Optimal Control of Multibattery Energy-Aware Systems

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***Abstract*—We study the problem of optimally controlling a set of nonideal rechargeable batteries that can be shared to perform a given amount of work over some specified time period. We seek to maximize the minimum residual energy among all batteries at the end of this period by optimally controlling the discharging and recharging process at each battery. Modeling a battery as a dynamic system, we adopt a kinetic battery model and formulate an optimal control problem under the constraint that discharging and recharging cannot occur at the same time. We show that the optimal solution must result in equal residual energies for all batteries as long as such a policy is feasible. This simplifies the task of subsequently deriving explicit solutions for the problem, which is accomplished by first analyzing the 2-battery case and then considering the general N-battery case ($N > 2$).**

***Index Terms*—Dynamic power management, energy-aware systems, kinetic battery model, nonideal batteries, optimal control.**

I. Introduction

WITH the increasing use and dependence on wireless and mobile devices, batteries are playing a critical role in areas such as communications, automotive, transportation, robotics, and consumer electronics. Due to their limited power capacity, especially for small and light devices, research on energy management of battery-powered systems has become increasingly active. The opportunity to recharge batteries through energy harvesting for small devices or connecting to the grid for electric vehicles adds an extra level of flexibility and power control. Energy-aware systems of this type have been studied with techniques such as dynamic voltage scheduling (DVS) [1]–[3] where a battery is modeled as a queueing system [4], usually based on the assumption that the battery is "ideal," it maintains a constant voltage throughout the discharge process and a constant capacity for all discharge profiles. However, because of the rate capacity effect [5] and the recovery effect [6], both characterizing real batteries, the voltage as well as energy amount delivered by the battery heavily rest on the discharge profile. Therefore, when dealing with energy optimization, it is necessary to take that into account along with nonlinear variations in a battery's capacity. As a result, there are several proposed models to describe a nonideal battery; a detailed overview is given in [7]. Accordingly, models are broadly classified as: 1) electrochemical [5], [8], [9]; 2) circuit-based [10], [11]; 3) stochastic [12]–[15]; and 4) analytical [16]–[18]. Electrochemical models possess the highest accuracy, but their complexity makes them impractical for most real-time applications. Comparatively, electrical-circuit models are much simpler and therefore computationally less expensive. However, they are generally less accurate, with errors of approximately 10% [7]. Recently, a more accurate electrical-circuit model was proposed in [11], reducing the error to less than 1% at the expense of added complexity. Stochastic models use a discrete time Markov chain with $N+1$ states to represent the number of charge units available in the battery. Since N is large (around $72 \cdot 10^7$), these models are also limited by high computational requirements. Last but not least, analytical models, including diffusion-based models [17], [19], [20] and the kinetic battery model (KBM) [15], [21], use only a few equations to capture the battery's main features. While diffusion-based models are hard to combine with a performance model [7], a KBM combines speed with sufficient accuracy, as reported, for instance, in embedded system applications [21]. It is also suitable for large-scale systems such as wireless sensor networks [22] where batteries are distributed over the nodes in the network.

With this motivation, in [23] we studied an optimal control problem based on a KBM with the added feature of a recharging capability so that the battery may be in either discharging or recharging mode at any time. We showed that an optimal policy maximizing the work performed by the battery over a given time interval while requiring that its energy is at a desired level at the end of this interval is of bang-bang type with an optimal time to switch from discharging to recharging within the constraints of the problem. This result was found to be consistent with a solution of the same problem using a much more elaborate linear state space model [19] derived from the popular RVW diffusion-based model [16]. In this paper, we study systems with multiple nonideal rechargeable batteries which can be shared in performing a certain amount of work, viewing this as a first step toward battery-powered networked systems with renewable energy. Along these lines, in [4], a dynamic node activation problem in networks of rechargeable sensors is addressed by modeling the battery as a queueing system processing energy tasks. In [24] an optimal control policy

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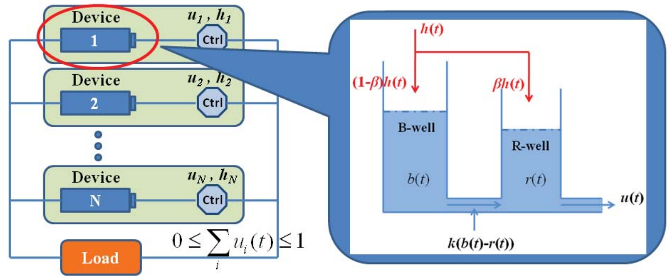
is presented for cross-layer resource allocation in wireless networks operating with rechargeable batteries. In [25] advantage is taken of battery energy storage in optimal power flow problems, while in [26] a network resource allocation problem is presented for energy-harvesting sensor platforms with time-varying battery recharging rates. However, in all these cases the battery models used are simple and assume ideal behavior. Recently, based on an electrochemical model, Moura *et al.* [27] used a deterministic dynamic programming formulation to derive a heuristic for controlling the charging processes so as to reduce film growth and thus improve battery pack lifetime. Despite different problem objectives, the optimal policy features in [27], such as unequal and delayed charging, are structurally consistent with our results based on the KBM.

In this paper, we use a KBM for multiple batteries that can be shared and are fully rechargeable. We seek to maximize the minimum residual energy among all batteries at the end of a given time interval $[0, T]$ with the requirement that the total battery output should reach a desired level at the end of $[0, T]$, subject to certain rechargeability constraints. We assume that recharging a battery is possible only while it is not being discharged, a requirement which is application-dependent. Relaxing this constraint is a special case of the more general problem we have analyzed and leads to a simpler solution. Our motivation mainly comes from: 1) multiple battery-powered robotic systems, which must periodically interrupt operation for recharging purposes; 2) wireless sensor nodes, which are usually battery-powered and must also be periodically recharged; and 3) electric vehicles, where the emerging "smart grid" provides considerable flexibility for controlling the timing of recharging intervals in between usage of the vehicle [28]. In many such applications, discharging and recharging processes of each battery-powered system cannot be conducted simultaneously. In the meantime, these multiple battery-powered systems are usually required to complete a common load of work over $[0, T]$ while ensuring the available functionality of the system at T so that the process may be repeated, possibly in periodic fashion. We first prove some properties of an optimal policy, the main one being the fact that it must result in equal residual energies for all batteries at time T. This enables us to subsequently derive explicit solutions for the problem. As already mentioned, we view this as a first step toward studying similar problems where the batteries are not all shared at a single location, but rather distributed over a network of devices with one or more batteries placed on board and powering each device. We may then address resource allocation and network lifetime maximization problems where the nonideal nature of the batteries is not only taken into account but also taken advantage of.

In Section II, a multibattery optimal control problem based on a KBM is formulated. Significant properties of the optimal solution are identified and proved in Section III. In Section IV, with the help of these properties, we are able to provide a full determination of the optimal control solution, starting with the 2-battery case, which is then extended to a solution of the N-battery case and its verification. Conclusions and a description of future research are given in Section V.



(a) (b)

Fig. 1. (a) Multibattery system based on KBM. (b) Modified KBM proposed in [29].

II. Problem Statement

We consider N battery-powered devices, each with an embedded battery and a corresponding controller, that may be shared to serve a common load, as shown in Fig. 1.

Each battery is modeled by a KBM, as in [29]. To briefly review, a KBM views each battery, indexed by $i = 1, \ldots, N$, as consisting of two communicating wells, a bound-charge well whose content (energy level) is $b_i(t)$ and an available-charge well whose content is $r_i(t)$ (details are given in [29]). The controllable input flow is denoted by $h(t)$ and, in general, it is distributed to both wells through a constant coefficient β $(0 \le \beta \le 1)$. As shown in [29], the case where $\beta = 0$ gives the same optimal solution structure as the generic case. Thus, to maintain some simplicity in the analysis, we will limit ourselves to $\beta = 0$ and obtain the KBM shown in (2) and (3) below. In our prior work involving a single battery we sought to control the discharging and recharging processes (executed by the controller) so as to maximize the battery output over a given interval while maintaining some required residual energy level. However, when dealing with multiple rechargeable batteries, we adopt an objective motivated by the goal of maximizing a system's "lifetime," often viewed as the time until the first battery is depleted (e.g., [22] and [30]). Thus, we seek to maximize the minimum residual energy after finishing a prescribed workload within a time interval $[0, T]$ (note that this provides the flexibility to repeat the system's operation over cycles of length T). Let S be the battery index set with $|S| = N$, and let $U(t) = (u_1(t), h_1(t), \ldots, u_N(t), h_N(t))^T$, where $u_i(t)$ and $h_i(t)$ for $i \in S$ denote the instantaneous discharge and recharge rate of battery i, respectively. We then formulate the problem as follows:

$$\max_{U(t)} \min_{i \in S} r_i(T) \tag{1}$$

$$\dot{r}_i(t) = -c_1 u_i(t) + k(b_i(t) - r_i(t)) \tag{2}$$

$$\dot{b}_i(t) = c_2 h_i(t) - k(b_i(t) - r_i(t)) \tag{3}$$

$$r_i(t) \ge 0, \quad b_i(t) \le B \tag{4}$$

$$u_i(t) h_i(t) = 0 \tag{5}$$

$$0 \le u_i(t) \le 1, \; 0 \le h_i(t) \le 1 \tag{6}$$

$$0 \le \sum_{i=1}^{N} u_i(t) \le 1 \tag{7}$$

$$\int_0^T \sum_i u_i(t) dt = Q. \tag{8}$$

Here, (2) and (3) capture the battery dynamics through the KBM, where k depends on the battery characteristics and c_1, c_2 are battery-specific influencing factors for discharge and recharge processes, satisfying $c_1 > c_2 \geq 0$ (this indicates that a battery discharges faster than it recharges). The constraint (5) requires that the discharging and recharging processes cannot occur simultaneously (this can be relaxed, depending on the application) and (6) imposes natural limits on the corresponding process rates. The state variables $r_i(t)$, $b_i(t)$ are physically constrained as in (4) with $b_i(0) \geq r_i(0)$. The overall load to be served can be supported by either one or multiple batteries at any time, as indicated in (7) and consistent with (6). Finally, (8) captures the fact that the load is required to complete a specific amount of work Q within $[0, T]$.

