State-Space Models

Bayesian State Space Model

$$Y_t = g\left(X_t | \phi\right)$$ **Data Model**

$$X_t = f\left(X_{t-1} | \theta\right)$$ **Process Model**

- Y = observed data
- X = <u>latent</u> time series
- ε = process error
- ω = observation error

Random Walk State Space Model

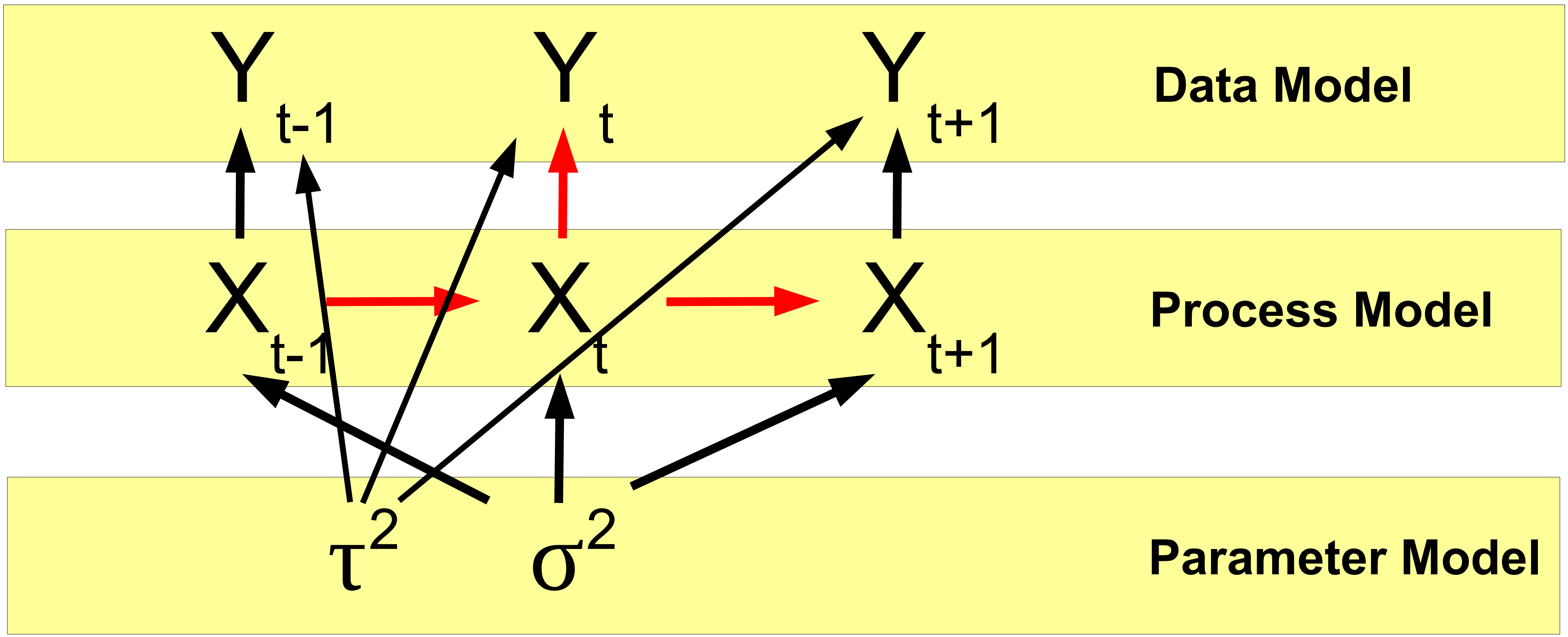
- What are the conditional distributions?

$$X_t \sim N(X_t | X_{t-1}, \tau^2_{add}) \times$$
$$N(X_{t+1} | X_t, \tau^2_{add}) \times$$
$$N(Y_t | X_t, \tau^2_{obs})$$

