Appendix 1

Commonly Used Mathematical Rules

A1.1 Rules for Algebraic Functions

The following rules help to simplify functions involving powers and fractions:

Rule A1.1: $a^x a^y = a^{x+y}$

Rule A1.2: $\frac{1}{a^y} = a^{-y}$

Rule A1.3: $\frac{a^x}{a^y} = a^{x-y}$

Rule A1.4: $(a^x)^y = a^{xy}$

Rule A1.5: $a^{1/x} = \sqrt[x]{a}$; in particular, $a^{1/2} = \sqrt{a}$

Rule A1.6: $(a^x)^{1/x} = a$

Rule A1.7: $(a\,c)^x = a^x c^x$

Rule A1.8: $\left(\frac{a}{c}\right)^x = \frac{a^x}{c^x}$

Rule A1.9: $\frac{1}{(a_1 + b_1 x)(a_2 + b_2 x)} = \frac{A}{(a_1 + b_1 x)} + \frac{B}{(a_2 + b_2 x)}$ (partial fractions)

where $A = -\frac{b_1}{(a_1 b_2 - a_2 b_1)}$ and $B = \frac{b_2}{(a_1 b_2 - a_2 b_1)}$

Rule A1.10: If $a\,x^2 + b\,x + c = 0$ then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (quadratic formula)

A1.2 Rules for Logarithmic and Exponential Functions

The following rules help to simplify functions involving logarithms. On the left are rules for logarithms in any base, b, defined by the fact that if $y = b^x$ then $\log_b(y) = x$. On the right are specific rules for natural logs in base $e = 2.71\ldots$, defined by the fact that if $y = e^x$ then $\ln(y) = \log_e(y) = x$.

Rule A1.11: $\log_b(a^t) = t\,\log_b(a)$ $\qquad$ $\ln(a^t) = t\,\ln(a)$

Rule A1.12: $\log_b(a\,c) = \log_b(a) + \log_b(c)$ $\qquad$ $\ln(a\,c) = \ln(a) + \ln(c)$

Rule A1.13: $\log_b\left(\frac{1}{c}\right) = -\log_b(c)$ $\qquad$ $\ln\left(\frac{1}{c}\right) = -\ln(c)$

Rule A1.14: $\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c)$ $\quad$ $\ln\left(\frac{a}{c}\right) = \ln(a) - \ln(c)$

Rule A1.15: $\log_b(b^x) = x$ $\quad$ $\ln(e^x) = x$

Rule A1.16: $b^{\log_b(x)} = x$ $\quad$ $e^{\ln(x)} = x$

A1.3 Some Important Sums

The following rules describe how certain sums can be evaluated and written in simpler terms. To interpret a sum, read $\Sigma_{i=1}^{n} f(i)$ as "the sum of the values of $f(i)$ starting with i equals one, then two, then three, etc., up until $i = n$." That is, $\Sigma_{i=1}^{n} f(i) = f(1) + f(2) + f(3) + \cdots + f(n)$. Sums starting with $i = 1$ are given on the left and starting with $i = 0$ on the right.

Rule A1.17: $\sum_{i=1}^{n} a = n\,a$ $\quad$ $\sum_{i=0}^{n} a = (n+1)\,a$

Rule A1.18: $\sum_{i=1}^{n} i = \frac{n\,(n+1)}{2}$ $\quad$ $\sum_{i=0}^{n} i = \frac{n\,(n+1)}{2}$

Rule A1.19: $\sum_{i=1}^{n} a^i = \frac{a - a^{n+1}}{1 - a}$ $\quad$ $\sum_{i=0}^{n} a^i = \frac{1 - a^{n+1}}{1 - a}$

Rule A1.20: $\sum_{i=1}^{\infty} a^i = \frac{a}{1 - a}$ if $|a| < 1$ $\quad$ $\sum_{i=0}^{\infty} a^i = \frac{1}{1 - a}$ if $|a| < 1$

Rule A1.21: $\frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$ $\quad$ (arithmetic mean)

Rule A1.22: $\frac{1}{(n-1)}\sum_{i=1}^{n} (x_i - \bar{x})^2 = \mathrm{Var}(x)$ $\quad$ (sample variance)

Note that, in Rule A1.22, if the variance is based on the true value of the mean (i.e., the mean is known without error), then the sum should be divided by (n) rather than $(n - 1)$. For example, if the x_i values are known for every member of a population rather than just a sample, then the variance is given by $(1/n)\Sigma_{i=1}^{n} (x_i - \bar{x})^2$. The order in which a sum is taken does not matter, so that $\Sigma_{i=1}^{n} (f(i) + g(i)) = \left(\Sigma_{i=1}^{n} f(i)\right) + \left(\Sigma_{i=1}^{n} g(i)\right)$, and constants can always be factored out of a sum, $\Sigma_{i=1}^{n} (a f(i)) = a\Sigma_{i=1}^{n} f(i)$.

