Block Referenced Spatial Models

Acknowledgement: Many slides based on / borrowed from Sudipto Banerjee

Block referenced data

- Data has an location, an attribute and an AREA
- Areas are usually contiguous
- Data often conceived of as being area integrals of some underlying continuous surface

$$z(B_i) = \frac{1}{|B_i|} \int_{B_i} z(s)\,ds$$

- Goals
  - Estimate surface z(s) or new blocks
  - Account for non-independence of adjacent blocks

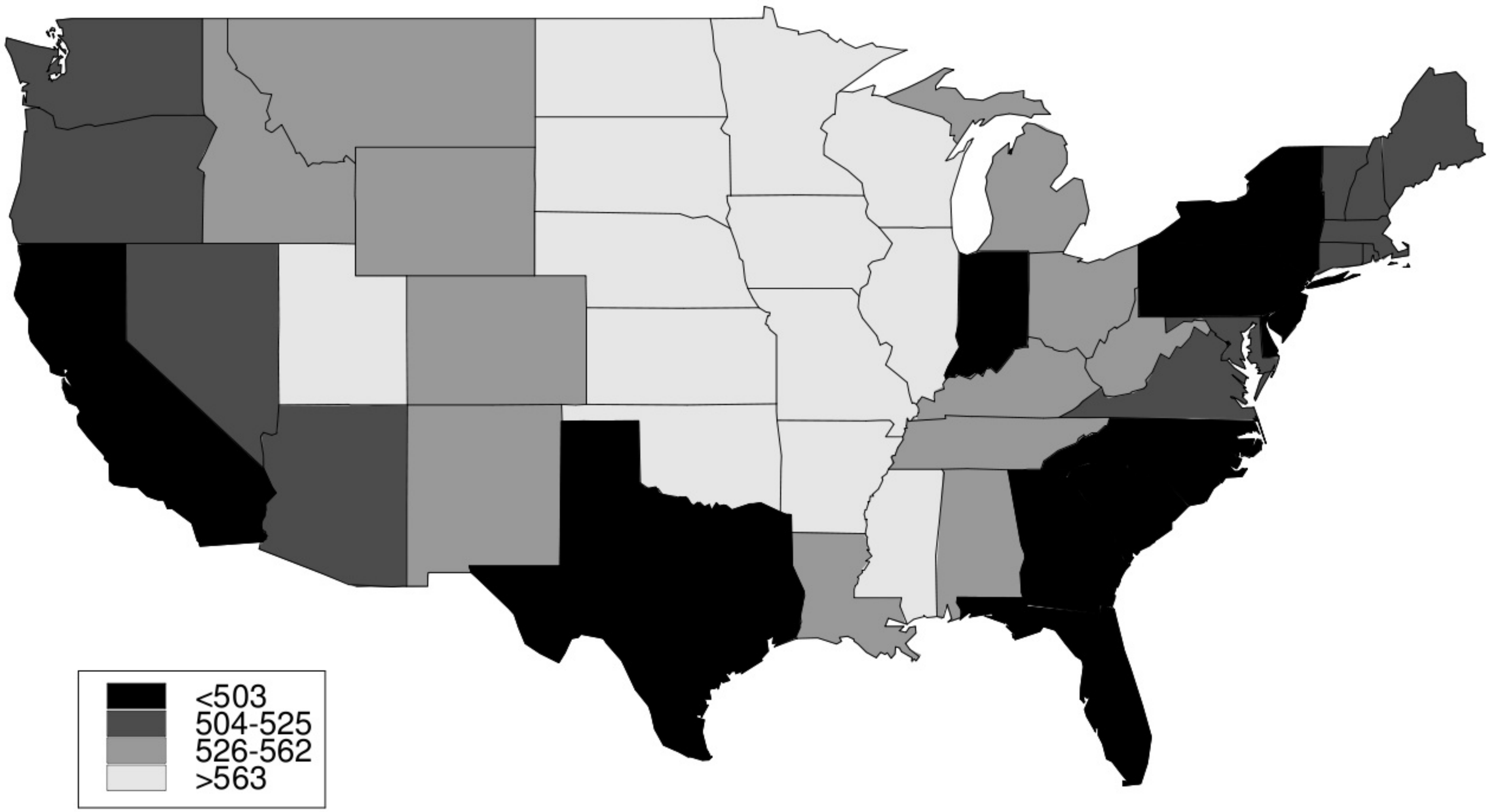

Figure 1: Choropleth map of 1999 average verbal SAT scores, lower 48 U.S. states.

