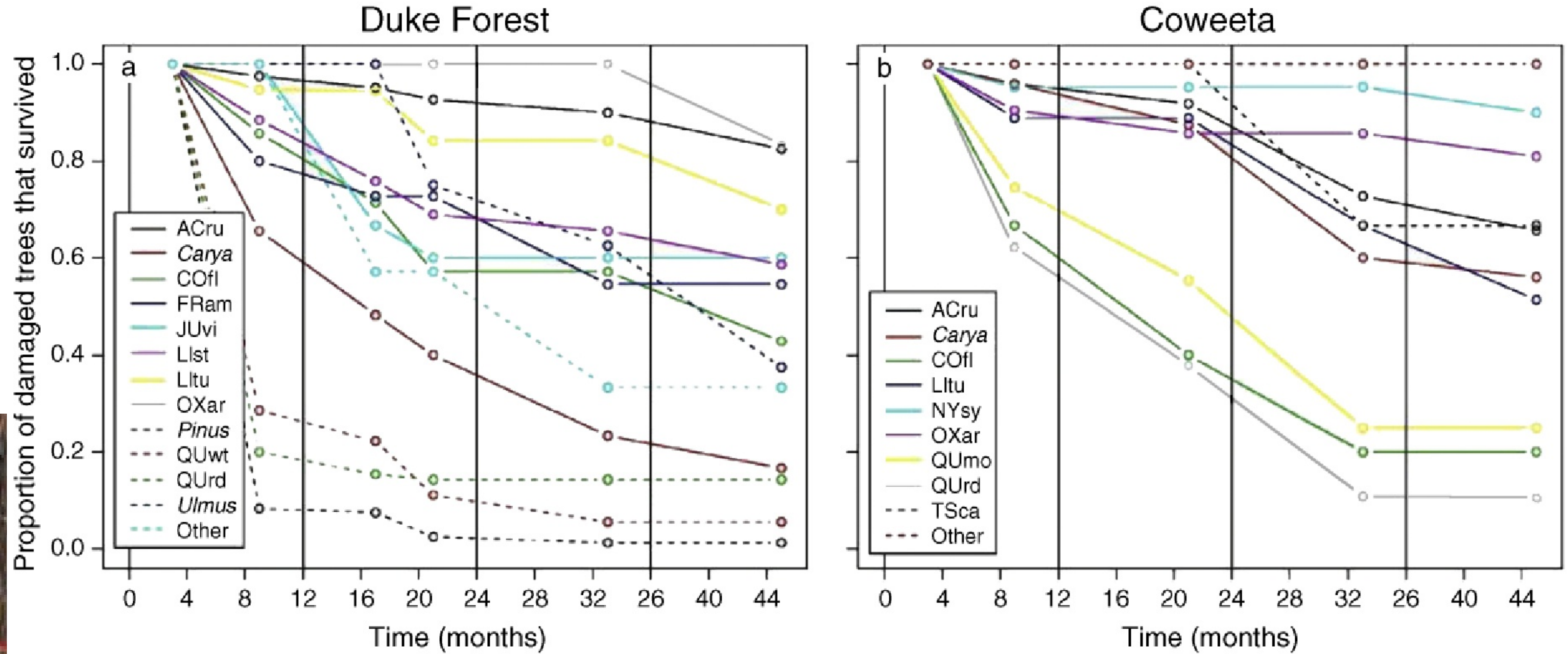
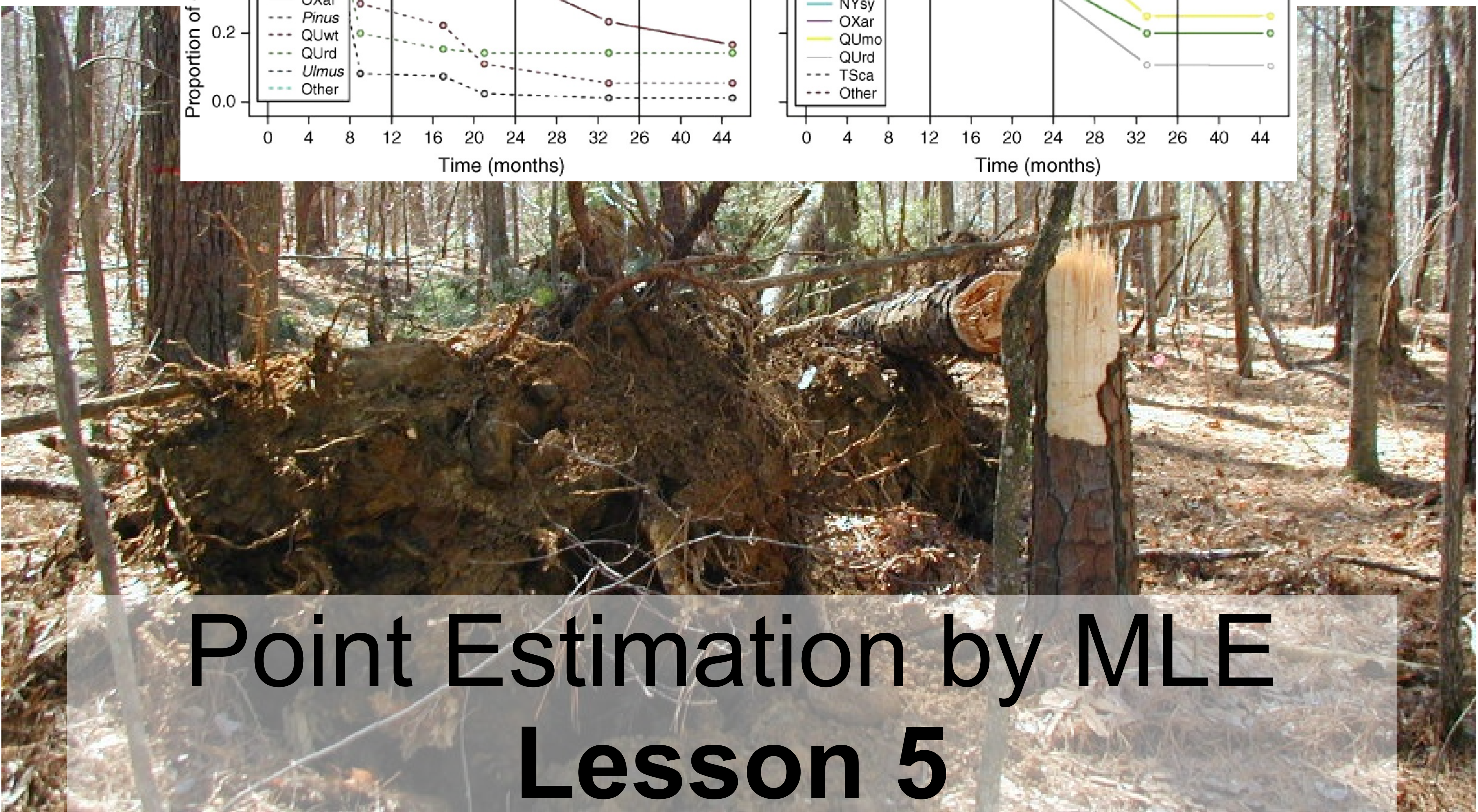