- Three special cases
  - First
  - Last
  - Missing Y

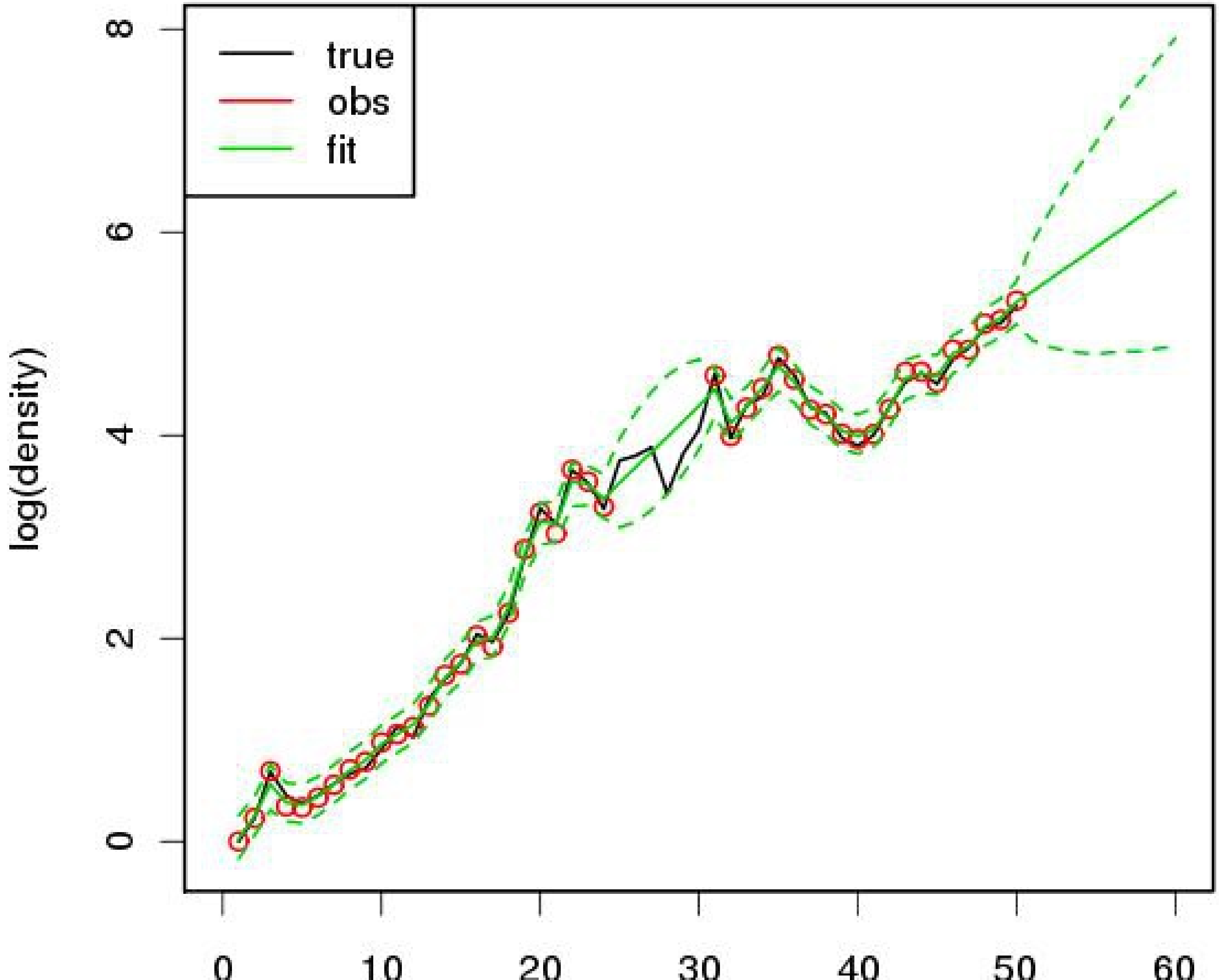
$$X_t \sim N(X_{t-1}, \tau^2_{add})$$
$$Y_t \sim N(X_t, \tau^2_{obs})$$
$$\tau^2_{obs} \sim IG(a_{obs}, r_{obs})$$
$$\tau^2_{add} \sim IG(a_{add}, r_{add})$$
$$X_0 \sim N(X_{ic}, \tau_{IC})$$