Proximity matrices

(block analog to distance matrix)

- W, entries w_{ij} (with $w_{ii} = 0$). Choices for w_{ij}:
  - $w_{ij} = 1$ if i, j share a common boundary (possibly a common vertex)
  - w_{ij} is an *inverse* distance between units
  - $w_{ij} = 1$ if distance between units is $\leq K$
  - $w_{ij} = 1$ for m nearest neighbors.
- W is typically symmetric, but need not be
- $\widetilde{W}$: standardize row i by $w_{i+} = \sum_j w_{ij}$
- W elements often called "weights"; interpretation
- Could also define first-order neighbors $W^{(1)}$, second-order neighbors $W^{(2)}$, etc.

Measures of spatial association

- Moran's I: essentially an "areal covariogram"

$$I = \frac{n \sum_i \sum_j w_{ij}(Y_i - \bar{Y})(Y_j - \bar{Y})}{(\sum_{i \neq j} w_{ij}) \sum_i (Y_i - \bar{Y})^2}$$

- Geary's C: essentially an "areal variogram"

$$C = \frac{(n-1) \sum_i \sum_j w_{ij}(Y_i - Y_j)^2}{(\sum_{i \neq j} w_{ij}) \sum_i (Y_i - \bar{Y})^2}$$

- Both are asymptotically normal if Y_i are i.i.d.; Moran has mean $-1/(n-1) \approx 0$, Geary has mean 1
- Significance testing by comparing to a collection of say 1000 random permutations of the Y_i

Measures of spatial association (cont'd)

- For these data, the Moran's I is computed as 0.5833, with associated standard error estimate 0.0920 $\Rightarrow$ very strong evidence against H_0 : no spatial correlation
- We obtain a Geary's C of 0.3775, with associated standard error estimate 0.1008 $\Rightarrow$ again, very strong evidence against H_0 (departure from 1)
- Warning: These data have not been adjusted for covariates, such as the proportion of students who take the exam (Midwestern colleges have historically relied on the ACT, not the SAT; only the best and brightest students in these states would bother taking the SAT)
- $\Rightarrow$ the map, I, and C all motivate the search for spatial covariates!

Spatial smoothers

- To smooth Y_i, replace with $\hat{Y}_i = \frac{\sum_i w_{ij} Y_j}{w_{i+}}$
- More generally, we could include the value actually observed for unit i, and revise our smoother to

$$(1-\alpha)Y_i + \alpha\hat{Y}_i$$

  For $0 < \alpha < 1$, this is a linear (convex) combination in "shrinkage" form

Finally, we could try model-based smoothing, i.e., based on $E(Y_i|Data)$, i.e., the mean of the predictive distribution. Smoothers then emerge as byproducts of the hierarchical spatial models we use to explain the Y_i's

Conditional Autoregressive (CAR) Model

$$y_i = \underbrace{\mu_i}_{\text{process model}} + \underbrace{\frac{1}{w_{i+}} \sum_{j \neq i} w_{ij} (y_j - \mu_j)}_{\text{spatial autocorrelation}} + \underbrace{\epsilon_i}_{\text{error}}$$

- If raster, equivalent to Markov Random Field
- Analogous to AR(1) or our general model for spatial point data

$$Z(s) = \underbrace{\mu(s|\beta)}_{\textit{trend}} + \underbrace{w(s|\phi)}_{\textit{spatial error}} + \underbrace{\epsilon(s)}_{\textit{residual error}}$$