A1.4 Some Important Products

The following rules describe how certain products can be evaluated. To interpret a product, read $\Pi_{i=1}^{n} f(i)$ as "the product of the values of $f(i)$ starting with i equals one, then two, then three, etc., up until $i = n$." That is, $\Pi_{i=1}^{n} f(i) = f(1)\, f(2)\, f(3) \cdots f(n)$.

Rule A1.23: $\prod_{i=1}^{n} a = a^n$

Rule A1.24: $\prod_{i=1}^{n} i = n!$ $\quad$ (n factorial)

Rule A1.25: $\prod_{i=1}^{n} a^i = a^{n(n+1)/2}$

Rule A1.26: $\left(\prod_{i=1}^{n} x_i\right)^{1/n} = \sqrt[n]{\prod_{i=1}^{n} x_i} = \bar{x}_h$ (geometric mean)

Note that the order in which a product is taken does not matter, so that $\prod_{i=1}^{n}(f(i)g(i)) = \left(\prod_{i=1}^{n} f(i)\right) \times \left(\prod_{i=1}^{n} g(i)\right)$, and that constants can be factored out of each term in a product, $\prod_{i=1}^{n}(a\, f(i)) = a^n \prod_{i=1}^{n} f(i)$.

A1.5 Inequalities

The following rules are used to simplify functions involving the inequalities "$<$" (less than) and "$>$" (greater than):

Rule A1.27: If $x + a > y + b$, then $x > y + b - a$. The direction of an inequality is unchanged by addition or subtraction.

Rule A1.28: If $x/a > y/b$, then $x > y\, a/b$ if a is positive, while $x < y\, a/b$ if a is negative. The direction of an inequality must be reversed when multiplying or dividing by a negative number.

Exercise A1.1: The following questions review algebraic techniques needed throughout the text.

(a) Solve $2x^2 - 7x + 3 = 0$ for x.
(b) Simplify $((a\, x)^2 - a^2)/(ax - a)$ as much as possible.
(c) Factor both sides of $x^3 - yx^2 = x - y$. What are the three possible values of x that ensure that this equation holds true?
(d) Solve $\ln(x^t) = 1/2$ for t.
(e) Write $\ln(a\, x) + \ln(b\, x) - \ln(c)$ as a single logarithmic function ln( ).
(f) Solve $x^{rt} = 100$ for t. [Hint: take the logarithm of both sides.]
(g) What does the sum $\sum_{i=1}^{n} 1$ equal?
(h) What does the product $\prod_{i=1}^{n} 1$ equal?
(i) Evaluate and simplify $\sum_{i=1}^{n} (2i - 1)$.
(j) If $x/(-3) + 5 > 15$, is x greater than some number or less than some number? What is that number?

Answers to Exercise

Exercise A1.1

(a) Using the quadratic formula (Rule A1.10) with $a = 2$, $b = -7$, and $c = 3$, the two solutions of $2x^2 - 7x + 3 = 0$ are $x = \left(7 \pm \sqrt{49 - 4(2)(3)}\right)/4 = (7 \pm 5)/4$. That is, $x = 3$ and $x = 1/2$. Alternatively, we could try to factor $2x^2 - 7x + 3$ in various combinations to show that it equals $(2x - 1)(x - 3)$, which gives the same answer.

(b) $$\frac{(a\,x)^2 - a^2}{a\,x - a} = \frac{a^2x^2 - a^2}{a\,x - a} = \frac{a^2(x^2 - 1)}{a\,(x - 1)} = \frac{a^2(x + 1)(x - 1)}{a\,(x - 1)}.$$
$$= a\,(x + 1)$$

It is worth remembering that $(x^2 - 1)$ can be factored as $(x + 1)(x - 1)$; alternatively, the quadratic formula (Rule A1.10) can be used to show that the two roots of $(x^2 - 1) = 0$ are $x = -1$ and $x = +1$, indicating that we can factor $(x^2 - 1)$ as $(x + 1)(x - 1)$.