III. Optimal Control Properties

We begin by solving (2) and (3) under the assumption that a control policy $\{u_i(t), h_i(t),\ i \in S\}$ is feasible over some interval $[t_1, t_2] \subseteq [0, T]$, including possible boundary arcs where $r_i(\tau) = 0$ or $b_i(\tau) = B$, $\tau \in [t_1, t_2]$. It is straightforward to derive this solution

$$\begin{aligned} r_i(\tau) &= \tfrac{1}{2}\left[b_i(t_1) + r_i(t_1) - (b_i(t_1) - r_i(t_1))e^{-2k(\tau - t_1)}\right] \\ &\quad -\tfrac{1}{2}\int_{t_1}^{\tau}\left[c_1 u_i(t)[1 + e^{2k(t-\tau)}] - c_2 h_i(t)[1 - e^{2k(t-\tau)}]\right]dt \end{aligned} \tag{9}$$

$$\begin{aligned} b_i(\tau) &= \tfrac{1}{2}\left[b_i(t_1) + r_i(t_1) + (b_i(t_1) - r_i(t_1))e^{-2k(\tau - t_1)}\right] \\ &\quad -\tfrac{1}{2}\int_{t_1}^{\tau}\left[c_1 u_i(t)[1 - e^{2k(t-\tau)}] - c_2 h_i(t)[1 + e^{2k(t-\tau)}]\right]dt. \end{aligned} \tag{10}$$

Setting $\rho_i(t_1, \tau)$ and $\beta_i(t_1, \tau)$ to denote the first term in (10) and (10) respectively, we can write for a solution over $[0, T]$

$$\begin{aligned} r_i(T) &= \rho_i(0, T) - \int_0^T c_1 u_i(r)\frac{1 + e^{2k(r-T)}}{2}dr \\ &\quad + \int_0^T c_2 h_i(r)\frac{1 - e^{2k(r-T)}}{2}dr \end{aligned} \tag{11}$$

$$\begin{aligned} b_i(T) &= \beta_i(0, T) - \int_0^{\tau} c_1 u_i(r)\frac{1 - e^{2k(r-T)}}{2}dr \\ &\quad + \int_0^{\tau} c_2 h_i(t)\frac{1 + e^{2k(r-T)}}{2}dr \end{aligned} \tag{12}$$

as long as the feasibility of $\{u_i(t), h_i(t)\}$ over $[0, T]$ is assumed. Since we have assumed $b_i(0) \geq r_i(0)$, obviously $\beta_i(0, T) > \rho_i(0, T) > 0$.

Let us denote an optimal control policy by $\{u_i^*(t), h_i^*(t),\ i \in S\}$. We can immediately observe that $\{0, 0,\ i \in S\}$ cannot be an optimal policy, i.e., a policy that maximizes $\min_{i \in S} r_i(T)$. This follows from the constraint (5) and the fact that $\{0, 0\}$ in (11) is dominated by any control $\{0, h_i(t)\}$ with $h_i(t) > 0$ which gives a larger value for $r_i(T)$. Moreover, (8) requires $u_i(t) > 0$, $h_i(t) = 0$ for some i and over some interval $[t_1, t_2] \subseteq [0, T]$. Thus, at least some $i \in S$ must include $u_i(t) > 0$; for any remaining $i \in S$ an optimal control would be $\{0, h_i(t)\}$ with $h_i(t) > 0$. Therefore, an optimal control for any $i \in S$ has the property that either $u_i^*(t) > 0, h_i^*(t) = 0$ or $u_i^*(t) = 0, h_i^*(t) > 0$ (with $h_i^*(t) = 1$ when $b_i(t) < B$).

The main result in this section (Theorem 1) is that, under optimal control, all $r_i^*(T)$, $i \in S$, are equal provided there is at least one feasible policy under which all $r_i(T)$, $i \in S$, are equal. In order to establish this result, we will make use of a perturbed policy $\{u_i'(t), h_i'(t),\ i \in S\}$ relative to any feasible one $\{u_i(t), h_i(t),\ i \in S\}$. We define such a policy by perturbing two of the controls indexed by i and $j \neq i$ respectively as follows:

$$\begin{cases} u_i'(t) = u_i(t),\ h_i'(t) = h_i(t) & t \in [0, T]/[\tau_i, \tau_i + \Delta_i] \\ u_i'(t) = u_i(t) - \Delta u_i,\ h_i'(t) = 0 & t \in [\tau_i, \tau_i + \Delta_i] \end{cases} \tag{13}$$

$$\begin{cases} u_j'(t) = u_j(t),\ h_j'(t) = h_j(t) & t \in [0, T]/[\tau_j, \tau_j + \Delta_j] \\ u_j'(t) = u_j(t) + \Delta u_j,\ h_j'(t) = 0 & t \in [\tau_j, \tau_j + \Delta_j] \end{cases} \tag{14}$$

where Δu_i, Δu_j, Δ_i, and Δ_j are all positive constants. For notational convenience, we shall refer to the perturbed control for i in (13) above as $\pi^-[u_i, \tau_i, \Delta_i, \Delta u_i]$ and the one for j in (14) as $\pi^+[u_j, \tau_j, \Delta_j, \Delta u_j]$. In simple terms, under $\pi^-[u_i, \tau_i, \Delta_i, \Delta u_i]$ the discharging control $u_i(t)$ is reduced by $\Delta u_i > 0$ over an interval $[\tau_i, \tau_i + \Delta_i]$, while $u_j(t)$ under $\pi^+[u_j, \tau_j, \Delta_j, \Delta u_j]$ is increased by $\Delta u_j > 0$ over an interval $[\tau_j, \tau_j + \Delta_j]$; in both cases, the recharging control over these intervals is 0 to satisfy (5) and the controls remain unchanged over the rest of $[0, T]$. Assuming for the moment the feasibility of $\{u_i'(t), h_i'(t)\}$ and $\{u_j'(t), h_j'(t)\}$, let $\Delta r_i(t) = r_i'(t) - r_i(t)$, $\Delta b_i(t) = b_i'(t) - b_i(t)$ and observe that for any $t \in [\tau_i, T]$ it follows from (9) and (10):

$$\Delta r_i(t) = \begin{cases} \frac{1}{2}c_1 \Delta u_i \int_{\tau_i}^{t}[1 + e^{2k(r-t)}]dr & t < \tau_i + \Delta_i \\ \frac{1}{2}c_1 \Delta u_i \int_{\tau_i}^{\tau_i + \Delta_i}[1 + e^{2k(r-t)}]dr & t \geq \tau_i + \Delta_i \end{cases} \tag{15}$$

$$\Delta b_i(t) = \begin{cases} \frac{1}{2}c_1 \Delta u_i \int_{\tau_i}^{t}[1 - e^{2k(r-t)}]dr & t < \tau_i + \Delta_i \\ \frac{1}{2}c_1 \Delta u_i \int_{\tau_i}^{\tau_i + \Delta_i}[1 - e^{2k(r-t)}]dr & t \geq \tau_i + \Delta_i \end{cases} \tag{16}$$

and note that $\Delta r_i(t) > \Delta b_i(t) > 0$. Similarly, for any $t \in [\tau_j, T]$

$$\Delta r_j(t) = \begin{cases} -\frac{1}{2}\int_{\tau_j}^{t}\big[c_1 \Delta u_i[1 + e^{2k(r-t)}] \\ \qquad + c_2 h_j(r)[1 - e^{2k(r-t)}]\big]dr & t < \tau_j + \Delta_j \\ -\frac{1}{2}\int_{\tau_j}^{\tau_j + \Delta_j}\big[c_1 \Delta u_i[1 + e^{2k(r-t)}] \\ \qquad + c_2 h_j(r)[1 - e^{2k(r-t)}]\big]dr & t \geq \tau_j + \Delta_j \end{cases} \tag{17}$$

$$\Delta b_i(t) = \begin{cases} -\frac{1}{2}\int_{\tau_j}^{t}\big[c_1 \Delta u_i[1 - e^{2k(r-t)}] \\ \qquad + c_2 h_j(r)[1 + e^{2k(r-t)}]\big]dr & t < \tau_j + \Delta_j \\ -\frac{1}{2}\int_{\tau_j}^{\tau_j + \Delta_j}\big[c_1 \Delta u_i[1 - e^{2k(r-t)}] \\ \qquad + c_2 h_j(r)[1 + e^{2k(r-t)}]\big]dr & t \geq \tau_j + \Delta_j \end{cases} \tag{18}$$

and note that $\Delta r_j(t) < \Delta b_j(t) < 0$. Regarding the feasibility of $\{u_i'(t), h_i'(t)\}$ and $\{u_j'(t), h_j'(t)\}$, we need to satisfy all problem constraints. This can be accomplished under certain conditions, as expressed in the next two lemmas.