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$$\downarrow$$

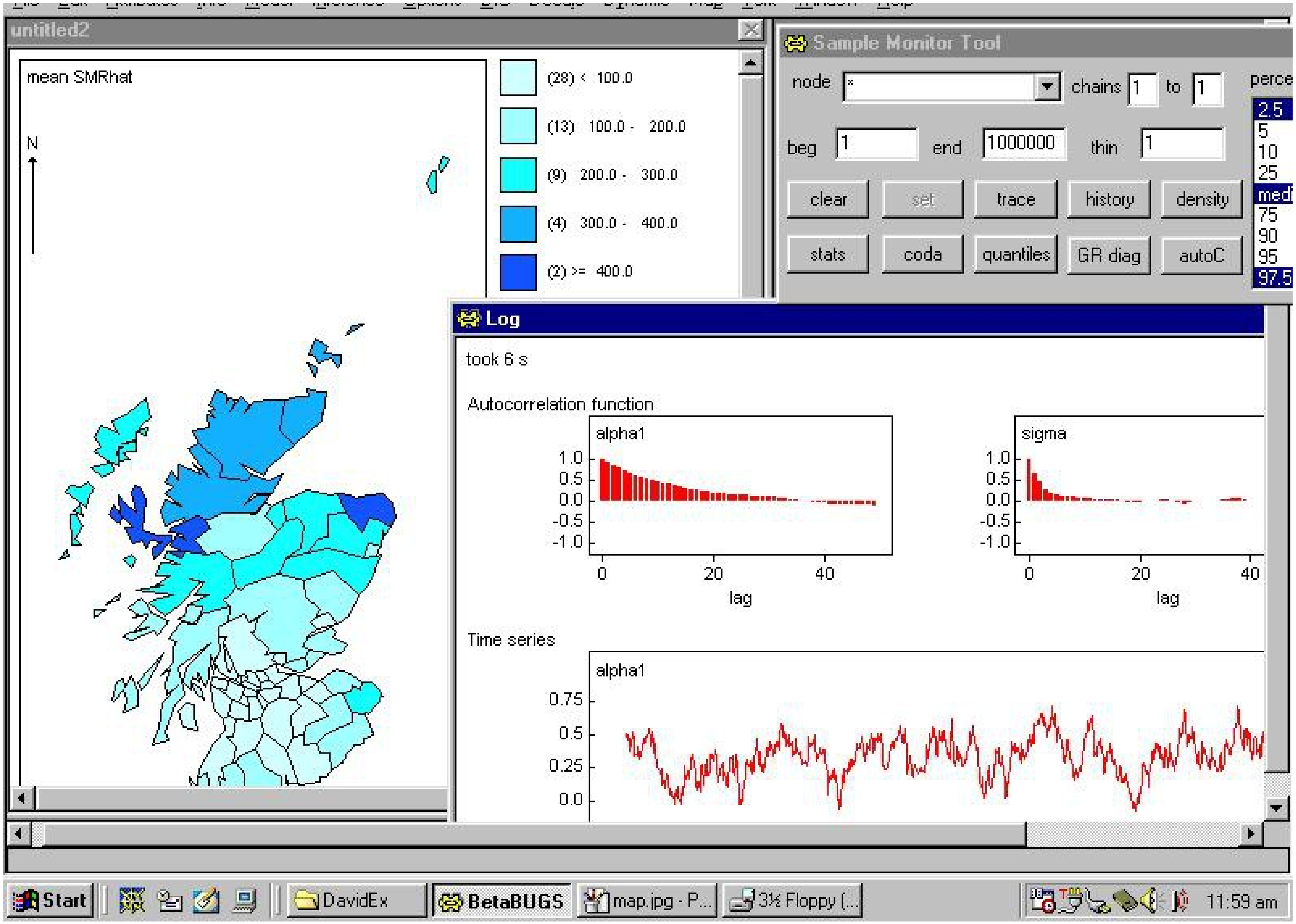
$$\vec{y} \sim N\left(\vec{\mu} \middle| (I - \tilde{W})^{-1} \sigma^2 I\right)$$

Analogous to time-series

$$Y_t = \mu + \sum_{i=1}^{p} \rho_i Y_{t-i} + \epsilon_t \rightarrow Y \sim N\left(\mu, \frac{\sigma^2}{1-\rho^2} R\right)$$

Computation of CAR models

- “GeoBUGS” extension of WinBUGS



Spatial Misalignment Problem

- “Change of support” problem
- Often need to compare / compute / infer spatial data of different types

  Observe Infer

  - Point – Point (Kriging)
  - Point – Block
  - Block – Point
  - Block – Block
- DON'T just interpolate/regrid
  - Misrepresents sample size & uncertainty

Point to Block

- Collect point data, want to infer the integral of the surface (e.g. county level biomass)
- Traditional approach: sample mean, var
  - Ignores autocorrelation, covariates, etc.
- Recommended Alternative:
  - Bayesian Kriging -> project to a fine grid
  - From each grid, numerically integrate