Point Estimation by MLE

Lesson 5

Review

- Defined Likelihood
- Maximum Likelihood Estimation
  - Step 1: Construct Likelihood
  - Step 2: Maximize function
    - Take Log of likelihood function
    - Take derivative of function
    - Set derivative = 0
    - Solve for parameter

Point estimation

- **Goal: 'fit' a model to data**
- Single "best" value of a parameter that finds most support in the data set
- Contrasts with interval estimation (e.g. 95% confidence interval)
- Examples:

$$\mu_{SS} = \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

$$\sigma_{SS}^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2$$

Maximum Likelihood

- Step 1: Write a likelihood function describing the likelihood of the observation
- Step 2: Find the value of the model parameter that maximized the likelihood

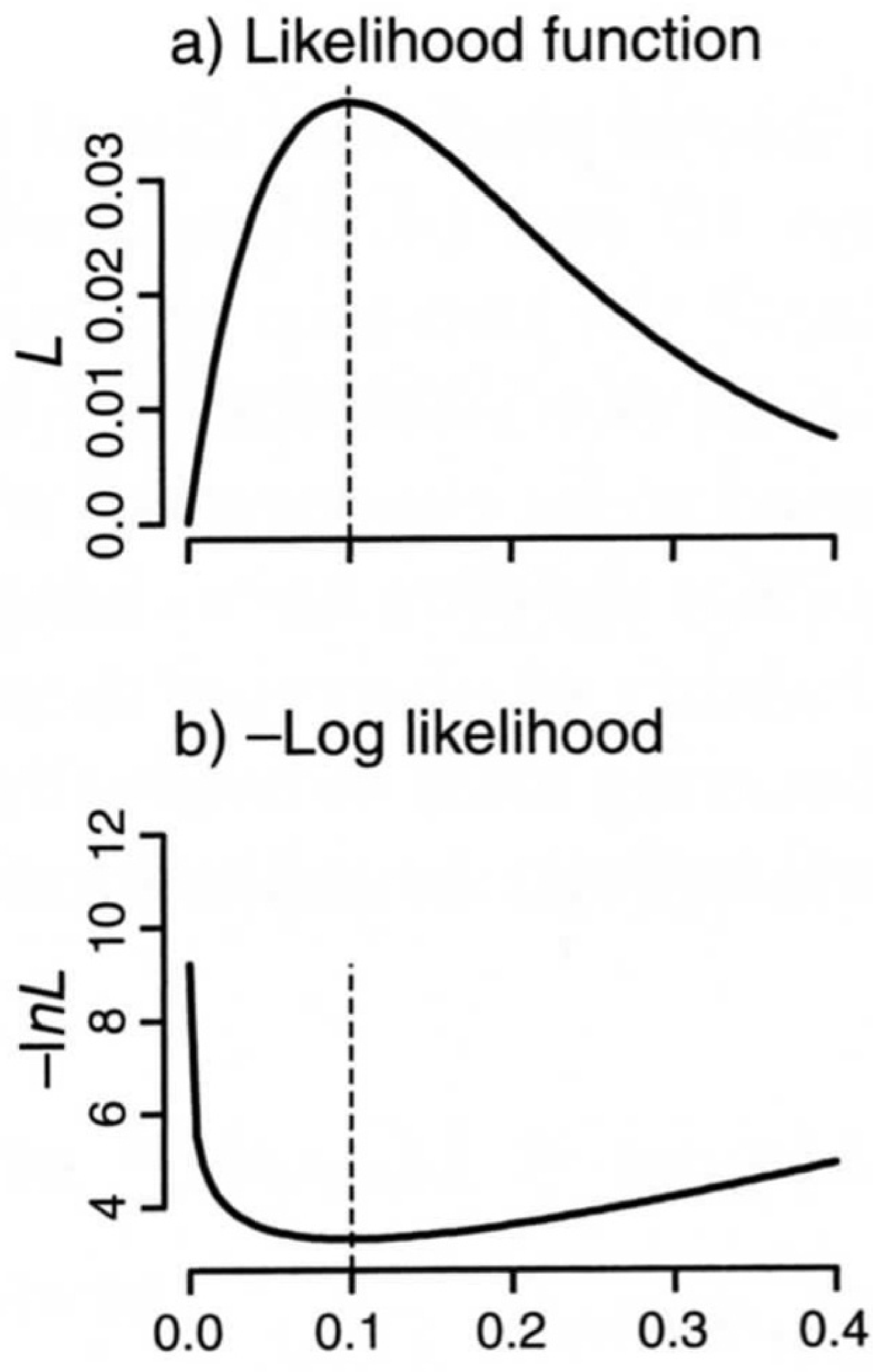
$$\frac{dL}{d\rho} = 0$$

$$L = \rho\, e^{-\rho a}$$

$$\ln L = \ln \rho - \rho a$$

$$\frac{\partial \ln L}{\partial \rho} = \frac{1}{\rho} - a = 0$$

$$\rho_{ML} = \frac{1}{a} = 0.1\, day^{-1}$$


a) Likelihood function
L
0.0
0.01
0.02
0.03
b) –Log likelihood
–lnL
4
6
8
10
12
0.0
0.1
0.2
0.3
0.4

A second data point

- Suppose a second plant dies at day 14
- Step 1: Define the likelihood

$$
\begin{aligned}
L &= Pr(a_{1,} a_2 | \rho) \\
&= Pr(a_2 | a_{1,} \rho) Pr(a_1 | \rho) \\
&= Pr(a_2 | \rho) Pr(a_1 | \rho) \\
&\propto \mathrm{Exp}(a_2 | \rho) \mathrm{Exp}(a_1 | \rho)
\end{aligned}
$$