Lemma 1: Let $\{u_i(t), h_i(t)\}$, $\{u_j(t), h_j(t)\}$ be controls for i, j in a feasible policy. If $r_j(t) > 0$ for all $t \in [0, T]$ under this policy, then the following conditions ensure that there are feasible perturbed controls $\{u_i'(t), h_i'(t)\}$, $\{u_j'(t), h_j'(t)\}$.

C1) There exists an interval $[\tau_i, \tau_i + \Delta_i]$ with $u_i(t) > 0$, $t \in [\tau_i, \tau_i + \Delta_i]$.

C2) There exists an interval $[\tau_j, \tau_j + \Delta_j]$ such that $\sum_{k\in S} u_k(t) < 1$, $t \in [\tau_j, \tau_j + \Delta_j]/[t_1, t_2]$, where $[t_1, t_2] = [\tau_i, \tau_i + \Delta_i] \cap [\tau_j, \tau_j + \Delta_j]$ and $[\tau_i, \tau_i + \Delta_i]$ satisfies C1).

Proof: In order to ensure the feasibility of $\{u_i'(t), h_i'(t)\}$, $\{u_j'(t), h_j'(t)\}$, we must satisfy the constraints (4) through (8). Note that (5) holds by construction. Next, to satisfy (6), we have $u_j'(t) > u_j(t) \geq 0$ in (14), but must also ensure that $u_i'(t) \geq 0$ in (13); this follows from C1) since we may select $\Delta u_i > 0$ arbitrarily small. Regarding (7), $u_i'(t) < u_i(t)$ in (13) preserves the inequality, but $u_j'(t)$ in (14) may violate it. However, under C2) we may select $\Delta u_j > 0$ arbitrarily small to satisfy $\sum_{k\in S} u_k'(t) \leq 1$ over $[\tau_j, \tau_j + \Delta_j]/[t_1, t_2]$. In the interval $[t_1, t_2]$ (if it is not empty), we can select $\Delta u_i \geq \Delta u_j$ to preserve (7). To satisfy (8), we require $\int_0^T [u_i(t) + u_j(t)]dt = \int_0^T [u_i'(t) + u_j'(t)]dt$, which implies $\Delta u_i \Delta_i = \Delta u_j \Delta_j$ a condition which may be satisfied by properly selecting Δ_i, Δ_j relative to the values of Δu_i, Δu_j.

Regarding the constraints in (4), from (15) and (17), we have

$$r_i'(t) > r_i(t), \quad b_i'(t) > b_i(t), \qquad t \in [\tau_i, T] \tag{19}$$

$$r_j'(t) < r_j(t), \quad b_j'(t) < b_j(t), \qquad t \in [\tau_j, T] \tag{20}$$

so that we only have to ensure that $b_i'(t) \leq B$ and $r_j'(t) \geq 0$ in (4) are satisfied under $\{u_i'(t), h_i'(t)\}$, $\{u_j'(t), h_j'(t)\}$. As far as the constraint $b_i'(t) \leq B$ is concerned, we can ensure it remains in force as follows. Suppose $b_i(t_1) = B$ for some $t_1 \in [\tau_i, T]$. Consider an interval $[\tau_h, \tau_h + \Delta_h]$ with $\tau_h + \Delta_h < t_1$, where $u_i(t) = 0$, $h_i(t) > 0$. The existence of $[\tau_h, \tau_h + \Delta_h]$ is guaranteed because $b_i(t_1) = B$ cannot be satisfied by (3) if $h_i(t) = 0$ over $[0, t_1]$. Let $h_i(t)$ be perturbed by $-\Delta h_i < 0$ over $[\tau_h, \tau_h + \Delta_h]$ and observe that from (9) and (10) we get corresponding perturbations

$$\Delta r_i^h(t) = -\tfrac{1}{2} c_2 \Delta h_i \int_{\tau_h}^{\tau_h+\Delta_h} [1 - e^{2k(r-t)}]dr \tag{21}$$

$$\Delta b_i^h(t) = -\tfrac{1}{2} c_2 \Delta h_i \int_{\tau_h}^{\tau_h+\Delta_h} [1 + e^{2k(r-t)}]dr \tag{22}$$

where $\Delta b_i^h(t) < \Delta r_i^h(t) < 0$. Now recalling that in (15) and (16) $\Delta r_i(t) > \Delta b_i(t) > 0$ over $[\tau_i, T]$, we can always select $\tau_h, \Delta_h, \Delta h_i$ to satisfy $\Delta r_i(t) + \Delta r_i^h(t) > 0$ and $\Delta b_i(t) + \Delta b_i^h(t) = 0$ over $[t_1, T]$. Consequently, for any i subject to $\pi^-[u_i, \tau_i, \Delta_i, \Delta u_i]$, we can guarantee $b_i'(t) \leq B$ by adequately perturbing $h_i(t)$ over $[\tau_h, \tau_h + \Delta_h]$.

Regarding the constraint $r_j'(t) \geq 0$, under the lemma's assumption $r_j(t) > 0$ for all $t \in [\tau_j, T]$, there exists $\epsilon > 0$ such that $r_j(t) > \epsilon > 0$. Then, from (17), we can always satisfy $r_j'(t) \geq 0$ by selecting $\Delta u_j > 0$, $\Delta_j > 0$ such that

$$\frac{1}{2}\int_{\tau_j}^{\tau_j+\Delta_j} \Big[c_1 \Delta u_i [1 + e^{2k(r-t)}] + c_2 h_j(r)[1 - e^{2k(r-t)}]\Big] dr \leq \epsilon$$

which completes the proof of the lemma. ■

Under certain conditions, C2) in Lemma 1 can be relaxed and the result requires only C1) as expressed in the following corollary.

Corollary 1: For the setting of Lemma 1, suppose $\tau_i = \tau_j$ and $\Delta_i = \Delta_j$. Then, C2) is not needed and the result holds under C1) only.

Proof: Condition C2) was needed to ensure (7) is satisfied. In this case, suppose $\sum_{k\in S} u_k(t) = 1$ over $[\tau_i, \tau_i + \Delta_i] \equiv [\tau_j, \tau_j + \Delta_j]$. Since, under $\{u_i'(t), h_i'(t)\}$, $\{u_j'(t), h_j'(t)\}$ the only changes occur over $[\tau_i, \tau_i + \Delta_i]$, we can ensure that $\sum_{k\in S} u_k'(t) = 1$ over $[\tau_i, \tau_i + \Delta_i]$ [hence, (7) still holds] by selecting $\Delta u_i = \Delta u_j$. If, on the other hand, $\sum_{k\in S} u_k(t) < 1$, the same construction obviously satisfies $\sum_{k\in S} u_k'(t) < 1$. ■

Before establishing our main result, we need one more lemma as follows, which ensures the existence of some j with $r_j(t) > 0$ whenever $\sum_{i\in S} u_i(t) = 1$.

Lemma 2: Suppose $\sum_{i\in S} u_i(t) = 1$ over some interval $[t_1, t_2] \subset (0, T]$. Then, among all j with $u_j(t) > 0$ over $[t_1, t_1 + \epsilon] \subseteq [t_1, t_2]$ for some $\epsilon > 0$, there exists at least one with $r_j(t) > 0$ over $[t_1, t_1 + \epsilon]$.

Proof: We first use a contradiction argument to prove that there exists $j \in S$ with $u_j(t) > 0$ over $[t_1, t_1 + \epsilon]$ such that $r_j(t_1) > 0$. Assume that for all j with $u_j(t) > 0$ over $[t_1, t_1 + \epsilon]$, we have $r_j(t_1) = 0$. Suppose there are n such batteries (arbitrarily indexed from 1 to n) with $u_j(t) > 0$, $h_j(t) = 0$ over $[t_1, t_1 + \epsilon]$ so that $U(t) \equiv \sum_{i=1}^n u_i(t) = 1$. Let $R(t) = \sum_{i=1}^n r_i(t)$ and $B(t) = \sum_{i=1}^n b_i(t)$ and observe that by summing (2) and (3) over all $i = 1, \ldots, n$ we get

$$\dot{R}(t) = -c_1 + k(B(t) - R(t)), \qquad t \in [t_1, t_1 + \epsilon] \tag{23}$$

$$\dot{B}(t) = -k(B(t) - R(t)) \tag{24}$$

where $R(t) \geq 0$ due to $r_i(t) \geq 0$, $i = 1, \ldots, n$, in (4). Moreover, under the assumption that $r_j(t_1) = 0$, $j = 1, \ldots, n$, we have $R(t_1) = 0$. We now invoke Lemma 2 in [29] which asserts that for the KBM model (2) and (3) if $t_1 \in (0, T)$ is such that $u_i(t_1) = 1$ and $\dot{r}_i(t_1) > 0$ regardless of $\{h_i(t)\}$, then, under feasible $\{u_i(t), h_i(t)\}$, we must have $\dot{r}_i(t) > 0$ for all $t \in [0, t_1]$. If $\dot{R}(\tau) > 0$, $\tau \in [t_1, t_1 + \epsilon]$, this property applies to (23) and (24), since $U(t) = 1$ over $[t_1, t_1 + \epsilon]$. Thus, suppose there exists $\tau \in [t_1, t_1 + \epsilon]$ such that $\dot{R}(\tau) > 0$. Then, $\dot{R}(t) > 0$ for all $t \in [0, \tau]$. However, this contradicts the fact that $R(t_1) = 0$ which requires $\dot{R}(t) < 0$ for at least some $t < t_1$, since $R(t) \geq 0$ over all $t \in [0, T]$. We conclude that $\dot{R}(t) \leq 0$ over $[t_1, t_1 + \epsilon]$. Since $R(t_1) = 0$, this implies that $R(t) \leq 0$ over $[t_1, t_1 + \epsilon]$. Still subject to $R(t) \geq 0$ over all $t \in [0, T]$, it follows that $R(t) = 0$ over all $t \in [t_1, t_1 + \epsilon]$, which also leads to $\dot{R}(t) = 0$ over $[t_1, t_1 + \epsilon]$. Then, (23) implies that $B(t) = \frac{c_1}{k}$ in such an interval, so that (24) implies that $\dot{B}(t_1) = -c_1 < 0$ contradicting the fact that $B(t) = \frac{c_1}{k}$ is constant. We conclude that $R(t_1) = 0$ cannot be true; hence, the assumption that $r_j(t_1) = 0$, $j = 1, \ldots, n$ cannot hold. Therefore, there exists $j \in S$, such that $r_j(t_1) > 0$. Since $r_j(t)$ is continuous, there exists $\epsilon > 0$ such that $r_j(t) > 0$, $t \in [t_1, t_1 + \epsilon]$, proving the lemma. ■