Block to Point

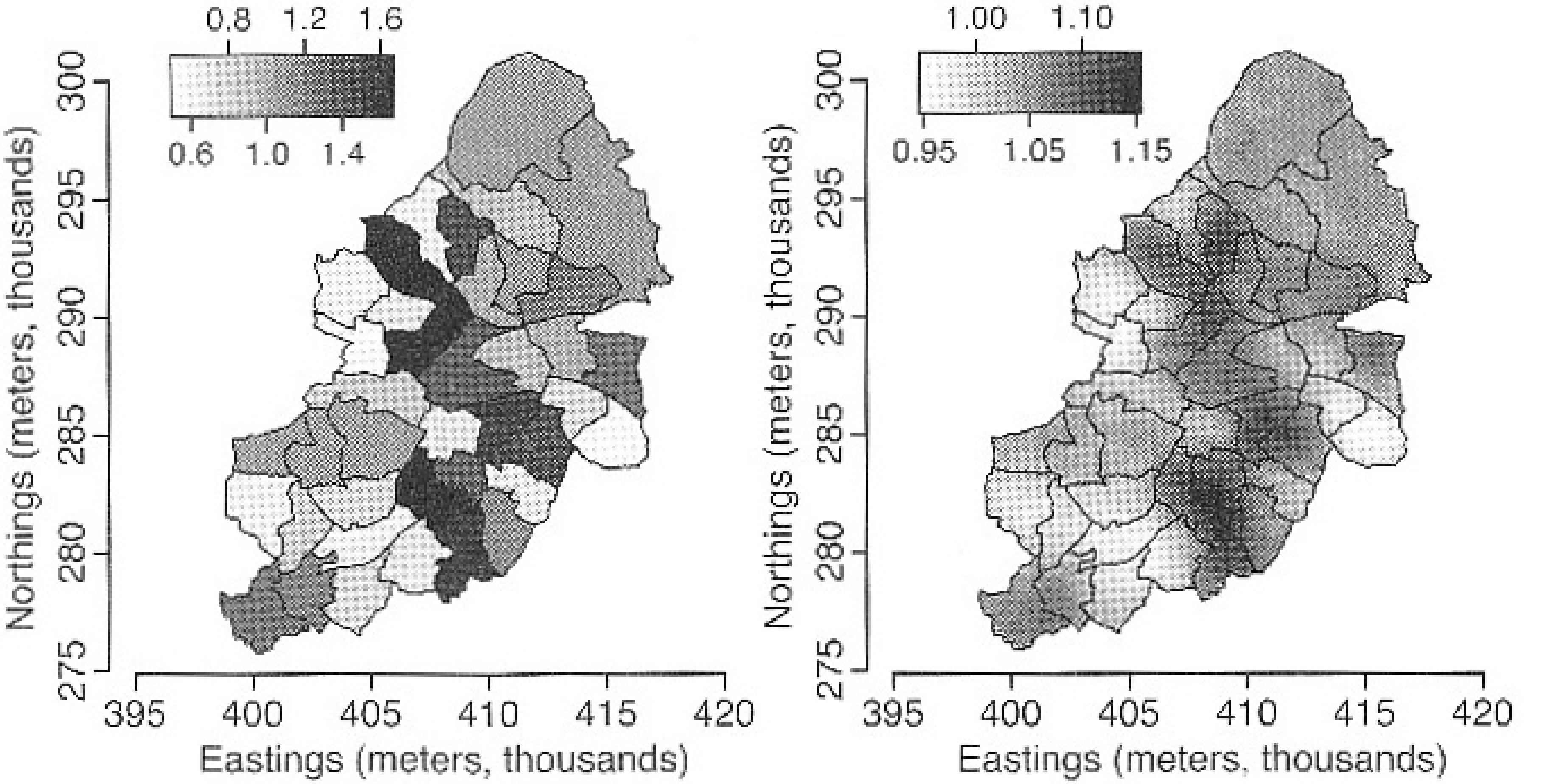