Random Walk State Space Model



Y's are conditionally independent given the X's

Prediction



Generality of the State Space Model

- Neither X nor Y need be Normal
- X and Y don't need to be the same type of data
- X and Y don't need to have the same time scale
- Easily handles missing data (gaps) and irregularly spaced data
- Easily handles multiple data sources (Y's), which don't need to be the same type or synchronous
- Easily handles time-integrated observations

**Note: "easy" in concept not always equal to short code or fast runtime

Unequal Observation Errors

- Suppose we have an *a prior* reason to believe observation errors were different in different years
  - Different methodology
  - Different sample size
  - QA/QC error estimate

$$\tau_t^2 \sim IG\left(\alpha_t + \frac{1}{2}, \beta_t + \frac{1}{2}(y_t - x_t)^2\right)$$

Defining prior differently by year

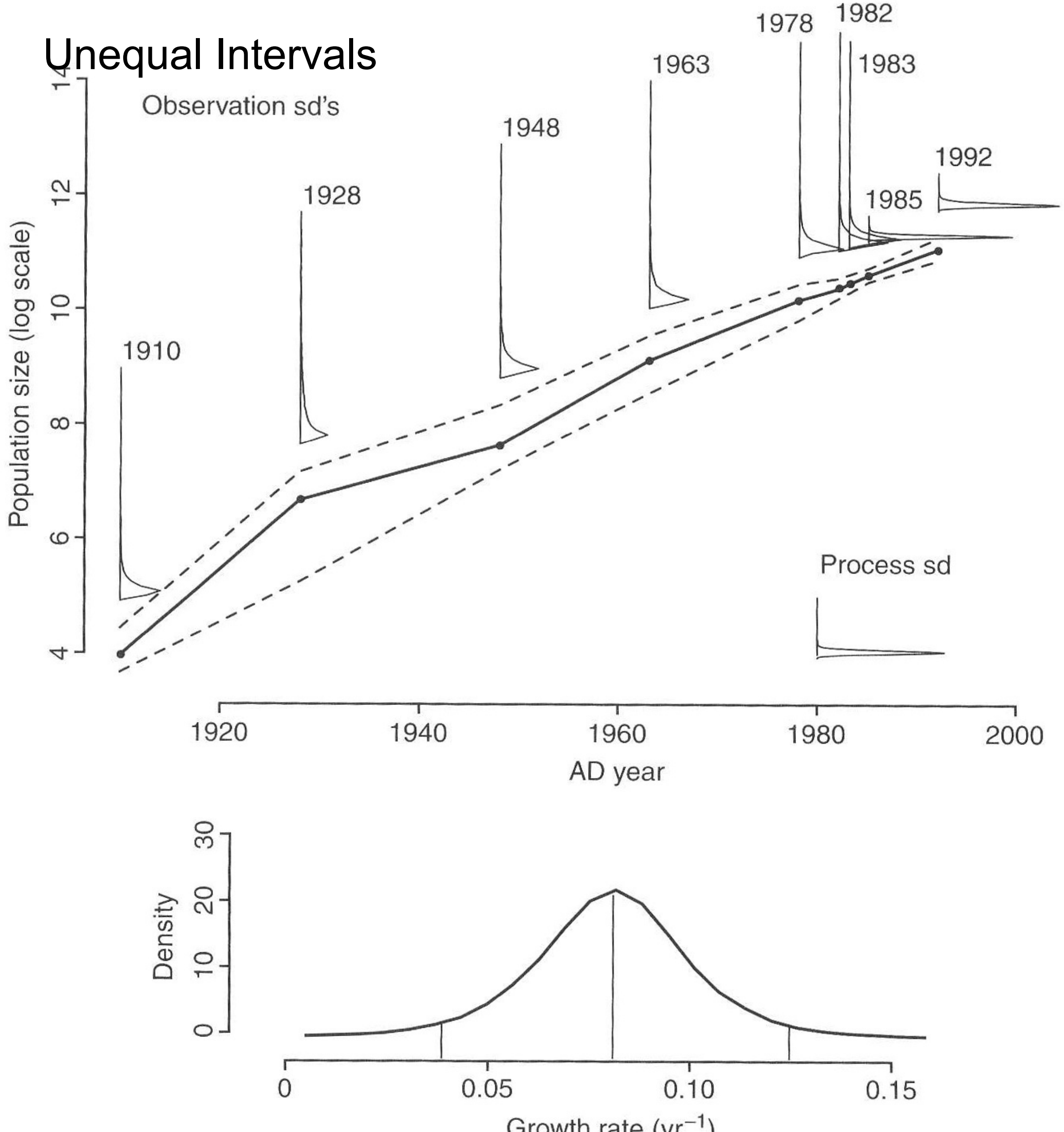
Unequal Sample Intervals

- Option 1: Treat as missing data
  - Generally applicable
- Option 2: Include time step in process model
  - Problem specific solution

$$X_t = X_{t-\Delta t_i} + (r + \epsilon_t) \cdot \Delta t_i$$

Example: Black Noddy (*Anous minutus*)