(c) Factoring both sides gives $x^2\,(x - y) = (x - y)$. For this equation to hold true, either x^2 must equal one or $(x - y)$ must equal zero. Three possible solutions for x are thus $x = -1$, $x = +1$, and $x = y$.

(d) Using Rule A1.11, $\ln(x^t) = t\ln(x) = 1/2$. Thus, $t = 1/(2\ln(x))$, which we can also write as $1/\ln(x^2)$ (both are correct).

(e) Using Rules A1.12 and A1.14, $\ln(a\,x) + \ln(b\,x) - \ln(c) = \ln(a\,b\,x^2/c)$.

(f) Solving for terms in the exponent is made easier by taking the logarithm (in any base) of both sides: $\ln(x^{rt}) = r\,t\ln(x) = \ln(100)$. Thus, $t = \ln(100)/(r\ln(x))$ or, equally, $t = \ln(100)/\ln(x^r)$.

(g) Using Rule A1.17, $\Sigma_{i=1}^{n} 1 = n$.

(h) Using Rule A1.23, $\Pi_{i=1}^{n} 1 = 1$.

(i) We can rewrite $\Sigma_{i=1}^{n}(2i - 1)$ as $\left(2\Sigma_{i=1}^{n} i\right) - \left(\Sigma_{i=1}^{n} 1\right)$, which according to Rules A1.17 and A1.18 equals $(n\,(n + 1)) - (n) = n^2$.

(j) Adding (-5) to both sides, we get $x/(-3) > 10$. Multiplying both sides by (-3), we get $x < -30$ (x must be less than -30). Note that this last operation required that we reverse the inequality.

Appendix 2

Some Important Rules from Calculus

A2.1 Concepts

In this appendix, we review basic concepts and formulae from calculus that are used repeatedly in the text. We assume that you have learned this material in the past and provide exercises to help refresh your memory. See Neuhauser (2003) for additional review.

Let us consider a function $f(x)$ of an independent variable x. The function can be represented as a curve drawn on a two-dimensional plot (Figure A2.1). The rate at which the height of $f(x)$ changes as x is varied is described by the *derivative* of the function. By definition, the derivative of a function $f(x)$ with respect to x is

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} \equiv \lim_{\Delta x \to 0}\left[\frac{f(x + \Delta x) - f(x)}{\Delta x}\right].$$

In words, the derivative of f with respect to x is defined as ("$\equiv$") the change in f over an interval Δx (that is, $f(x + \Delta x) - f(x)$), divided by the length of the interval (Δx), as the interval is reduced to zero ("$\lim_{\Delta x \to 0}$"). A graphical way to think about the derivative is that it equals the slope of the line tangent to the function at point, x. For example, in Figure A2.1, we plot the function $f(x) = x^3 - 12x^2 + 36x - 20$, whose derivative is $\mathrm{d}f(x)/\mathrm{d}x = 3x^2 - 24x + 36$. At $x = 1$, the slope of the tangent line ("rise over run") would be 15 (thin line). Whenever the function $f(x)$ is flat, for example at a local maximum or minimum, the derivative is zero. In Figure A2.1, the function has a derivative of zero at both $x = 2$ and $x = 6$. Another way to think about derivatives is that they measure the sensitivity of the height $f(x)$ to changes in x. This mental picture helps explain why derivatives are so important in biology, because we often want to describe how sensitive a quantity of interest (e.g., the growth of a population) is to some other quantity (e.g., the current population size).

If the derivative of a function $f(x)$ is $g(x)$, then the *antiderivative* of $g(x)$ is $f(x)$. Thus, antidifferentiation ("*integration*") undoes the process of taking the derivative of a function. We can represent the antiderivative of $g(x)$ with respect to x using an indefinite integral:

$$\int g(x)\,\mathrm{d}x = f(x).$$

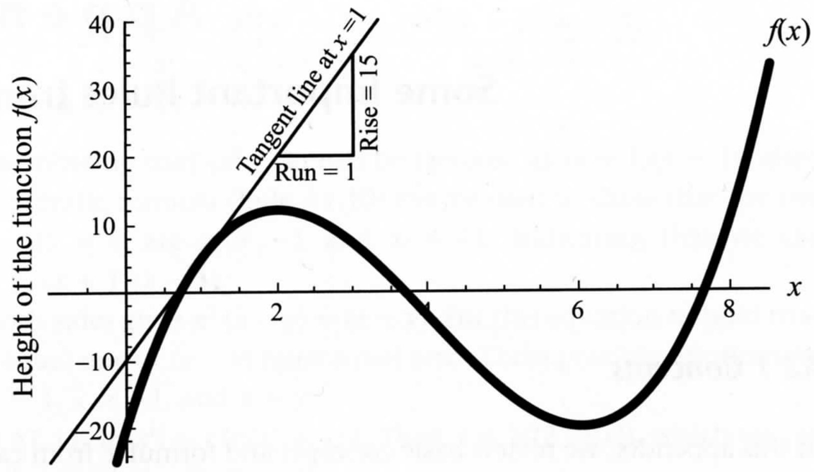

Figure A2.1: Slopes and derivatives. A plot of the function $f(x) = x^3 - 12x^2 + 36x - 20$ (thick curve) and its tangent line at $x = 1$ (thin line). The slope of the tangent line equals 15, which is described by the derivative of $f(x)$ at $x = 1$ and which can be confirmed by dividing the "rise" in the tangent (15) over a "run" in x of 1.

But not only is $f(x)$ the antiderivative of $g(x)$, so too is $f(x) + c$. That is, indefinite integrals are not unique. Thus, when taking an indefinite integral, we add a "constant of integration" (c) to the result to indicate that any possible value of c would work:

$$\int g(x)\,dx = f(x) + c.$$

For example, if $g(x) = 2x$, then its antiderivative would be any function $x^2 + c$, including both $f(x) = x^2$ and $f(x) = x^2 - 20$ (as can be confirmed by taking the derivative of both possibilities for $f(x)$).

If derivatives are measures of rates of change ("slopes"), antiderivatives are measures of "areas." Indefinite integrals represent the area under the curve $g(x)$ without specifying the range of values of x that we want to consider. If we take an indefinite integral evaluated at $x = b$ and subtract off the indefinite integral evaluated at $x = a$, then we get the definite integral:

$$\int_{x=a}^{b} g(x)\,dx;$$

this definite integral is the area under the function $g(x)$ between points a and b. This result is known as "the fundamental theorem of calculus." Figure A2.2 illustrates the integration of the function $g(x) = -x^2 + 8x - 12$, whose indefinite integral is $\int g(x)\,dx = -(1/3)x^3 + 4x^2 - 12x + c$. The area under the curve between $x = 5$ and $x = 6$ is $\int_{x=5}^{6} g(x)\,dx = 5/3$, while the area under the curve between $x = 6$ and $x = 8$ is $\int_{x=6}^{8} g(x)\,dx = -32/3$. The total area under the curve between $x = 5$ and $x = 8$ can be found either by evaluating the definite integral $\int_{x=5}^{8} g(x)\,dx = -9$ or by adding together the two areas between $x = 5$ and 6 and between $x = 6$ and 8; either way the answer is -9. The fact that this area is a negative number indicates that the curve lies mainly below the horizontal axis.

It is helpful to think of an area as a sum of rectangles whose width is very small and whose height is given by $g(x)$ (inset in Figure A2.2). This mental