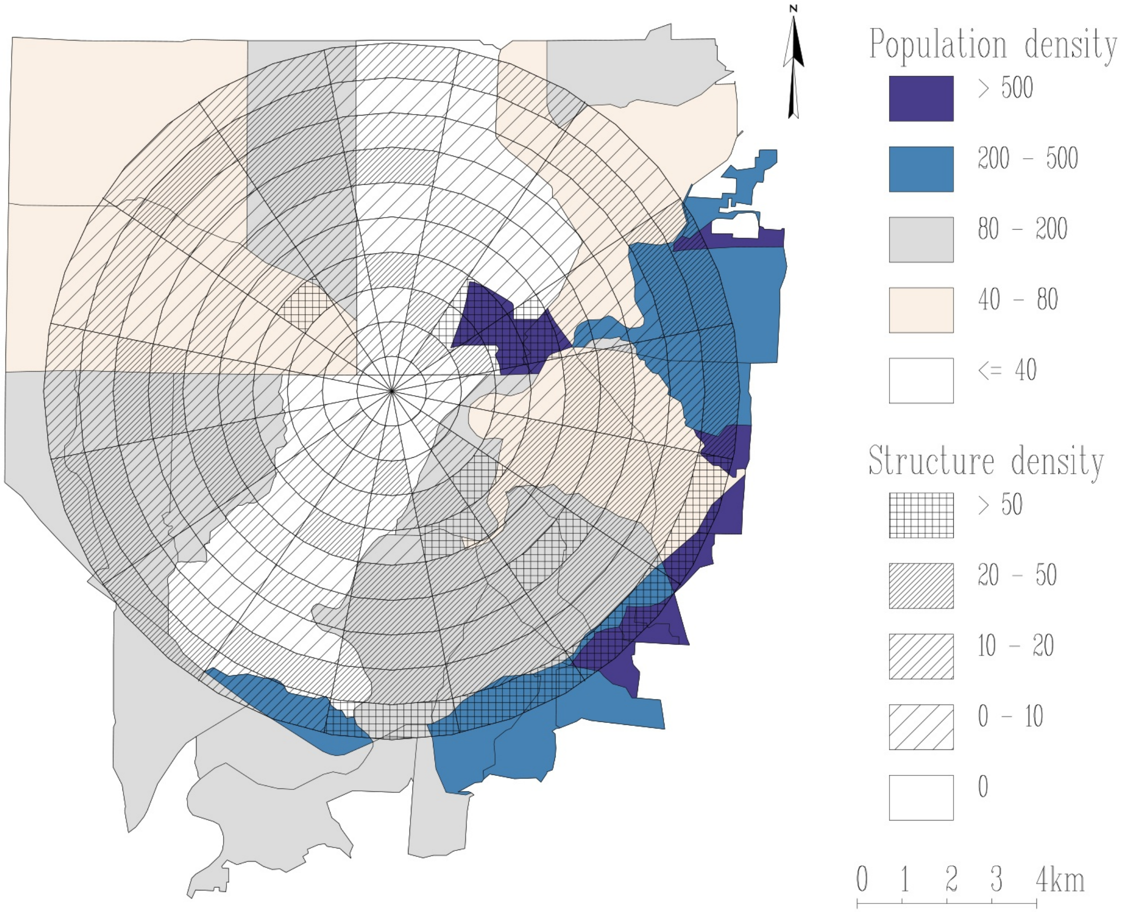