Theorem 1: Let Π be the set of feasible policies for the problem (1)–(8). If there exists $\pi_0 \in \Pi$ under which $r_i(T) = r_j(T)$ for all $i, j \in S$, then there exists an optimal policy $\pi^* \in \Pi$ such that $r_i^*(T) = r_j^*(T)$ for all $i, j \in S$.

Proof: For any optimal policy, let

$$S_1 = \{i : r_i^*(T) = \bar{r}\}, \quad S_2 = \{i : r_i^*(T) > \bar{r}\} \tag{25}$$

where $S = S_1 \cup S_2$, and $\bar{r}$ is the optimal value of the objective function in (1). Moreover, let

$$l = \arg\min_{i \in S_2}\{r_i^*(T)\} \tag{26}$$

so that for all $j \in S_1$ we have $\bar{r} = r_j^*(T) < r_l^*(T)$. Note that if $S_2 = \varnothing$, then $S_1 = S$, i.e., $r_i^*(T) = \bar{r}$ for all $i \in S$, which proves the theorem. Thus, suppose $S_2 \neq \varnothing$ and consider two possible cases.

Case 1: For all $j \in S_1$, $\int_0^T u_j^*(t)dt > 0$.

Case 2: For at least one $a \in S_1$, $\int_0^T u_a^*(t)dt = 0$.

Let us consider Case 1 first.

Under Case 1, we will use a contradiction argument to prove the assumed optimal policy is not optimal. In particular, since $S_2 \neq \varnothing$ for such a policy, then $r_i^*(T) \neq r_j^*(T)$ for some $i, j \in S$. We will show that there is a feasible perturbed policy that provides a higher objective value. There are two possible subcases to consider, as follows.

Case 1a: For all $j \in S_1$, $\int_0^T u_j^*(t)dt > 0$ and $r_l^*(t) > 0$ for all $t \in [0, T]$. In this case, the assumption of Lemma 1 applies for $l \in S_2$. In addition, there exists an interval $[\tau_j, \tau_j + \Delta_j] \subseteq [0, T]$ such that $u_j^*(t) > 0$ for all $t \in [\tau_j, \tau_j + \Delta_j]$ [this may include a boundary arc $r_j^*(t) = 0$ where $u_j^*(t) = [(kb_j^*(t))/c_1] > 0$ as seen in (2)]. Therefore, condition C1) of Lemma 1 holds for all $j \in S_1$. We now construct a perturbed policy where the controls of all $j \in S_1$ and of l in (26) are perturbed to $\{u_j'(t), h_j'(t)\}$ for $j \in S_1$ and to $\{u_l'(t), h_l'(t)\}$ as follows. We sequentially apply $\pi^-[u_j, \tau_j, \Delta_j, \Delta u_j]$ in (13) to each $j \in S_1$ and each time this is done we also apply $\pi^+[u_l, \tau_j, \Delta_j, \Delta u_l]$ to $l \in S_2$ over each individual $[\tau_j, \tau_j + \Delta_j]$. By Corollary 1, the resulting perturbed policy after each application of $\pi^-[\cdot]$, $\pi^+[\cdot]$ is feasible. In particular, let us arbitrarily re-index S_1 by $m = 1, \ldots, M$ where $M = |S_1|$. Then, let $\{u_m'(t), h_m'(t)\}$ and $\{u_l^{(m)}(t), h_l^{(m)}(t)\}$ denote the control resulting from the mth application of $\pi^-[\cdot]$ and $\pi^+[\cdot]$ respectively. By Corollary 1, $\pi^-[u_1, \tau_1, \Delta_1, \Delta u_1]$ and $\pi^+[u_l, \tau_1, \Delta_1, \Delta u_1]$ result in feasible $\{u_1'(t), h_1'(t)\}$ and $\{u_l^{(1)}(t), h_l^{(1)}(t)\}$. Repeating this process for $m = 2, \ldots, M$, applying $\pi^-[u_m, \tau_m, \Delta_m, \Delta u_m]$ and $\pi^+[u_l^{(m-1)}, \tau_m, \Delta_m, \Delta u_m]$ results in feasible $\{u_m'(t), h_m'(t)\}$ and $\{u_l^{(m)}(t), h_l^{(m)}(t)\}$. At the final step, we set $\{u_l^{(M)}(t), h_l^{(M)}(t)\} \equiv \{u_l'(t), h_l'(t)\}$ which is a feasible control and it is easy to check that $\sum_{k \in S} u_k'(t) = \sum_{k \in S} u_k(t)$ for all $t \in [\tau_m, \tau_m + \Delta_m]$, $m = 1, \ldots, M$.

By the construction of the perturbed policy, (15) applies to all $j \in S_1$, so that $r_j'(T) > \bar{r}$. Moreover, (17) applies to $l \in S_2$ so that $r_l'(T) < r_l^*(T)$. By selecting Δu_j, Δ_j sufficiently small, however, we can guarantee $|\Delta r_l(T)|$ in (17) is sufficiently small to ensure that $r_l'(T) > \bar{r}$. Therefore, under the feasible perturbed policy, the objective value is

$$\bar{r}' = \min_{i \in S_1 \cup S_2} r_i'(T) > \bar{r} \tag{27}$$

which contradicts the optimality of a policy with $r_i^*(T) \neq r_j^*(T)$ for at least some $i, j \in S$.

Case 1b: For all $j \in S_1$, $\int_0^T u_j^*(t)dt > 0$ and $r_l^*(t) = 0$ for some $t \in [0, T]$. In this case, let $t_e = \sup\{t : t \in (0, T),\ r_l^*(t) = 0\}$. Note that $t_e < T$ since $r_l^*(T) > \bar{r} \geq 0$ from (25). Then, we have $r_l^*(t) > 0$ over $(t_e, T]$ and $r_l^*(t_e) = 0$. Therefore, for any $j \in S_1$ such that $u_j^*(t) > 0$ over $[\tau_j, \tau_j + \Delta_j] \subset [t_e, T]$, the construction of perturbed controls is the same as in Case 1 since $r_l^*(t) > 0$ over $(t_e, T]$ and we can obtain $r_j'(T) > \bar{r}$ and $r_l^*(T) > r_l'(T) > \bar{r}$ by adequately selecting Δu_j, Δ_j.

Thus, we only need to consider $j \in S_1$ such that $u_j^*(t) = 0$ over $[t_e, T]$ in the sequel. In this case, however, since $\int_0^T u_j^*(t)dt > 0$ for all $j \in S_1$, there must exist some interval $[\tau_j, \tau_j + \Delta_j] \subseteq [0, t_e)$ with $u_j^*(t) > 0$ for all $t \in [\tau_j, \tau_j + \Delta_j]$. If there exists some interval $[\tau_l, \tau_l + \Delta_l] \subseteq [t_e, T]$ over which $\sum_{i \in S} u_i^*(t) < 1$, then condition C2) holds and Lemma 1 applies, so that we can again obtain a feasible perturbed policy with $r_j'(T) > \bar{r}$ and $r_l^*(T) > r_l'(T) > \bar{r}$ satisfying (27) by adequately selecting Δu_j, Δ_j.

Consequently, the only remaining case to consider is that of $j \in S_1$ such that both $u_j^*(t) = 0$ and $\sum_{i \in S} u_i^*(t) = 1$ for all $t \in [t_e, T]$. By Lemma 2, there exists some $m \in S$ such that $u_m^*(t) > 0$ and $r_m^*(t) > 0$ over a finite-length interval $[t_e, t_e + \epsilon] \subseteq [t_e, T]$. Then, regarding the interval $(0, t_e)$, we consider two possible cases: 1) $r_m^*(t) > 0$ for all $t \in (0, t_e)$, and 2) $r_m^*(t) = 0$ at some $t \in (0, t_e)$.