Unequal Intervals



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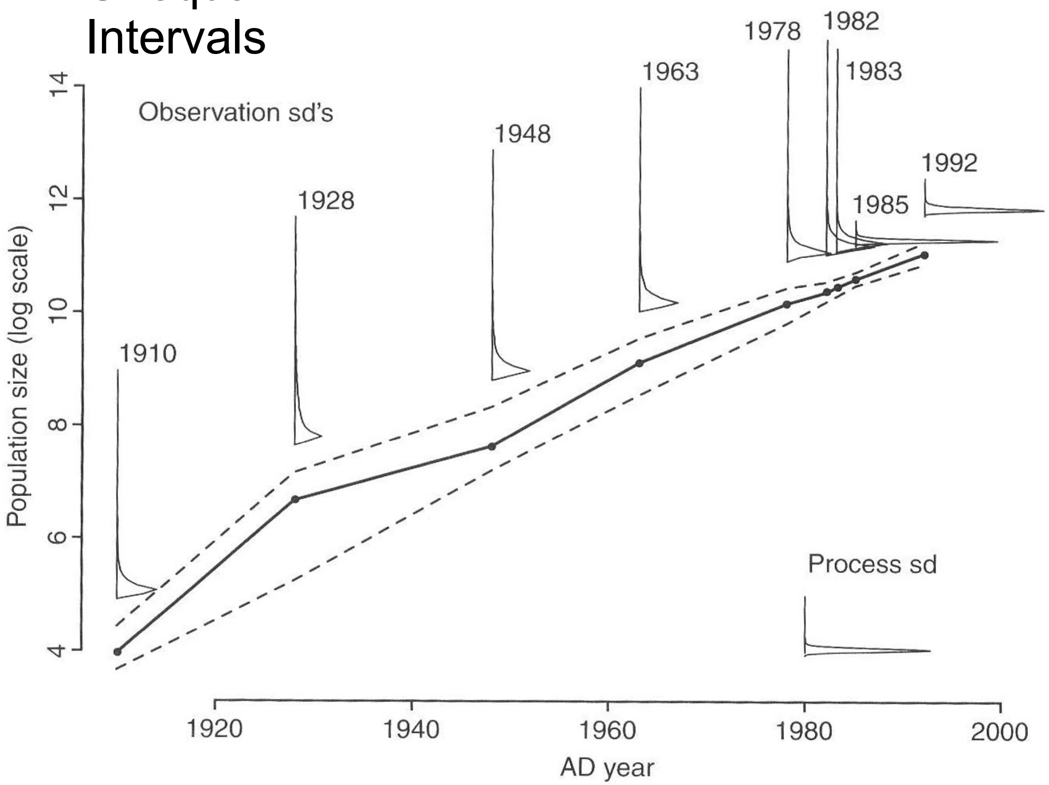