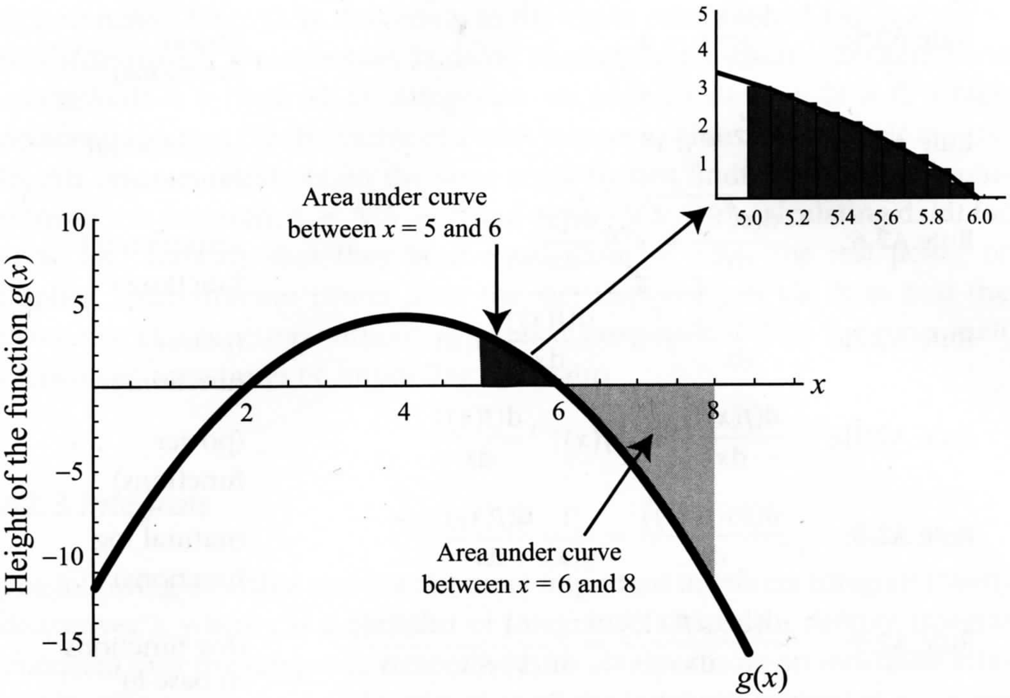

Figure A2.2: Areas and integration. A plot of the function $g(x) = -x^2 + 8x - 12$. The area under the curve can be found by integration and is 5/3 between $x = 5$ and $x = 6$ (dark shaded area) and $-32/3$ between $x = 6$ and $x = 8$ (light shaded area). The inset shows how the area under the curve can be approximated by filling in the area with a series of rectangles whose height is given by $g(x)$. The sum of these rectangles is an approximation to the area found by integration. This approximation improves when using more rectangles of smaller width.

image helps to explain the importance of integration in biology, because we often want to describe the sum total effect of a process. For example, we might want to determine the sum total effect of past selection or the sum total change to the size of a population over an interval of time.

A2.2 Derivatives

The following rules describe some of the more important derivatives:

Rule A2.1: $$\frac{da}{dx} = 0$$ (derivative of a constant)

Rule A2.2: $$\frac{d(a\,f(x))}{dx} = a\,\frac{df(x)}{dx}$$ (factoring out a constant)

Rule A2.3: $$\frac{d(f(x) + g(x))}{dx} = \frac{df(x)}{dx} + \frac{dg(x)}{dx}$$ (linearity property)

Rule A2.4: $\frac{d(a\,x)}{dx} = a$ (linear functions)

Rule A2.5: $\frac{dx^a}{dx} = a\,x^{a-1}$ (polynomial functions)

Rule A2.6: $\frac{d(e^{f(x)})}{dx} = e^{f(x)}\frac{df(x)}{dx}$ (exponential functions)

Rule A2.7a: $\frac{d(a^{f(x)})}{dx} = a^{f(x)}\frac{d(f(x))}{dx}\ln(a)$ (power functions)

Rule A2.7b: $\frac{d(f(x)^a)}{dx} = a\,f(x)^{a-1}\frac{d(f(x))}{dx}$ (power functions)

Rule A2.8: $\frac{d(\ln(f(x)))}{dx} = \frac{1}{f(x)}\frac{d(f(x))}{dx}$ (natural log functions)

Rule A2.9: $\frac{d(\log_b(f(x)))}{dx} = \frac{1}{\ln(b)\,f(x)}\frac{d(f(x))}{dx}$ (log functions in base b)

Rule A2.10: $\frac{d(\sin(f(x)))}{dx} = \cos(f(x))\frac{d(f(x))}{dx}$ (sine functions)

Rule A2.11: $\frac{d(\cos(f(x)))}{dx} = -\sin(f(x))\frac{d(f(x))}{dx}$ (cosine functions)

Rule A2.12: $\frac{d(f(x)\,g(x))}{dx} = \frac{d(f(x))}{dx}g(x) + f(x)\frac{d(g(x))}{dx}$ (product rule)

Rule A2.13: $\frac{d\left(\frac{f(x)}{g(x)}\right)}{dx} = \frac{\frac{d(f(x))}{dx}g(x) - f(x)\frac{d(g(x))}{dx}}{g(x)^2}$ (quotient rule)

Rule A2.14: $\frac{d(f(x(t)))}{dt} = \frac{df}{dx}\frac{dx}{dt}$ (chain rule: 1 variable)

Rule A2.15: $\frac{d(f(x(t), y(t)))}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$ (chain rule: 2 variables)