Assume measurements are independent

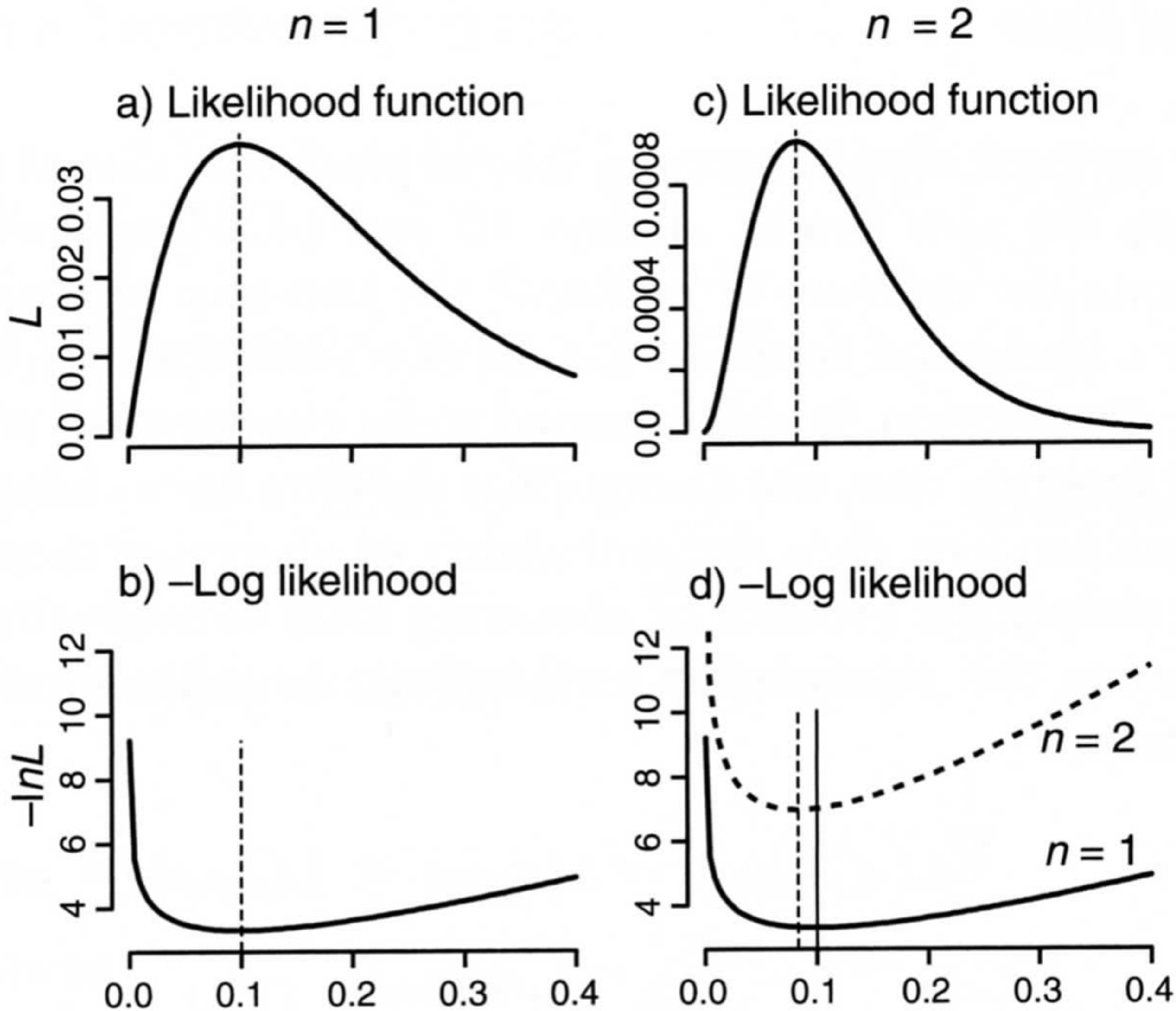
- Step 2: Find the maximum