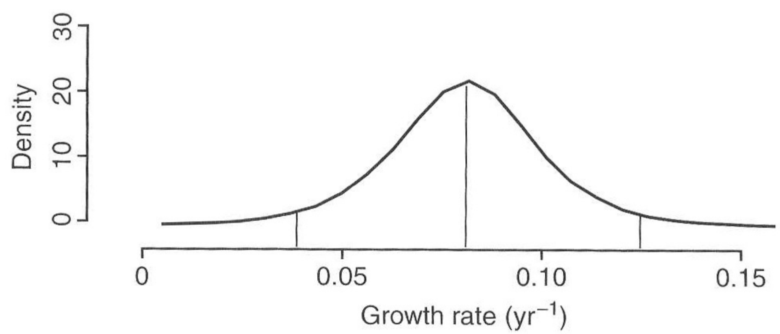
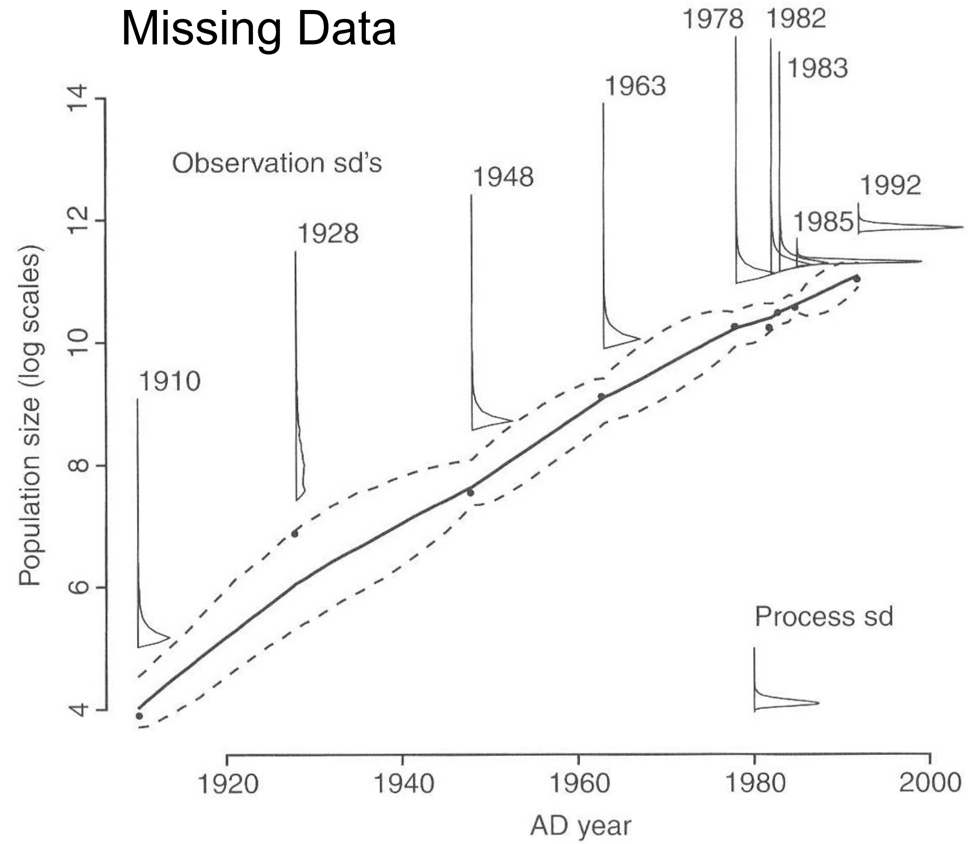
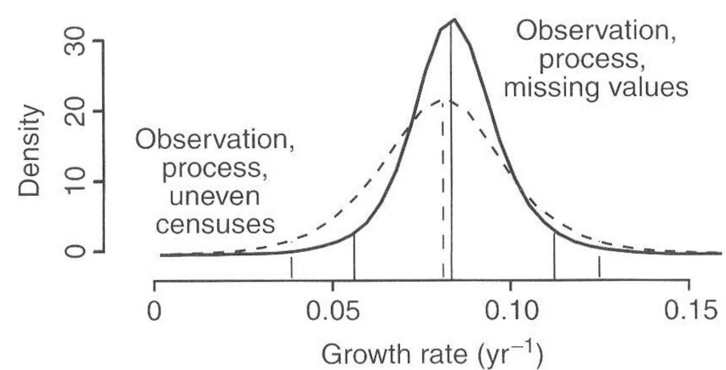
Unequal Intervals