Implicit differentiation is an important method that builds upon the chain rule. An *explicit* function describes how a variable x depends on another variable t in terms of an equation $x = f(t)$. Thus, when t changes, we have an explicit expression that tells us how x changes. In contrast, an *implicit* function has the form $f(x, t) = c$. Again we can view this as an equation governing how x must change whenever t changes; when t is varied, x must change in such a way that the function $f(x, t)$ remains equal to c. For example, the equation $x^2 + t^2 = 5$ implicitly describes x as a function of t. When t varies, x must also vary so that the sum $x^2 + t^2$ equals 5. Taking the derivative of implicit functions is called implicit differentiation; it can be carried out using the rules of calculus described above, and it can be a useful method for finding the derivative, dx/dt. For example, the derivative of $x^2 + t^2 = 5$ with respect to t is $d(x^2)/dt + d(t^2)/dt =$

d(5)/dt (using Rule A2.3). According to the chain rule (Rule A2.14), d(x^2)/dt = (d(x^2)/dx) (dx/dt), which equals 2x dx/dt. Furthermore, d(t^2)/dt = 2t (Rule A2.5) and d(5)/dt = 0 (Rule A2.1). Altogether, we have 2x dx/dt + 2t = 0, which demonstrates that the derivative of x with respect to t must equal dx/dt = $-t/x$. In this case, we could obtain the same result by first finding the explicit solutions to this function, $x = \sqrt{5 - t^2}$ and $x = -\sqrt{5 - t^2}$, calculating dx/dt for each, and showing that they both equal dx/dt = $-t/x$. The real power of implicit differentiation comes from the fact that you can use it to find the derivative of a function without an explicit expression for the function itself (which can sometimes be impossible to obtain).

A2.3 Integrals

The following describes some of the more important indefinite integrals ("antiderivatives"), where c is a constant of integration. Because a definite integral evaluated over the range a to b can always be obtained from an indefinite integral by plugging in $x = b$ and subtracting off the indefinite integral at $x = a$, we provide rules for indefinite integrals only.

Rule A2.16: $\int a\,dx = a\,x + c$ (integral of a constant)

Rule A2.17: $\int a\,f(x)\,dx = a \int f(x)\,dx$ (factoring out a constant)

Rule A2.18: $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$ (linearity property)

Rule A2.19: $\int a\,x\,dx = \dfrac{a\,x^2}{2} + c$ (linear functions)

Rule A2.20: $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$ (polynomial functions)

Rule A2.21: $\int \dfrac{a}{x}\,dx = a\ln(|x|) + c$ (fractional functions)

Rule A2.22: $\int \dfrac{1}{(a_1 + b_1 x)(a_2 + b_2 x)}\,dx = \dfrac{\ln\left(\left|\dfrac{a_1 + b_1 x}{a_2 + b_2 x}\right|\right)}{a_2 b_1 - a_1 b_2} + c$ (fractional functions)

Rule A2.23: $\int e^{ax}\,dx = \dfrac{e^{ax}}{a} + c$ (exponential functions)

Rule A2.24: $\int a^{bx}\,dx = \dfrac{a^{bx}}{b\ln(a)} + c$ for $a > 0$ (power functions)

Rule A2.25: $\int \ln(x)\,dx = x\ln(x) - x + c$ for $x > 0$ (natural log functions)

Rule A2.26: $\int \sin(x)\,dx = -\cos(x) + c$ (sine functions)

Rule A2.27: $\int \cos(x)\,dx = \sin(x) + c$ (cosine functions)

Rule A2.28: $\int f(x)\,dx = \int h(g(x))\frac{dg(x)}{dx}\,dx = \int h(u)\,du$
where $u = g(x)$ (integration by substitution)

Integration by substitution can be useful when the original function, $f(x)$, can be factored into the product of two terms, $h(g(x))\,dg(x)/dx$, where the first term depends on x only through the function, $g(x)$, and the second is the derivative of $g(x)$ with respect to x.

Rule A2.29: $\int u\,dv = u\,v - \int v\,du$ (integration by parts)

or, alternatively,

$$\int f(x)\,dx = \int \frac{d(g(x))}{dx} h(x)\,dx = g(x)\,h(x) - \int g(x)\frac{d(h(x))}{dx}\,dx$$

Integration by parts can be useful if the original function, $f(x)$, can be factored into the product of two terms $(d(g(x))/dx)\,h(x)$, where the first term is the derivative of another function $g(x)$. This method helps integrate functions whenever $g(x)\,d(h(x))/dx$ is easier to integrate than the original function.