In case 1), since all remaining $j \in S_1$ satisfy $u_j^*(t) > 0$ over some interval $[\tau_j, \tau_j + \Delta_j] \subseteq [0, t_e)$, Corollary 1 may be used over $[0, t_e + \epsilon]$ for j, m and we can apply $\pi^-[u_j, \tau_j, \Delta_j, \Delta u_j]$ and $\pi^+[u_m, \tau_j, \Delta_j, \Delta u_m]$ to all j and m by proceeding exactly as in Case 1 above. Although the resulting policy is feasible over $[0, t_e + \epsilon]$, we do not know whether $m \in S_1$ or $m \in S_2$ and whether $r_m^*(t) > 0$ over $(t_e + \epsilon, T]$. As a result, we cannot ensure that $r_m'(T) > \bar{r}$ or that the constraint $r_m'(t) \geq 0$ is satisfied over $(t_e + \epsilon, T]$. We can still achieve this, however, by applying $\pi^-[u_m, t_e, \epsilon, \Delta u_m]$ and $\pi^+[u_l, t_e, \epsilon, \Delta u_m]$ since $u_m^*(t) > 0$ over $[t_e, t_e + \epsilon]$ and $r_l^*(t) > 0$ over $(t_e, T]$ so Corollary 1 can be used again for m, l. In this way, by appropriately selecting $\Delta u_j, \Delta_j, \Delta u_m$, we can construct a perturbed policy that satisfies (27).

In case 2), let $t_1 = \sup\{t : t \in (0, t_e),\ r_m^*(t) = 0\}$. Since $r_m^*(t) > 0$ over $(t_1, t_e]$, we can still apply Corollary 1 for all $j \in S_1$ such that $\int_{t_1}^{t_e} u_j^*(t)dt > 0$ since $u_j^*(t) > 0$ over some $[\tau_j, \tau_j + \Delta_j] \subseteq (t_1, t_e]$. We apply $\pi^-[u_j, \tau_j, \Delta_j, \Delta u_j]$ and $\pi^+[u_m, \tau_j, \Delta_j, \Delta u_m]$ to all j and m and proceed exactly as in Case 1 above.

At this point, we have found perturbed policies which satisfy (27) for all $j \in S_1$ such that $\int_{t_1}^T u_j^*(t) > 0$. We can now proceed backward in time and repeat the exact same argument over $[t_2, t_1]$ where $t_2 = \sup\{t{:}t \in (0, t_1),\ r_{n_1}^*(t) = 0\}$ since, by Lemma 2, there exists some $n_1 \in S$ such that $u_{n_1}^*(t) > 0$ and $r_{n_1}^*(t) > 0$ over $[t_1, t_1 + \epsilon]$. If $r_{n_1}^*(t) > 0$ for all $t \in [0, t_1]$, we apply $\pi^-[u_j, \tau_j, \Delta_j, \Delta u_j]$ to all remaining $j \in S_1$ and $\pi^+[u_{n_1}, \tau_j, \Delta_j, \Delta u_{n_1}]$; otherwise, we limit ourselves to those $j \in S_1$ such that $u_j^*(t) > 0$ for some interval in $(t_2, t_1]$ and repeat the process over $[t_3, t_2]$ where $t_3 = \sup\{t{:}t \in (0, t_2),\ r_{n_2}^*(t) = 0\}$ where $n_2 \in S$ such that $u_{n_2}^*(t) > 0$ and $r_{n_2}^*(t) > 0$ over $[t_2, t_2 + \epsilon]$ and the existence of n_2 is still guaranteed by Lemma 2. Clearly, this iterative process ends when all $j \in S_1$ are exhausted, at which point a feasible perturbed policy has been constructed satisfying (27) and establishing a contradiction with the assumption that the policy described by (25) is optimal. Therefore, there exists an optimal policy π^* such that $r_i^*(T) = r_j^*(T)$ for all $i, j \in S$.

Next, we consider Case 2, in which there exists at least one $a \in S_1$ such that $\int_0^T u_a^*(t)dt = 0$, therefore $h_a^*(t) > 0$ for all $t \in [0, T]$ and $h_a^*(t)$ attains its maximum feasible value subject to all constraints so as to satisfy $r_a^*(T) = \bar{r}$. In this case, we can find another optimal policy π' such that $r_i'(T) = r_j'(T)$ for all $i, j \in S$ as follows. We construct a perturbed policy in which $\{u_j'(t), h_j'(t)\} = \{u_j^*(t), h_j^*(t)\}$ for all $j \in S_1$ and adjust $\{u_k^*(t), h_k^*(t)\}$ to $\{u_k'(t), h_k'(t)\}$for all $k \in S_2$, so that $r_k'(T) = \bar{r}$. Since, by assumption, $\bar{r}$ is the largest value $r_a(T)$ can attain, under all feasible policies we have $r_a(T) \le \bar{r}$. Then, since we have assumed there exists some $\pi_0 \in \Pi$ with $r_i(T) = r_j(T)$ for all $i, j \in S$, it follows that $r_i(T) = r_e \le \bar{r}$ for all $i \in S$ under π_0. This means that the optimal control that yields $r_i^*(T) \ge \bar{r} \ge r_e$ can be perturbed to a feasible control that yields $r_i(T) = r_e \le \bar{r}$. Looking at (11), since $r_i(T)$ is continuous, all $r_i(T)$ values in $[r_e, \bar{r}]$ are attainable, including $\bar{r}$. As a result, there is a policy π' that yields $r_i'(T) = \bar{r}$ for all $i \in S$, which is also optimal, implying the existence of an optimal policy satisfying $r_i^*(T) = r_j^*(T)$ for all $i, j \in S$. ∎

Remark 1: It is clear from the proof of Theorem 1 that not all optimal policies have the property $r_i^*(T) = r_j^*(T)$ for all $i, j \in S$. While this is true in Case 1, under Case 2 it is possible to have an optimal policy in which one or more batteries are never actually discharged over $[0, T]$. In the latter case, however, there is always a policy satisfying $r_i^*(T) = r_j^*(T)$ for all $i, j \in S$ which is also optimal.

We will now tackle the situation where there exists no $\pi_0 \in \Pi$ under which $r_i(T) = r_j(T)$ for all $i, j \in S$. Let us start by defining $\bar{r}_i(T)$ as the maximum reachable value in (11) based on the initial condition $\rho_i(0, T)$ defined in (11) and setting $h_i(t)$ to its maximum feasible value subject to $b_i(t) \le B$. Let

$$\bar{r}_L(T) = \min_{i \in S}\{\bar{r}_i(T)\}, \quad L = \operatorname*{argmin}_{i \in S}\{\bar{r}_i(T)\}. \tag{28}$$

We will show in Theorem 2 that $\bar{r}_L(T)$ is the optimal value of the objective function in (1). We will accomplish this with the help of the following lemma.

Lemma 3: If there exists no feasible policy $\pi_0 \in \Pi$ such that $r_i(T) = r_j(T)$ for all $i, j \in S$, then under an optimal control policy π^*, there exists $k \in S$ such that $r_k^*(T) > \bar{r}_L(T)$.

Proof: We will use a contradiction argument and assume that under π^* we have $r_i^*(T) \le \bar{r}_L(T)$ for all $i \in S$. We have already established that in an optimal policy we have either $u_i^*(t) > 0, h_i^*(t) = 0$ or $u_i^*(t) = 0, h_i^*(t) > 0$ for any $t \in [0, T]$. Therefore, if $r_L^*(T) = \bar{r}_L(T)$ we have $\int_0^T u_i^*(t)dt > 0$ for all $i \in S/\{L\}$, and if $r_L^*(T) < \bar{r}_L(T)$ we have $\int_0^T u_i^*(t)dt > 0$ for all $i \in S$. Since the case where $r_i^*(T) = \bar{r}_L(T)$ for all $i \in S$ is excluded by the assumption that a policy π_0 is not feasible, let us define two sets $S_1 = \{i : r_i^*(T) < \bar{r}_L(T)\}$, $S_2 = \{i : r_i^*(T) = \bar{r}_L(T)\}$. Note that regardless of whether $L \in S_1$ or $L \in S_2$, we have $\int_0^T u_j^*(t)dt > 0$ for all $j \in S_1$. Next, there are two cases to consider.

First, suppose $S_2 \neq \emptyset$. Then, we can perturb the controls of all $j \in S_1$ and all $k \in S_2$ to increase $r_j(T)$ and decrease $r_k(T)$ respectively. Since $\bar{r}_L(T) = \min_{i \in S}\{\bar{r}_i(T)\}$, it is feasible for each $r_j(T)$ to increase and reach the value $r_j'(T) = \bar{r}_L(T)$. Moreover, from (11), $r_i(T)$ can be continuously perturbed for all $i \in S$. Therefore, we can fix a value $r_i'(T) < \bar{r}_L(T)$ which is attainable by all $i \in S$. This contradicts the assumption that π_0 does not exist. Consequently, it is not possible to satisfy $r_i^*(T) \le \bar{r}_L(T)$ for all $i \in S$ under π^* and it follows that $r_k^*(T) > \bar{r}_L(T)$ for some $k \in S$.

Second, suppose $S_2 = \emptyset$, i.e., $S_1 = S$. Then, we can always find some $l = \arg\max_{i \in S}\{r_i^*(T)\}$ and similarly perturb the controls of all $j \in S/\{l\}$ and of l so as to increase $r_j(T)$ and decrease $r_l(T)$ through (11). Since $\int_0^T u_i^*(t)dt > 0$ for all $i \in S$, it follows that $r_l^*(T)$ is not the smallest value that l can reach. In addition, each $r_j(T)$ can be increased to $\bar{r}_L(T)$ since $r_j^*(T) \le \bar{r}_L(T)$. Thus, we can fix a value $r_i'(T) < \bar{r}_L(T)$ which is attainable by all $i \in S$. This again contradicts the assumption that π_0 does not exist. Consequently, it is not possible to satisfy $r_i^*(T) \le \bar{r}_L(T)$ for all $i \in S$ under π^* and it follows that $r_k^*(T) > \bar{r}_L(T)$ for some $k \in S$. ∎

Theorem 2: If there exists no feasible policy $\pi_0 \in \Pi$ such that $r_i(T) = r_j(T)$ for all $i, j \in S$, then the optimal value of the objective function is $\bar{r}_L(T)$ in (28).

