVE THE PROPER TRANSFER FUNCTION. SOLVING

$$c_1\left(15 + 8s + s^2\right) + c_2\left(5 + 6s + s^2\right) + c_3\left(3 + 4s + s^2\right)$$

$$= 6s + 8$$

YIELDS $c_1 = 1/4$, $c_2 = 5/2$, $c_3 = -11/4$.

CE/ME/SE Hour Quiz Grade Distribution 2024