$$L = \rho e^{-\rho a_1} \cdot \rho e^{-\rho a_2}$$

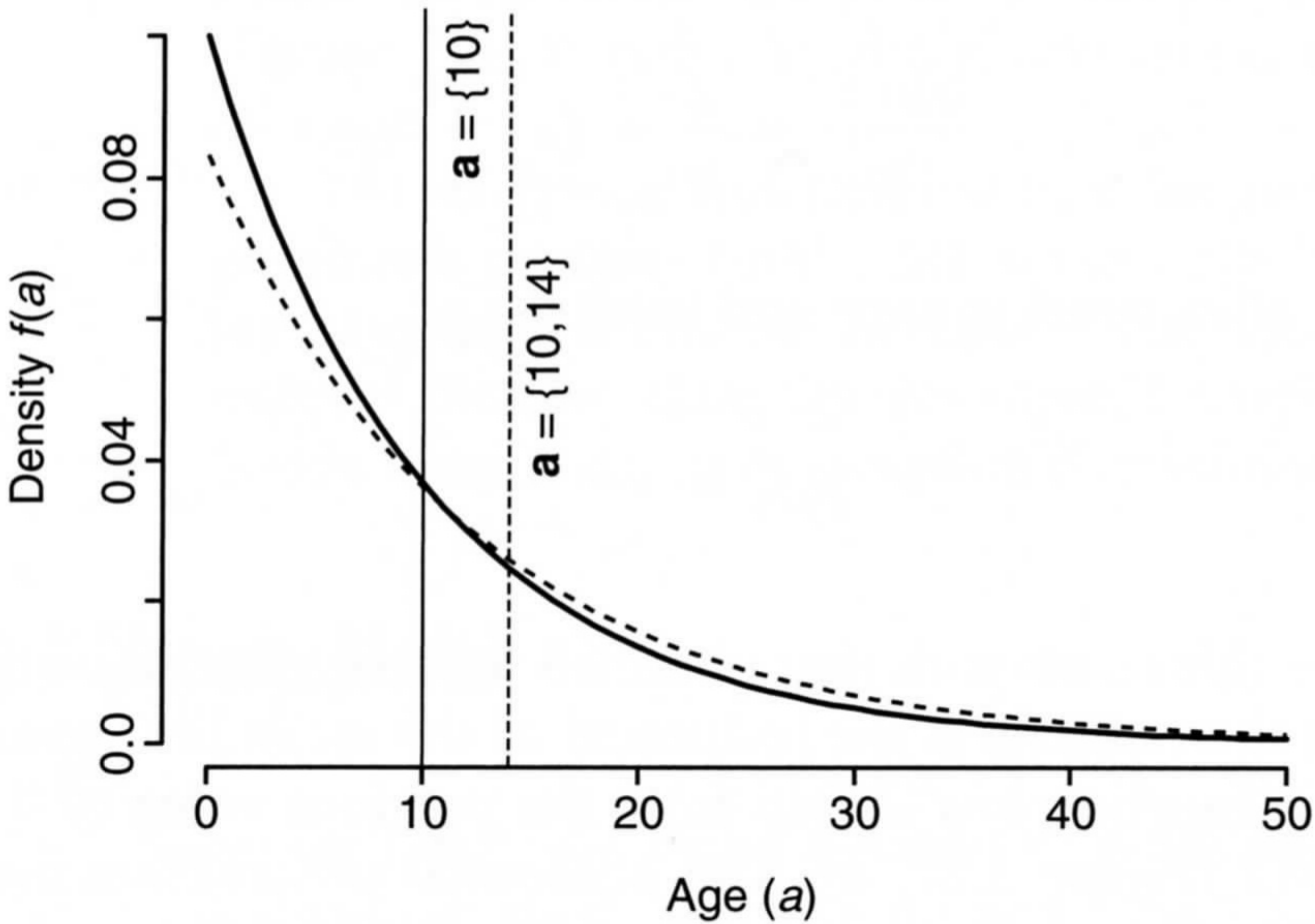
$$\ln L = 2 \ln \rho - \rho a_1 - \rho a_2$$