Missing Data





Nonlinear State Space: Density Dependence

Exponential

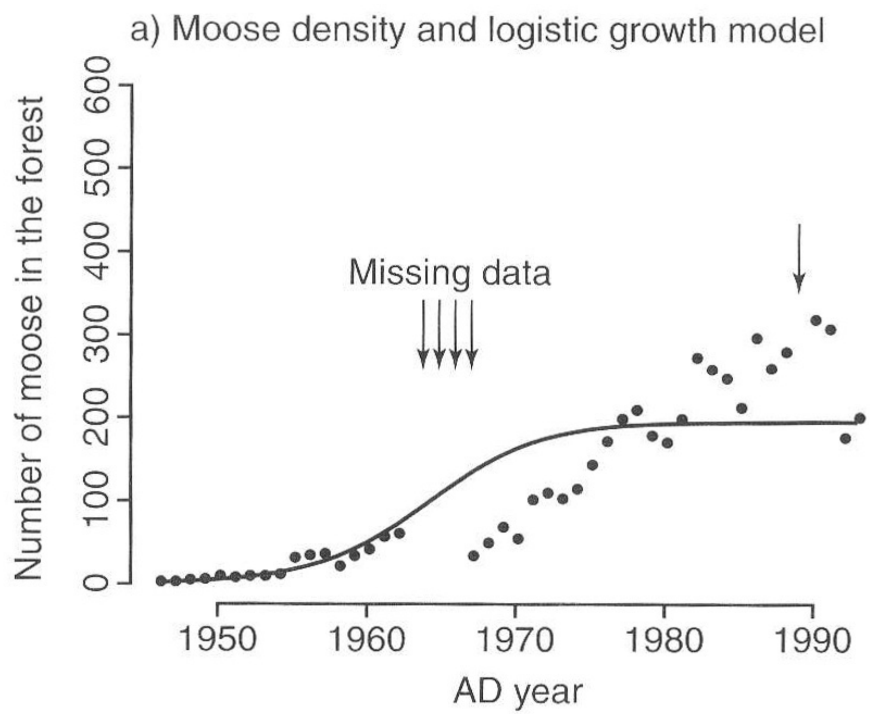
$$N_{t+1} = N_t \cdot e^{r+\epsilon_t} \qquad X_{t+1} = X_t + r + \epsilon_t$$