Proof: We will use a contradiction argument. Assume the optimal value of the objective function is $r^* < \bar{r}_L(T)$. Let us define three sets

$$S_1 = \{i : r_i^*(T) < \bar{r}_L(T)\}, \quad S_2 = \{i : r_i^*(T) = \bar{r}_L(T)\}$$
$$S_3 = \{i : r_i^*(T) > \bar{r}_L(T)\}.$$

By Lemma 3, $S_3 \neq \emptyset$. On the other hand, if $S_1 = \emptyset$, then it directly contradicts the assumption $r^* < \bar{r}_L(T)$. Therefore, $S_1 \neq \emptyset$ in the following argument. Since we have established that in an optimal policy we have either $u_i^*(t) > 0, h_i^*(t) = 0$ or $u_i^*(t) = 0, h_i^*(t) > 0$ and in view of the definition of $\bar{r}_L(T)$ in (28), we have $\int_0^T u_j^*(t)dt > 0$ for all $j \in S_1$. We can now proceed similar to the argument used in Cases 1 and 2 in the proof of Theorem 1. We can always find some $l \in S_3$ to increase $r_j(T)$ for all $j \in S_1$ through perturbations $\pi^-[u_j, \tau_j, \Delta_j, \Delta u_j]$ for all $j \in S_1$ and decrease $r_l(T)$ through $\pi^+[u_l, \tau_j, \Delta_j, \Delta u_l]$ for $l \in S_3$ as long as $r_j'(T) \le \bar{r}_L(T)$ and $l \in S_3$. If $r_l'(T)$ decreases to a value $r_l'(T) = \bar{r}_L(T)$, i.e., $l \in S_2$, and not all $r_j'(T)$ increase to $r_j'(T) = \bar{r}_L(T)$, i.e., $S_1 \neq \emptyset$, then we can select some other $m \in S_3$ to repeat the process. By Lemma 3, S_3 will never be empty. However, we will eventually reach the point where all $r_j'(T) = \bar{r}_L(T)$ for all $j \in S_1$, thus emptying S_1. Then, we contradict the assumption that $r^* < \bar{r}_L(T)$ and thus prove the theorem. ∎

IV. Optimal Control Solution

In this section, we provide a complete solution to the problem (1)–(8) by making use of the two main results in Section III. If there exists no feasible policy $\pi_0 \in \Pi$ such that $r_i(T) = r_j(T)$ for all $i, j \in S$, we can directly determine the optimal objective function value by Theorem 2. Therefore, let us concentrate on the case where $\pi_0 \in \Pi$ exists. Then, by Theorem 1, we can add a terminal state constraint to the problem (1)–(8) without affecting its solution

$$r_i(T) = r_j(T) \qquad \forall i, j \in S \tag{29}$$

so that in the original objective function (1) we have $\min_{i\in S} r_i(T) = r_i(T)$ for any $i \in S$. Since $\max_{U(t)} r_i(T) = \max_{U(t)} \sum_{i=1}^{N} r_i(T)$ in light of (29), we can rewrite (1) as

$$\min_{U(t)} \; -\sum_{i=1}^{N} r_i(T). \tag{30}$$

As for the integral constraint (8), we define an additional state variable $q(t)$ and replace (8) by

$$\dot{q}(t) = \sum_{i=1}^{N} u_i(t), \quad q(0) = 0, \quad q(T) = Q. \tag{31}$$

Now the original max-min problem becomes a typical optimal control problem with terminal state constraints. However, there are $2N + 1$ states in total such that the problem is not easy to solve if N is large. Thus, we will start with the $N = 2$ case which provides insights allowing us to tackle the higher-dimensional cases.

A. Solution of the $N = 2$ Case

When $N = 2$ we index the batteries so that $\rho_1(0, T) \geq \rho_2(0, T)$. Accordingly, the control is $U(t) = (u_1(t), h_1(t), u_2(t), h_2(t))^T$. Moreover, in order to satisfy (29) it follows from (11) that $U(t)$ must be such that

$$\rho_1(0,T) - \rho_2(0,T) = \int_0^T \left(c_1(u_1(r) - u_2(r)) \frac{1 + e^{2k(r-T)}}{2} - c_2(h_1(r) - h_2(r)) \frac{1 - e^{2k(r-T)}}{2} \right) dr.$$

Based on the definition of Π in Theorem 1, we denote the set of feasible policies in Π satisfying (29) by Π_0. Subject to the control constraints (5)–(8), no feasible solution exists if $\rho_1(0, T) - \rho_2(0, T) > \bar{\alpha}$ where $\bar{\alpha}$ is determined from the above equation

$$\bar{\alpha} = \max_{\pi \in \Pi_0} \int_0^T \left(c_1(u_1(r) - u_2(r)) \frac{1 + e^{2k(r-T)}}{2} - c_2(h_1(r) - h_2(r)) \frac{1 - e^{2k(r-T)}}{2} \right) dr.$$

Since $1 + e^{2k(t-T)}$ is monotonically increasing in t, $\bar{\alpha}$ is attained by letting $u_1(t) = 0$ over $[0, T - Q)$ and $u_1(t) = 1$ over $[T - Q, T]$, $h_1(t) = 0$, $u_2(t) = 0$, $h_2(t) = 1$ over $[0, T]$

$$\bar{\alpha} = \int_{T-Q}^{T} c_1 \frac{1 + e^{2k(r-T)}}{2} dr + \int_0^T c_2 \frac{1 - e^{2k(r-T)}}{2} dr. \tag{32}$$

Then, $\rho_1(0, T) - \rho_2(0, T) \leq \bar{\alpha}$ must be satisfied to ensure a feasible solution.

1) Unconstrained Case: In order to obtain an explicit optimal control $U^*(t)$, we proceed as in [29] by first analyzing the unconstrained case in which (4) is relaxed and the optimal state trajectories for both batteries consist of an interior arc over the entire interval $[0, T]$. Let $\mathbf{x}(t) = (r_1(t), b_1(t), r_2(t), b_2(t), q(t))^T$ and $\lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t), \lambda_5(t))^T$ denote the state and costate vector respectively. The Hamiltonian for this problem is then

$$\begin{aligned} H(\mathbf{x}, \lambda, u_1, h_1, u_2, h_2) &= \lambda(t)^T \dot{\mathbf{x}}(t) \\ &= [-c_1\lambda_1(t) + \lambda_5(t)]u_1(t) + c_2\lambda_2(t)h_1(t) \\ &\quad + [-c_1\lambda_3(t) + \lambda_5(t)]u_2(t) + c_2\lambda_4(t)h_2(t) \\ &\quad + k[\lambda_1(t) - \lambda_2(t)][b_1(t) - r_1(t)] \\ &\quad + k[\lambda_3(t) - \lambda_4(t)][b_2(t) - r_2(t)]. \end{aligned} \tag{33}$$

The costate equations $\dot{\lambda} = -\frac{\partial H}{\partial \mathbf{x}}$ give

$$\begin{aligned} &\dot{\lambda}_1(t) = k(\lambda_1(t) - \lambda_2(t)), \quad \dot{\lambda}_2(t) = -k(\lambda_1(t) - \lambda_2(t)) \\ &\dot{\lambda}_3(t) = k(\lambda_3(t) - \lambda_4(t)), \quad \dot{\lambda}_4(t) = -k(\lambda_3(t) - \lambda_4(t)) \\ &\dot{\lambda}_5(t) = 0 \end{aligned} \tag{34}$$

and, due to (29) and (31), we must satisfy $\lambda(T) = (\partial\Phi(\mathbf{x}(T)))/\partial\mathbf{x}$ where $\Phi(\mathbf{x}(T)) = -r_1(T) - r_2(T) + \nu_1(r_1(T) - r_2(T)) + \nu_2(q(T) - Q)$ and ν_1, ν_2 are unknown multipliers, so that

$$\begin{aligned} &\lambda_1(T) = -1 + \nu_1, \quad \lambda_2(T) = 0 \\ &\lambda_3(T) = -1 - \nu_1, \quad \lambda_4(T) = 0, \quad \lambda_5(T) = \nu_2. \end{aligned} \tag{35}$$

Solving (34) with the boundary conditions (35), we get

$$\begin{cases} \lambda_1(t) = \frac{\nu_1 - 1}{2}[1 + e^{2k(t-T)}] \\ \lambda_2(t) = \frac{\nu_1 - 1}{2}[1 - e^{2k(t-T)}] \\ \lambda_3(t) = \frac{-\nu_1 - 1}{2}[1 + e^{2k(t-T)}] \\ \lambda_4(t) = \frac{-\nu_1 - 1}{2}[1 - e^{2k(t-T)}] \\ \lambda_5(t) = \nu_2. \end{cases} \tag{36}$$