$$\frac{\partial \ln L}{\partial \rho} = \frac{2}{\rho} - (a_1 + a_2) = 0$$

$$\rho_{ML} = \frac{2}{a_1 + a_2} = 0.0833\, day^{-1}$$



n = 1
n = 2
a) Likelihood function
c) Likelihood function
L
0.0 0.01 0.02 0.03
0.0 0.0004 0.0008
b) –Log likelihood
d) –Log likelihood
–lnL
4 6 8 10 12
n = 2
n = 1
0.0 0.1 0.2 0.3 0.4



Density f(a)
0.0
0.04
0.08
a = {10}
a = {10,14}
0
10
20
30
40
50
Age (a)

A whole data set

- Step 1: Define Likelihood

$$L = Pr\left(a_{1,} a_{2,} \cdots, a_n | \rho\right)$$

Assume measurements are independent

$$= \prod_{i=1}^{n} Pr\left(a_i | \rho\right)$$

$$= \prod_{i=1}^{n} \mathrm{Exp}\left(a_i | \rho\right)$$

- Step 2:
  Find the maximum

$$L = \prod_{i=1}^{n} \rho\, e^{-\rho\, a_i}$$

$$\ln L = \sum_{i=1}^{n} \left(\ln \rho - \rho\, a_i \right)$$

$$= n \ln \rho - \rho \sum_{i=1}^{n} a_i$$

$$\frac{\partial \ln L}{\partial \rho} = \frac{n}{\rho} - \sum_{i=1}^{n} a_i = 0$$

$$\rho_{ML} = \frac{n}{\sum_{i=1}^{n} a_i} = 1/\bar{a}$$

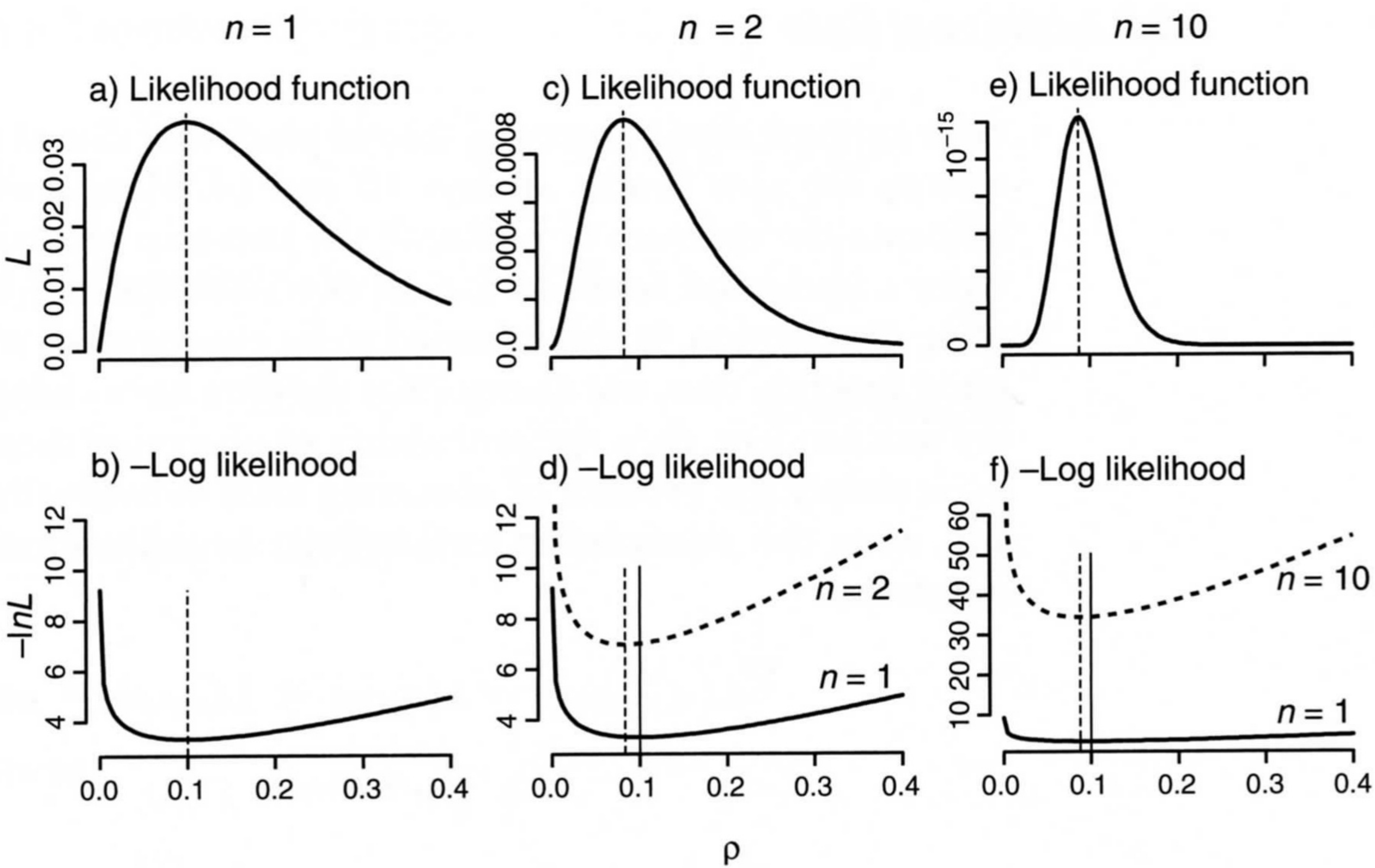


FIGURE 3.2. Likelihood functions for the exponential model with three different sample sizes. Note the different scales on the vertical axes.