A2.4 Limits

The rules of calculus are also useful for determining the *limit* of a function as a variable approaches a specific value. To denote the limit of $f(x)$ as x goes to a, we write $\lim_{x\to a} f(x)$. For many functions, the limit is straightforward to determine. For example, the limit of $f(x) = 2 + x^2$ as x goes to one is three, and the limit of $f(x) = e^{rx}$ as x goes to zero is one. In some cases, the limit as x approaches a depends on whether x starts above a ($\lim_{x\to a+} f(x)$) or below a ($\lim_{x\to a-} f(x)$). For example, the limit of $f(x) = 1/x$ as x goes to zero is $+\infty$ if x is initially positive but $-\infty$ if x is initially negative (Figure A2.3).

In certain cases, however, the limit is not obvious. For example, what is the limit of $(e^{rx} - e^{sx})/x$ as x goes to zero? The answer is unclear because both the numerator and the denominator are zero at $x = 0$. In such cases, we can use L'Hôpital's rule to determine the limit:

Rule A2.30: $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ (L'Hôpital's rule)

L'Hôpital's rule requires that both $f(x)$ and $g(x)$ are zero at $x = a$ (or both are infinite), that both $f'(x) = df/dx$ and $g'(x) = dg/dx$ exist at $x = a$, and that $g(a)$ is not zero.

L'Hôpital's rule allows us to calculate limits of quotients such as $\lim_{x\to 0}((e^{rx} - e^{sx})/x)$. Specifically, Rule A2.30 tells us that $(e^{rx} - e^{sx})/x$ has the same limit as $f'(x)/g'(x) = (re^{rx} - se^{sx})/1$, whose limit as x goes to zero is easy to calculate: $(r - s)$. Occasionally, you must rearrange a function to apply L'Hôpital's

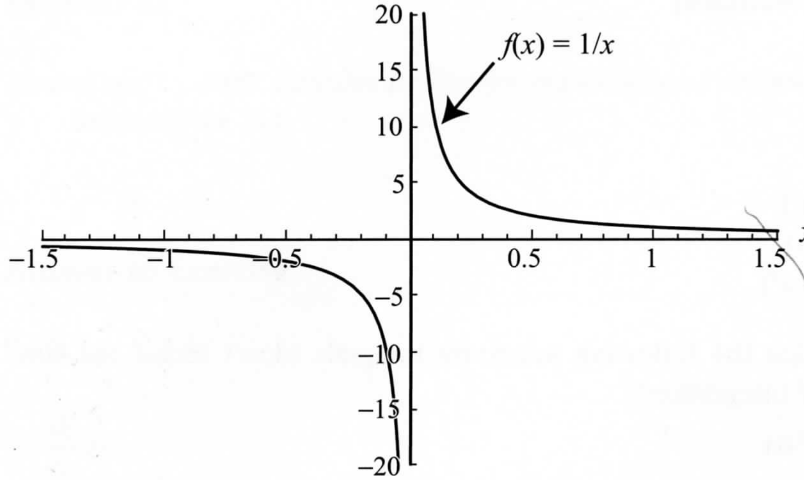

Figure A2.3: Limit of the inverse function. A plot of the function $f(x) = 1/x$, whose limit as x goes to zero depends on whether x is initially positive or negative.

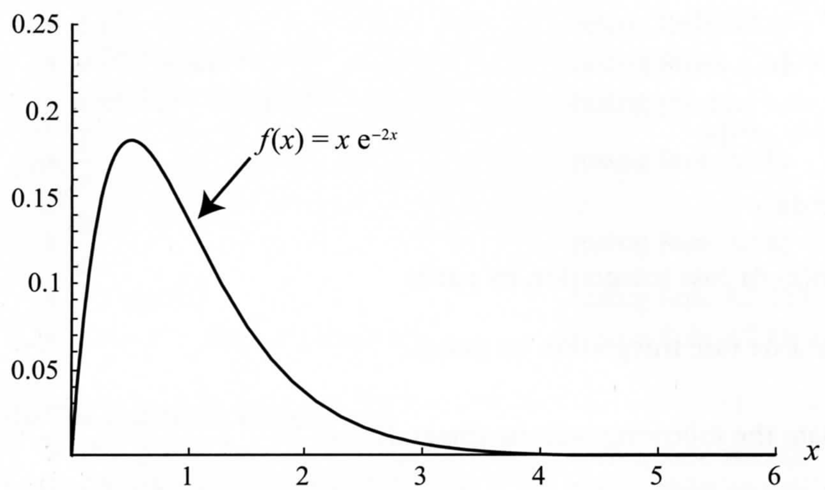

Figure A2.4: An application of L'Hôpital's rule. A plot of the function $f(x) = x\, e^{-rx}$ using $r = 2$, whose limit as x goes to infinity is zero, as can be shown using L'Hôpital's rule (A2.30).

rule. For example, to evaluate $\lim_{x\to\infty}(x\, e^{-rx})$ for positive r using L'Hôpital's rule, we must first write it as a quotient: $\lim_{x\to\infty}(x/e^{rx})$. In this example, both the numerator and denominator approach infinity, rather than zero, in the limit. L'Hôpital's rule can still be applied in such cases, allowing us to determine the limit of x/e^{rx} from the limit of $f'(x)/g'(x) = 1/(r\, e^{rx})$, which approaches zero as x goes to infinity (Figure A2.4).