Looking at (33), we define the switching functions $s_1(t), s_2(t)$ and $s_3(t), s_4(t)$ corresponding to $u_1(t), h_1(t)$ and $u_2(t), h_2(t)$ respectively

$$\begin{aligned} &s_1(t) = -c_1\lambda_1(t) + \lambda_5(t), \quad s_2(t) = c_2\lambda_2(t) \\ &s_3(t) = -c_1\lambda_3(t) + \lambda_5(t), \quad s_4(t) = c_2\lambda_4(t) \end{aligned} \tag{37}$$

and apply the Pontryagin minimum principle: $H(\mathbf{x}^*, \lambda^*, u_i^*, h_i^*) = \min_{(u_i, h_i)} H(\mathbf{x}, \lambda, u_i, h_i)$, where $u_i^*(t)$, $h_i^*(t)$ for $i = 1, 2$, $t \in [0, T)$, denote the optimal controls. We can then see that

$$u_1^*(t) = \begin{cases} 1 & s_1(t) < 0 \\ 0 & s_1(t) > 0, \end{cases} \quad h_1^*(t) = \begin{cases} 1 & s_2(t) < 0 \\ 0 & s_2(t) > 0 \end{cases}$$

$$u_2^*(t) = \begin{cases} 1 & s_3(t) < 0 \\ 0 & s_3(t) > 0, \end{cases} \quad h_2^*(t) = \begin{cases} 1 & s_4(t) < 0 \\ 0 & s_4(t) > 0. \end{cases}$$

Singular cases may arise when $\nu_2 = 0$ and $\nu_1 = 1$ or -1, making $s_1(t) = s_2(t) = 0$ or $s_3(t) = s_4(t) = 0$ respectively. Let us proceed by setting these aside for the time being. Given the constraint $u_i(t)h_i(t) = 0$, as well as the already excluded $u_i^*(t) = h_i^*(t) = 0$ (see Section III), we can set $h_i^*(t) = 1 - u_i^*(t)$ in this unconstrained case and rewrite $H(\mathbf{x}, \lambda, u_i, h_i)$ as follows:

$$\begin{aligned} H(\mathbf{x}, \lambda, u_i, h_i) = \; &\sigma_1(t)u_1(t) + \sigma_2(t)u_2(t) + c_2\lambda_2(t) \\ &+ c_2\lambda_4(t) + k[\lambda_1(t) - \lambda_2(t)][b_1(t) - r_1(t)] \\ &+ k[\lambda_3(t) - \lambda_4(t)][b_2(t) - r_2(t)] \end{aligned} \tag{38}$$

where $\sigma_1(t) = -c_1\lambda_1(t) + \lambda_5 - c_2\lambda_2(t)$, $\sigma_2(t) = -c_1\lambda_3(t) + \lambda_5 - c_2\lambda_4(t)$ are the new switching functions of u_1, u_2 respectively. Using (36), σ_1, σ_2 become

$$\sigma_1(t) = \tfrac{1-\nu_1}{2}\left[c_1 + c_2 + (c_1 - c_2)e^{2k(t-T)}\right] + \nu_2 \quad (39)$$

$$\sigma_2(t) = \tfrac{1+\nu_1}{2}\left[c_1 + c_2 + (c_1 - c_2)e^{2k(t-T)}\right] + \nu_2. \quad (40)$$

Thus, to minimize (38), the optimal control on the interior arc is

$$\begin{cases} u_i^*(t) = 0,\ h_i^*(t) = 1 & \text{if } \sigma_i(t) > 0 \\ u_i^*(t) = 1,\ h_i^*(t) = 0 & \text{if } \sigma_i(t) < 0 \end{cases} \quad (41)$$

for $i = 1, 2$. We immediately observe in (41) that $u_1^*(t) = u_2^*(t) = 1$ when $\sigma_1(t) < 0$ and $\sigma_2(t) < 0$, which violates the constraint (7). In this case, 1) $u_1^*(t) = 1, u_2^*(t) = 0$ if $\sigma_1(t) < \sigma_2(t) < 0$; 2) $u_1^*(t) = 0, u_2^*(t) = 1$ if $\sigma_2(t) < \sigma_1(t) < 0$; and 3) either $u_1^*(t) = 1, u_2^*(t) = 0$ or $u_1^*(t) = 0, u_2^*(t) = 1$ if $\sigma_1(t) = \sigma_2(t) < 0$. Correspondingly, $h_i^*(t) = 1 - u_i^*(t)$, $i = 1, 2$. In other words, the optimal control in the interior arc depends on the sign of $\sigma_1(t) - \sigma_2(t)$ when $\sigma_1(t) < 0$, $\sigma_2(t) < 0$. By (39) and (40)

$$\sigma_1(t) - \sigma_2(t) = -\nu_1\left(c_1 + c_2 + (c_1 - c_2)e^{2k(t-T)}\right). \quad (42)$$

Therefore, along with (41), the optimal control can be summarized as

$$U^*(t) = (0, 1, 0, 1)^T \quad \text{if} \quad \sigma_1(t) > 0,\ \sigma_2(t) > 0 \quad (43)$$

$$U^*(t) = (0, 1, 1, 0)^T \quad \text{if} \quad \begin{matrix}\sigma_2(t) < 0 < \sigma_1(t) \text{ or} \\ \sigma_2(t) < \sigma_1(t) < 0\end{matrix} \quad (44)$$

$$U^*(t) = (1, 0, 0, 1)^T \quad \text{if} \quad \begin{matrix}\sigma_1(t) < 0 < \sigma_2(t) \text{ or} \\ \sigma_1(t) < \sigma_2(t) < 0.\end{matrix} \quad (45)$$

Note that by (39) and (40), $\sigma_1(t) = \sigma_2(t)$ when $\nu_1 = 0$, but one can see that $\sigma_1(t) = \sigma_2(t) = 0$ is not possible for any finite-length time interval. Thus, when $\sigma_1(t) = \sigma_2(t)$, we only need to consider the solution with $\sigma_1(t) = \sigma_2(t) > 0$ or $\sigma_1(t) = \sigma_2(t) < 0$. The solution to the former is given in (43) and for the latter it is either $(1, 0, 0, 1)^T$ or $(0, 1, 1, 0)^T$ as already analyzed earlier for the case where $\sigma_1(t) < 0$ and $\sigma_2(t) < 0$. Now, in view of (39) and (40), we can determine the optimal solution by considering all possible values of the unknown constants ν_1, ν_2.

Case 1, $\nu_1 = 0$: By (42), $\sigma_1(t) - \sigma_2(t) = 0$, implying that the optimal control $U^*(t)$ can be either $(0, 1, 1, 0)^T$ or $(1, 0, 0, 1)^T$ if $\sigma_1(t) = \sigma_2(t) < 0$; and $(0, 1, 0, 1)^T$ if $\sigma_1(t) = \sigma_2(t) > 0$ according to (43). Moreover, in terms of (39) and (40) we have

$$\sigma_1(t) = \sigma_2(t) = \frac{1}{2}\left((c_1 + c_2) + (c_1 - c_2)e^{2k(t-T)}\right) + \nu_2.$$

Now, by analyzing ν_2, we can determine the solution as follows.

1) If $\nu_2 \geq -\frac{1}{2}\left((c_1 + c_2) + (c_1 - c_2)e^{-2kT}\right)$, then $\sigma_1(t) = \sigma_2(t) > 0$ over $(0, T]$ such that $U^*(t) = (0, 1, 0, 1)^T$ over $[0, T]$. However, this solution violates the constraint (8) and thus can be excluded.

2) If $-c_1 < \nu_2 < -\frac{1}{2}\left((c_1 + c_2) + (c_1 - c_2)e^{-2kT}\right)$, then $\sigma_1(t) = \sigma_2(t) < 0$ over $[0, t_s)$ and $\sigma_1(t) = \sigma_2(t) > 0$ over $(t_s, T]$, where t_s is the switching time, such that

$$U^*(t) = \begin{cases} (1, 0, 0, 1)^T \text{ or } (0, 1, 1, 0)^T & t \in [0, t_s] \\ (0, 1, 0, 1)^T & t \in (t_s, T]. \end{cases}$$

Since $u_1^*(t)+u_2^*(t) = 1$ over $[0, t_s)$ and $u_1^*(t)+u_2^*(t) = 0$ over $(t_s, T]$, the constraint (8) requires $t_s = Q$. Despite the nonuniqueness of the solution over $[0, Q]$, $U^*(t)$ over $[0, Q]$ should be such that (29) is satisfied, i.e., $r_1^*(T) = r_2^*(T)$.

3) If $\nu_2 \leq -c_1$, then $\sigma_1(t) = \sigma_2(t) < 0$ over $[0, T)$, implying $U^*(t) = (1, 0, 0, 1)^T$ or $(0, 1, 1, 0)^T$ at any time $t \in [0, T)$. Furthermore, since $u_1^*(t) + u_2^*(t) = 1$ over $[0, T)$, this implies that $T = Q$ so as to satisfy (8) while $U^*(t)$ is still subject to (29). Since $T = Q$ is a special case, this solution is of little interest.