Mortality – Alternate data

- First example based on knowing times individuals died
- What if instead we only knew the final outcome
  - N = number of individual plants
  - y = number that survived
- Want to find the MLE for
  - θ = probability of surviving to the end of the expm't

- Step 1: Write the likelihood

$$L = \text{Binom}(y|N, \theta) \propto \theta^{y}(1-\theta)^{N-y}$$

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- Step 2: Find the maximum

$$lnL = y\ln\theta + (N-y)\ln(1-\theta)$$

- Step 1: Write the likelihood

$$L = \mathrm{Binom}(y|N,\theta) \propto \theta^{y}(1-\theta)^{N-y}$$

- Step 2: Find the maximum

$$lnL = y\ln\theta + (N-y)\ln(1-\theta)$$

$$\frac{\partial\, lnL}{\partial\theta} = \frac{y}{\theta} - \frac{(N-y)}{(1-\theta)}$$

- Step 1: Write the likelihood

$$L = \text{Binom}(y|N,\theta) \propto \theta^{y}(1-\theta)^{N-y}$$

- Step 2: Find the maximum

$$lnL = y\ln\theta + (N-y)\ln(1-\theta)$$

$$\frac{\partial\, lnL}{\partial\,\theta} = \frac{y}{\theta} - \frac{(N-y)}{(1-\theta)}$$

$$(1-\theta)\,y = \theta(N-y)$$

- Step 1: Write the likelihood

$$L = \text{Binom}(y|N,\theta) \propto \theta^{y}(1-\theta)^{N-y}$$

- Step 2: Find the maximum

$$\begin{aligned}
lnL &= y\ln\theta + (N-y)\ln(1-\theta) \\
\frac{\partial\, lnL}{\partial\theta} &= \frac{y}{\theta} - \frac{(N-y)}{(1-\theta)} \\
(1-\theta)y &= \theta(N-y) \\
\theta_{ML} &= \frac{y}{N}
\end{aligned}$$

Mortality – combining age of death and survival till end

- Step 1 – Write the likelihood

$$L = \mathrm{Binom}(y|N,\theta) \propto \theta^{y}(1-\theta)^{N-y}$$

Mortality – combining age of death and survival till end

- Step 1 – Write the likelihood

$$L = \mathrm{Binom}(y|N, \theta) \propto \theta^{y}(1-\theta)^{N-y}$$

$$\theta = P(\text{survive to T}|\rho) = 1 - P(\text{dead by T}|\rho)$$

Mortality – combining age of death and survival till end

- Step 1 – Write the likelihood

$$L = \text{Binom}(y|N,\theta) \propto \theta^{y}(1-\theta)^{N-y}$$

$$\theta = P(\text{survive to T}|\rho) = 1 - P(\text{dead by T}|\rho)$$

$$= 1 - (1 - e^{-\rho T}) = e^{-\rho T}$$

Exponential CDF

Mortality – combining age of death and survival till end

- Step 1 – Write the likelihood

$$L = \text{Binom}(y|N,\theta) \propto \theta^{y}(1-\theta)^{N-y}$$

$$\theta = P(\text{survive to T}|\rho) = 1 - P(\text{dead by T}|\rho)$$

$$= 1-(1-e^{-\rho T}) = e^{-\rho T}$$

$$L = \text{Binom}(y|N,\rho) \propto e^{-\rho T y}(1-e^{-\rho T})^{N-y}$$

- Step 2: Find the maximum

$$L = e^{-\rho T y}\left(1 - e^{-\rho T}\right)^{N-y}$$

- Step 2: Find the maximum

$$L = e^{-\rho T y}\left(1-e^{-\rho T}\right)^{N-y}$$

$$lnL \quad = \quad -y\rho T+(N-y)\ln\left(1-e^{-\rho T}\right)$$

- Step 2: Find the maximum

$$L = e^{-\rho T y}\left(1 - e^{-\rho T}\right)^{N-y}$$

$$lnL = -y\rho T + (N-y)\ln\left(1 - e^{-\rho T}\right)$$

$$\frac{\partial\, lnL}{\partial \rho} = -yT - (N-y)\frac{Te^{-\rho T}}{1 - e^{-\rho T}}$$