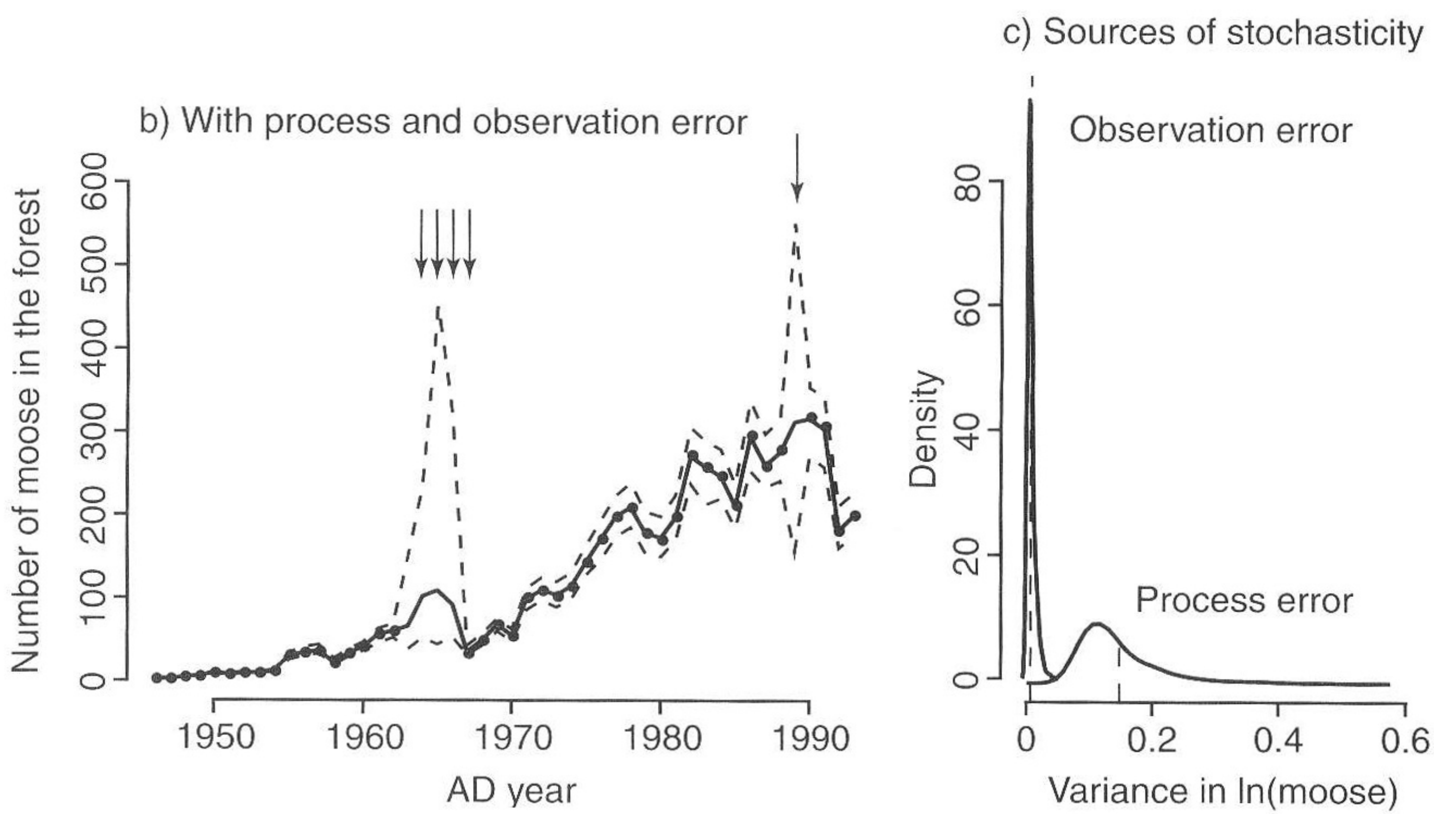
Ricker Discrete Logistic

$$N_{t+1} = N_t \cdot e^{r\left(1-N_t/K\right)+\epsilon_t} \qquad X_{t+1} = X_t + r\left(1-N_t/K\right) + \epsilon_t$$

- K needs a positive continuous prior (e.g. lognormal, gamma)
- Metropolis-Hastings sampling for K

Bialowieza Primeval Forest (BPF) Moose





Capture-Recapture

- Individuals captured, marked, and released
- Over repeated censuses will recapture some fraction of the population
- Recapture is random and <100%
- Interested in demography (survival, reproduction, growth) and population size
- Very common with animal data

Missing Data

- Suppose an individual record consists of the following capture data
$$Y_i = [1,0,1,0,0]$$
- This is compatible with the following survival
$$Z_i = [1,1,1,0,0]$$
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- Don't know the exact time of death
- DO know the second census was just a failure to recapture

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Allow estimation of capture probability

- Don't know the exact time of death
- DO know the second census was just a failure to recapture

Basic Mark-Recapture State Space

- Process model

$$P(X_t=1|X_{t-1}=1)=s_t$$
$$P(X_t=1|X_{t-1}=0)=0$$
$$P(X_t=0|X_{t-1}=1)=1-s_t$$
$$P(X_t=0|X_{t-1}=0)=1$$