Exercise A2.1: The following questions review calculus techniques needed throughout the text (treat everything but x as a constant).

(a) Find the derivative with respect to x of the following functions $f(x)$:

- $-\,x^2 + 8x - 12$
- $\dfrac{1}{x^2 + 1}$

Continued

Exercise A2.1 (*Continued*)

- e^{3x}
- $x^n + e^{a\,x}$
- $x^n\,e^{a\,x}$
- $\ln(x)$
- $\ln(a\,x^2)$
- $\cos(5x)$
- $\sin^2(b\,x^a)$

(b) Determine the following indefinite integrals (don't forget the constant of integration):

- $\int 3x^2\,dx$
- $\int (4x + 5)\,dx$
- $\int \frac{2}{x}\,dx$
- $\int e^{2x}\,dx$
- $\int x^n + e^{ax}\,dx$
- $\int 2^x\,dx$
- $\int \ln(x)\,dx$ (use integration by parts)
- $\int e^x\,x\,dx$ (use integration by parts)

(c) Calculate the following definite integrals:

- $\int_{x=1}^{2} 8\,x^3\,dx$
- $\int_{x=0}^{3} (2x + 1)\,dx$
- $\int_{x=0}^{1} e^{-r\,x}\,dx$
- $\int_{x=0}^{\infty} e^{-r\,x}\,dx$ (assume that r is positive)

(d) Calculate the following limits:

- $\lim_{x\to 0}\frac{\sin(x)}{1 + x}$
- $\lim_{x\to 0}\frac{\sin(x)}{x}$

References

Neuhauser, C. 2003. *Calculus for Biology and Medicine.* Prentice-Hall, Upper Saddle River, N.J.

Answer to Exercise

Exercise A.2.1

(a) $\frac{df}{dx}$ is

- $-2x + 8$ (using Rule A2.5)
- $\frac{-2x}{(x^2+1)^2}$ (using quotient Rule A2.13)
- $3\,e^{3x}$ (using Rule A2.6)
- $n\,x^{n-1} + a\,e^{a\,x}$ (using Rules A2.3, A2.5, and A2.6)
- $n\,x^{n-1}\,e^{a\,x} + a\,x^n\,e^{a\,x}$ (using product rule A2.12)
- $\frac{1}{x}$ (using Rule A2.8)
- $\frac{2}{x}$ (using Rule A2.8)
- $-5\sin(5x)$ (using Rule A2.11)
- $2\,b\,a\,x^{a-1}\sin(b\,x^a)\cos(b\,x^a)$ (using Rule A2.7b and A2.10)

(b) The indefinite integrals are

- $x^3 + c$ (using Rule A2.20)
- $2x^2 + 5x + c$ (using Rule A2.20)
- $2\ln(|x|) + c$ (using Rule A2.21)
- $\frac{e^{2x}}{2} + c$ (using Rule A2.23)
- $\frac{x^{n+1}}{n+1} + \frac{e^{ax}}{a} + c$ (using Rules A2.18, A2.20, A2.23)
- $\frac{2^x}{\ln(2)} + c$ (using Rule A2.24)
- Letting $u = \ln(x)$ and $dv = dx$ so that $du = 1/x\,dx$ and $v = x$,
  $$\int \ln(x)\,dx = x\ln(x) - \int 1\,dx = x\ln(x) - x + c \quad \text{(using Rule A2.29)}$$
- Letting $u = x$ and $dv = e^x\,dx$ so that $du = 1\,dx$ and $v = e^x$,
  $$\int e^x x\,dx = e^x x - \int e^x\,dx = e^x x - e^x + c \quad \text{(using Rule A2.29)}$$

(c) The definite integrals are

- $$\int_{x=1}^{2} 8\,x^3\,dx = 2\,x^4\Big|_{x=1}^{2} = 32 - 2 = 30 \quad \text{(using Rule A2.20)}$$