- Step 2: Find the maximum

$$L = e^{-\rho T y}\left(1 - e^{-\rho T}\right)^{N-y}$$

$$lnL = -y\rho T + (N-y)\ln\left(1-e^{-\rho T}\right)$$

$$\frac{\partial lnL}{\partial \rho} = -yT - (N-y)\frac{Te^{-\rho T}}{1-e^{-\rho T}}$$

$$\rho_{ML} = \frac{-\ln(y/N)}{T}$$

Survival Analysis

$$L = \text{Binom}(y|N,\theta) \propto \theta^{y}(1-\theta)^{N-y}$$

Died **Survived**

Survival Analysis

$$L = \mathrm{Binom}(y|N,\theta) \propto \theta^{y}(1-\theta)^{N-y}$$

$$L = \prod_{i=1}^{y} \theta_i \times \prod_{i=1}^{n-y} (1-\theta_i)$$

Survival Analysis

$$L = \text{Binom}(y|N, \theta) \propto \theta^{y}(1-\theta)^{N-y}$$

$$L = \prod_{i=1}^{y} \theta_i \times \prod_{i=1}^{n-y} (1-\theta_i)$$

Mortality at time a

$$L = \prod_{i=1}^{y} \rho e^{-\rho a_i} \times \ldots$$

Survival Analysis

$$L = \text{Binom}(y|N,\theta) \propto \theta^{y}(1-\theta)^{N-y}$$

$$L = \prod_{i=1}^{y}\theta_i \times \prod_{i=1}^{n-y}(1-\theta_i)$$

Mortality at time a

Censored at time c

$$L = \prod_{i=1}^{y}\rho\, e^{-\rho a_i} \times \prod_{i=1}^{k} e^{-\rho c_i} \times \ldots$$

Survival Analysis

$$L = \text{Binom}(y|N,\theta) \propto \theta^{y}(1-\theta)^{N-y}$$

$$L = \prod_{i=1}^{y} \theta_i \times \prod_{i=1}^{n-y} (1-\theta_i)$$

Mortality at time a

Censored at time c

Survived to time T

$$L = \prod_{i=1}^{y} \rho e^{-\rho a_i} \times \prod_{i=1}^{k} e^{-\rho c_i} \times e^{-\rho T(n-y-k)}$$

$$L = \prod_{i=1}^{y} \rho e^{-\rho a_i} \times \prod_{i=1}^{k} e^{-\rho c_i} \times e^{-\rho T(n-y-k)}$$

$$L = \prod_{i=1}^{y} \rho e^{-\rho a_i} \times \prod_{i=1}^{k} e^{-\rho c_i} \times e^{-\rho T(n-y-k)}$$

$$lnL = y \ln \rho - \rho \sum_{i=1}^{y} a_i - \rho \sum_{i=1}^{k} c_i - \rho T(n-y-k)$$

$$L = \prod_{i=1}^{y} \rho e^{-\rho a_i} \times \prod_{i=1}^{k} e^{-\rho c_i} \times e^{-\rho T(n-y-k)}$$

$$lnL = y \ln \rho - \rho \sum_{i=1}^{y} a_i - \rho \sum_{i=1}^{k} c_i - \rho T(n-y-k)$$

$$\frac{\partial lnL}{\partial \rho} = \frac{y}{\rho} - \sum_{i=1}^{y} a_i - \sum_{i=1}^{k} c_i - T(n-y-k) = 0$$

$$L = \prod_{i=1}^{y} \rho e^{-\rho a_i} \times \prod_{i=1}^{k} e^{-\rho c_i} \times e^{-\rho T(n-y-k)}$$

$$lnL = y \ln \rho - \rho \sum_{i=1}^{y} a_i - \rho \sum_{i=1}^{k} c_i - \rho T(n-y-k)$$

$$\frac{\partial lnL}{\partial \rho} = \frac{y}{\rho} - \sum_{i=1}^{y} a_i - \sum_{i=1}^{k} c_i - T(n-y-k) = 0$$

$$\rho_{ML} = \frac{y}{\sum_{i=1}^{y} a_i + \sum_{i=1}^{k} c_i + T(n-y-k)}$$

MLE for the Normal

$$L = \prod_{i=1}^{n} \mathrm{N}(y_i | \mu, \sigma^2)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(y_i - \mu)^2}{2\sigma^2}\right]$$

$$\mathrm{L} = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[\frac{-(y_i-\mu)^2}{2\sigma^2}\right]$$

$$lnL = -\frac{n}{2}\ln(2\pi) - n\ln\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2$$

$$\mathrm{L} = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(y_i-\mu)^2}{2\sigma^2}\right]$$

$$lnL = -\frac{n}{2}\ln(2\pi) - n\ln\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2$$

$$\frac{\partial \ln L}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i-\mu) = 0$$

$$L = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(y_i - \mu)^2}{2\sigma^2}\right]$$

$$lnL = -n\ln\sigma - \frac{n}{2}\ln(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2$$

$$\frac{\partial \ln L}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - \mu) = 0$$

$$\sum_{i=1}^{n} y_i = n\mu$$

$$\mathrm{L} = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(y_i-\mu)^2}{2\sigma^2}\right]$$

$$lnL = -n\ln\sigma - \frac{n}{2}\ln(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2$$

$$\frac{\partial \ln L}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i-\mu) = 0$$

$$\sum_{i=1}^{n} y_i = n\mu$$

$$\mu_{ML} = \bar{y}$$

Normal Variance

$$lnL \quad = \quad -n\ln\sigma - \frac{n}{2}\ln(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2$$