FIGURE 10.20. Standardized mortality ratios for thirty-nine wards in Birmingham, England, calculated as observed versus expected cases (*left*), and posterior median relative risk $y(s)$ (*right*). From Kelsall and Wakefield (2002).

Block-Block Misalignment

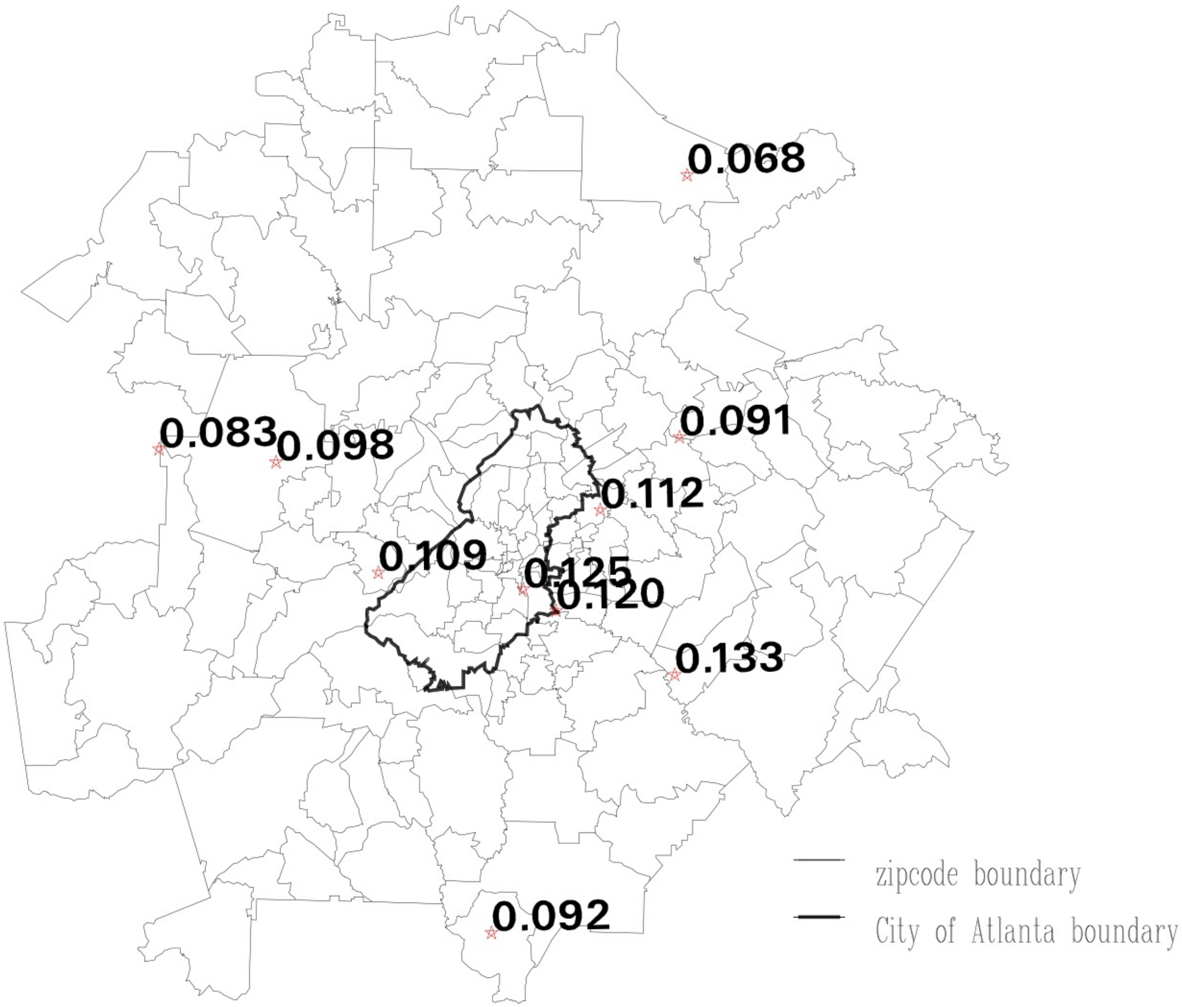
Population by census tract; residential structures by “cell”:



- “Areal Allocation”
- Hierarchical Modeling (e.g. CAR)

Bivariate misalignment

Ozone measurements at fixed sites; counts of pediatric asthma cases by zip code in Atlanta, GA:



Bivariate misalignment issues

- When we have two spatially referenced variables, interest often lies in spatial regression.
- But we cannot fit a regression if the two variables are misaligned:
  - X at point level, Y at other points
  - X at point level, Y at block level
  - X at block level, Y at point level
  - X at block level, Y at a different block level
- Solution: Bring the X's to the scale of the Y's, then fit the model (BCG, Sec 6.4)
- With more than two variables, bring all the variables to a common scale. Highest resolution is obviously preferred, but may be computationally infeasible!

Next Steps

If you are interested in spatial modeling I recommend:
"Hierarchical Modeling and Analysis for Spatial Data" 2003 by Sudipto Banerjee, Alan E. Gelfand, Bradley. P. Carlin

Cressie & Wikle 2011 "Statistics for Spatio-Temporal Data"

Wikle, Zammit-Mangion, & Cressie. 2019. Spatio-Temporal Statistics with R, Chapman and Hall/CRC.
http://spacetimewithr.org/ (free pdf)