Bernoulli Survival Probability

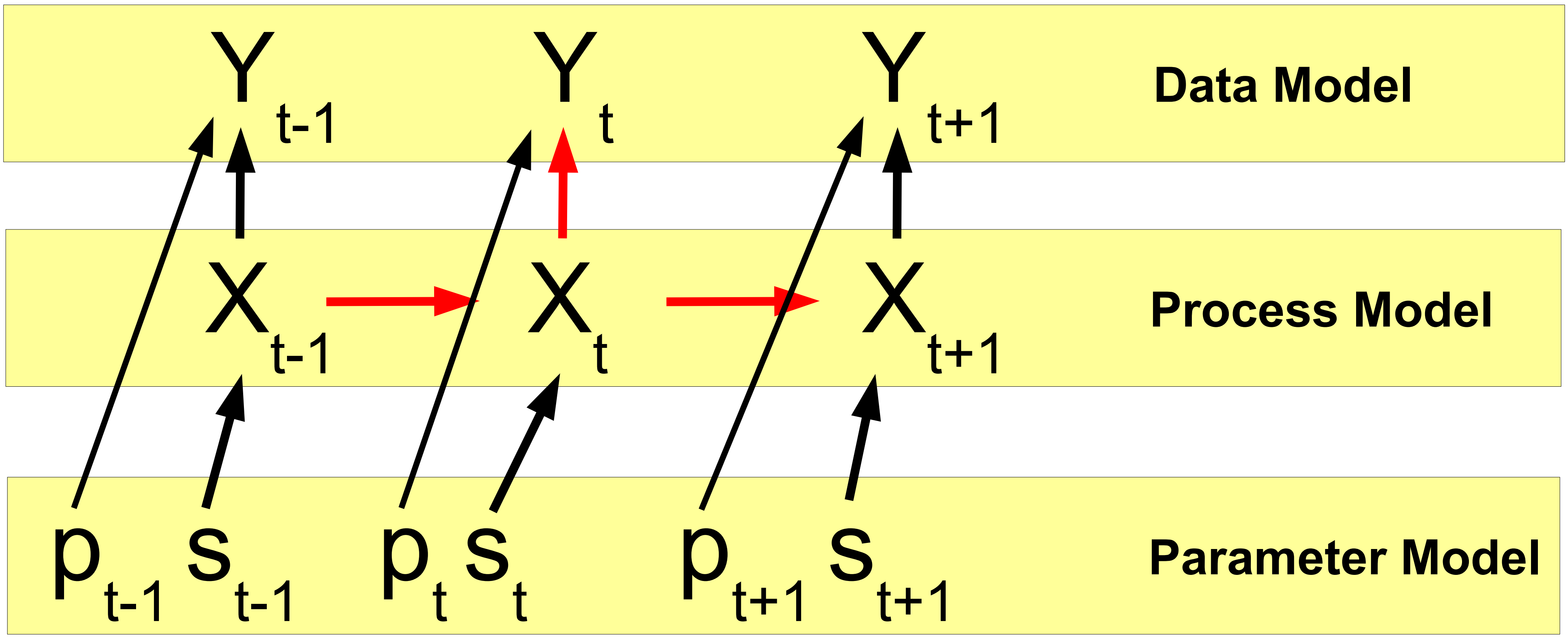
- Observation model

$$P(Y_t=1|X_t=1)=p_t$$
$$P(Y_t=1|X_t=0)=0$$
$$P(Y_t=0|X_t=0)=1$$
$$P(Y_t=0|X_t=1)=1-p_t$$

Bernoulli Detection Probability

- Priors on p and s (e.g. Beta)

Mark Recapture State Space



Sampling

- As with previous State-Space, update state variables sequentially based on previous X, next X, and current Y.
  - Don't need to update values if state is known
  - State is binomial [0,1]
- Survival and capture probabilities are Beta-Binomial (Gibbs)
  - Don't double count dead (only die once)

Extensions

- Current model assumes p and s vary with time
- Could assume a common, time-invariant p and/or s
- Could assume a hierarchical p and/or s
- Could make either a function of covariates (e.g. GLMM)

Note: no J

$$logit(s) = Z\beta + \alpha$$

Covariates

Fixed effects

Random effects

Mark Recapture State Space