Normal Variance

$$lnL = -n \ln \sigma - \frac{n}{2} \ln(2\pi) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2$$

$$\frac{\partial \ln L}{\partial \sigma} = \frac{-n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} (y_i - \mu)^2 = 0$$

Normal Variance

$$lnL = -n\ln\sigma - \frac{n}{2}\ln(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2$$

$$\frac{\partial \ln L}{\partial \sigma} = \frac{-n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(y_i - \mu)^2 = 0$$

$$\frac{n}{\sigma} = \frac{1}{\sigma^3}\sum_{i=1}^{n}(y_i - \mu)^2$$

Normal Variance

$$lnL = -n\ln\sigma - \frac{n}{2}\ln(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2$$

$$\frac{\partial \ln L}{\partial \sigma} = \frac{-n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(y_i-\mu)^2 = 0$$

$$\frac{n}{\sigma} = \frac{1}{\sigma^3}\sum_{i=1}^{n}(y_i-\mu)^2$$

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i-\mu)^2$$

Population Growth Rate

$$n_{t+1} = n_t \lambda = n_t e^{r}$$

$$n_{t+1} = n_t e^{r+\epsilon_t}$$

$$\ln n_{t+1} = \ln n_t + r + \epsilon_t$$

$$x_{t+1} = x_t + r + \epsilon_t$$

$$\epsilon_t \sim N(0, \sigma^2)$$

$$x_{t+1} \sim N(x_t + r, \sigma^2)$$

- Step 1: Write Likelihood

$$L = \prod_{t=1}^{T} N(x_{t+1} | x_t + r, \sigma^2)$$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma} \right)^T \exp\left[\frac{-1}{2\sigma^2} \sum_{t=1}^{T} (x_{t+1} - x_t - r)^2 \right]$$

- Step 1: Write Likelihood

$$L = \prod_{t=1}^{T} N\left(x_{t+1} | x_t + r, \sigma^2\right)$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^T \exp\left[\frac{-1}{2\sigma^2}\sum_{t=1}^{T}\left(x_{t+1} - x_t - r\right)^2\right]$$

- Step 2: Maximize Likelihood
  - Take logs

$$\ln L = -T \ln \sigma - \frac{1}{2\sigma^2}\sum_{t=1}^{T}\left(x_{t+1} - x_t - r\right)^2$$

Solve for mean

$$\ln L = -T \ln \sigma - \frac{1}{2\sigma^2} \sum_{t=1}^{T} (x_{t+1} - x_t - r)^2$$

$$\frac{\partial \ln L}{\partial r} = \frac{1}{\sigma} \sum_{t=1}^{T} (x_{t+1} - x_t - r) = 0$$

Solve for mean

$$\ln L = -T \ln \sigma - \frac{1}{2\sigma^2} \sum_{t=1}^{T} (x_{t+1} - x_t - r)^2$$

$$\frac{\partial \ln L}{\partial r} = \frac{1}{\sigma} \sum_{t=1}^{T} (x_{t+1} - x_t - r) = 0$$

$$T r = \sum_{t=1}^{T} (x_{t+1} - x_t)$$

Solve for mean

$$\ln L = -T \ln \sigma - \frac{1}{2\sigma^2} \sum_{t=1}^{T} (x_{t+1} - x_t - r)^2$$

$$\frac{\partial \ln L}{\partial r} = \frac{1}{\sigma} \sum_{t=1}^{T} (x_{t+1} - x_t - r) = 0$$

$$T r = \sum_{t=1}^{T} (x_{t+1} - x_t)$$

$$r_{ML} = \frac{x_T - x_0}{T}$$

Solve for variance

$$\ln L = -T \ln \sigma - \frac{1}{2\sigma^2} \sum_{t=1}^{T} (x_{t+1} - x_t - r)^2$$

Solve for variance

$$\ln L = -T \ln \sigma - \frac{1}{2\sigma^2} \sum_{t=1}^{T} \left(x_{t+1} - x_t - r \right)^2$$

$$\frac{\partial \ln L}{\partial \sigma} = \frac{-T}{\sigma} + \frac{1}{\sigma^3} \sum_{t=1}^{T} \left(x_{t+1} - x_t - r \right)^2 = 0$$

Solve for variance

$$\ln L = -T \ln \sigma - \frac{1}{2\sigma^2} \sum_{t=1}^{T} (x_{t+1} - x_t - r)^2$$

$$\frac{\partial \ln L}{\partial \sigma} = \frac{-T}{\sigma} + \frac{1}{\sigma^3} \sum_{t=1}^{T} (x_{t+1} - x_t - r)^2 = 0$$

$$\sigma_{ML}^2 = \frac{1}{T} \sum_{t=1}^{T} (x_{t+1} - x_t - r)^2$$

Homework

- You want to know the density of fish in a set of experimental ponds
- You observe the following counts in ten ponds: 5,6,7,3,6,5,8,4,4,3
- What is your process model?
- What is your data model?
- Solve for the analytical MLE
- What is the estimate for this population?