Remark 2: Cases (b) and (c) determine a class of solutions in which the two batteries cooperatively discharge in order to satisfy the total load requirement specified by Q. We shall refer to this as a type I solution. Moreover, according to (11), in order to satisfy (29) $\rho_i(0, T)$ must be such that

$$\rho_1(0, T) - \rho_2(0, T) = \int_0^T c_1(u_1(t) - u_2(t))\tfrac{1+e^{2k(t-T)}}{2} - c_2(h_1(t) - h_2(t))\tfrac{1-e^{2k(t-T)}}{2}dt. \quad (46)$$

Regarding the solutions in cases (b) and (c), $h_i^*(t) = 1 - u_i^*(t)$ over $[0, T]$ and $h_1^*(t) - h_2^*(t) = 0$ over $(Q, T]$ for both cases [for case (c), $(Q, T]$ is a null set since $T = Q$]. Accordingly, for this type of solution, (46) becomes

$$\rho_1(0, T) - \rho_2(0, T) = \tfrac{1}{2}\int_0^Q (c_1 + c_2)(u_1^*(t) - u_2^*(t)) + (c_1 - c_2)e^{2k(t-T)}dt.$$

Furthermore, for all possible type I solutions, $u_1^*(t) - u_2^*(t) = \pm 1$ over $[0, Q]$. If $u_1^*(t) - u_2^*(t) = 1$ over $[0, Q]$, implying $u_2^*(t) = 0$ all the time, then $\rho_1(0, T) - \rho_2(0, T) = \underline{\alpha}$, where $\underline{\alpha} = \frac{1}{2}\int_0^Q (c_1+c_2)+(c_1-c_2)e^{2k(t-T)}dt$. Otherwise, $\rho_1(0, T) - \rho_2(0, T) < \underline{\alpha}$.

Case 2, $0 < \nu_1 < 1$: Given (42), $\sigma_1(t) < \sigma_2(t)$ over $[0, T]$, which makes the solution only depend on the sign of $\sigma_1(t)$ by (43) and (45). The optimal control in this case can be derived by examining all possible values of ν_2.

1) If $\nu_2 \geq \frac{\nu_1-1}{2}\left((c_1 + c_2) + (c_1 - c_2)e^{-2kT}\right)$, then $0 < \sigma_1(t) < \sigma_2(t)$ over $(0, T]$ such that $U^*(t) = (0, 1, 0, 1)^T$ over $(0, T]$ by (43). Thus, this solution can be excluded for the same reason as Case 1(a).

2) If $(\nu_1 - 1)c_1 < \nu_2 < \frac{\nu_1-1}{2}\left((c_1 + c_2) + (c_1 - c_2)e^{-2kT}\right)$, then $\sigma_1(t) < 0$ over $[0, t_s)$ and $\sigma_1(t) > 0$ over $(t_s, T]$. Accordingly, referring to (43) and (45), when $\sigma_1(t) < \sigma_2(t)$, we have

$$U^*(t) = \begin{cases} (1, 0, 0, 1)^T & t \in [0, t_s] \\ (0, 1, 0, 1)^T & t \in (t_s, T] \end{cases} \quad (47)$$

where $t_s = Q$ due to (8). Moreover, substituting (47) into (11), we require $\rho_i(0, T)$ to meet the following equation in order to satisfy $r_1^*(T) = r_2^*(T)$

$$\rho_1(0, T) - \rho_2(0, T) = \frac{1}{2}\int_0^Q (c_1 + c_2) + (c_1 - c_2)e^{2k(r-T)}dr = \underline{\alpha}. \quad (48)$$

3) If $\nu_2 \leq (\nu_1 - 1)c_1$, then $\sigma_1(t) < 0$ over $[0, T)$. Since $\sigma_1(t) < \sigma_2(t)$, $U^*(t) = (1, 0, 0, 1)^T$ throughout $[0, T]$

by (45). This is a special case of (47) where $t_s = T$. Hence, $\rho_i(0, T)$ should satisfy (48) with $Q = T$.

Case 3, $\nu_1 = 1$: In this case, $\sigma_1(t) = \nu_2$ and $\sigma_1(t) < \sigma_2(t)$ over $[0, T]$. Similar to case (2), we only need to consider the sign of $\sigma_1(t)$ to determine the optimal solution in terms of (43) and (45).

1) If $\nu_2 > 0$, then $\sigma_1(t) > 0$ over $[0, T]$, implying $u_1^*(t) = u_2^*(t) = 0$ throughout. Thus, due to the constraint (8), this case is excluded.
2) If $\nu_2 = 0$, then referring to (36) and (37), $s_1(t) = s_2(t) = 0$ over $[0, T]$, which is the singular case for $u_1^*(t), h_1^*(t)$ as seen in (33). Since the entire optimal state trajectory of battery 1 is a singular arc, then $u_1^*(t), h_1^*(t)$ can be any feasible control satisfying the control constraints (6) and (5). On the other hand, since $\nu_1 = 1, \nu_2 = 0$, then $\sigma_2(t) > 0$ over $[0, T]$, which indicates $u_2^*(t) = 0, h_2^*(t) = 1$ throughout. This requires $\int_0^T u_1^*(t)dt = Q$ due to (8). Therefore, the optimal control $U^*(t) = (u_1^*(t), h_1^*(t), 0, 1)^T$ over $[0, T]$ where $u_1^*(t), h_1^*(t)$ is any feasible control satisfying (5) and (6), $\int_0^T u_1^*(t)dt = Q$ and $r_1^*(T) = r_2^*(T)$.
   In this case, still owing to (29), we also need to determine the range of $\rho_1(0, T) - \rho_2(0, T)$ preserving the feasibility of this solution. Since $U^*(t) = (u_1^*(t), h_1^*(t), 0, 1)^T$ over $[0, T]$, then $\bar{\alpha}$ determined in (32) automatically becomes the upper bound of $\rho_1(0, T) - \rho_2(0, T)$. Moreover, as a special case of $(u_1^*(t), h_1^*(t), 0, 1)^T$, (47) achieves the lower bound of this solution type as in (48), because when $\rho_1(0, T) - \rho_2(0, T) < \underline{\alpha}$, the solution becomes of type I (see Remark 2.) Consequently, $\underline{\alpha} \le \rho_1(0, T) - \rho_2(0, T) \le \bar{\alpha}$ for $U^*(t) = (u_1^*(t), h_1^*(t), 0, 1)^T$ over $[0, T]$.
3) If $\nu_2 < 0$, then $\sigma_1(t) < 0$ over $[0, T]$, implying $U^*(t) = (1, 0, 0, 1)^T$ throughout, which is the same as Case 2(c).

Case 4, $\nu_1 > 1$: According to (42), $\sigma_1(t) < \sigma_2(t)$ over $[0, T]$. Similarly, the solution only depends on the sign of $\sigma_1(t)$.

1) If $\nu_2 \ge (\nu_1 - 1)c_1$, then $\sigma_1(t) > 0$ over $(0, T]$, which makes $U^*(t) = (0, 1, 0, 1)^T$ over $[0, T]$ by (43). As before, this solution can be immediately excluded due to (8).
2) If $\frac{\nu_1 - 1}{2}\left[(c_1 + c_2) + (c_1 - c_2)e^{-2kT}\right] < \nu_2 < (\nu_1 - 1)c_1$, then $\sigma_1(t) > 0$ over $[0, t_s)$ and $\sigma_1(t) < 0$ over $(t_s, T]$. Thus, since $\sigma_1(t) < \sigma_2(t)$, the optimal control can be derived by (43) and (45) as
$$U^*(t) = \begin{cases} (0, 1, 0, 1)^T & t \in [0, t_s) \\ (1, 0, 0, 1)^T & t \in (t_s, T]. \end{cases} \tag{49}$$
   Similar to the case (b) of $0 < \nu_1 < 1$, battery 2 recharges at full rate throughout $[0, T]$. In the meantime, battery 1 recharges at full rate first and then fully discharges until the end so as to attain the required workload Q and achieve $r_1^*(T) = r_2^*(T)$. Therefore, in order to make this solution feasible, we not only require $t_s = T - Q$ in light of (8), but also need to substitute (49) into (11) to satisfy $r_1^*(T) = r_2^*(T)$ so that
$$\rho_1(0, T) - \rho_2(0, T) = \int_{T-Q}^{T} \frac{1}{2}\left[(c_1 + c_2) + (c_1 - c_2)e^{2k(r-T)}\right] dr.$$
   This solution is included in Case 3(b).
3) If $\nu_2 \le \frac{\nu_1 - 1}{2}\left[(c_1 + c_2) + (c_1 - c_2)e^{-2kT}\right]$, then $\sigma_1(t) < 0$ over $(0, T]$, which renders $U^*(t) = (1, 0, 0, 1)^T$ throughout $[0, T]$ and become the same as Case 2(c).

Remark 3: The optimal solutions derived in Case 2–4 can be classified as type II solutions, in which $\underline{\alpha} \le \rho_1(0, T) - \rho_2(0, T) \le \bar{\alpha}$ and battery 2 recharges at full rate all the time while battery 1 serves the load alone by any feasible control $u_1(t), h_1(t)$ satisfying (5) and (6), $\int_0^T u_1(t)dt = Q$ and $r_1(T) = r_2(T)$. Moreover, in all type II solutions, a fixed value of $r_2(T)$ is achieved based on a given $\rho_2(0, T)$ due to $u_2^*(t) = 0$, $h_2^*(t) = 1$ over $[0, T]$.

As for the analysis of $\nu_1 < 0$, since the dynamics of batteries 1 and 2 are identical and σ_1 and σ_2 are symmetric with respect to $\nu_1 = 0$ by (39) and (40), then the result of analyzing $\nu_1 < 0$ will lead to the same solution as $\nu_1 > 0$ but with the roles of batteries 1 and 2 reversed. Thus, a detailed analysis is omitted.

To sum up, following Remarks 2 and 3, the optimal solution to the unconstrained case can be expressed as follows:

type I
$$\text{if } 0 \le \rho_1(0, T) - \rho_2(0, T) \le \underline{\alpha} \tag{50}$$
then
$$U^*(t) = \begin{cases} (1, 0, 0, 1)^T \text{ or } (0, 1, 1, 0)^T & t \in [0, Q] \\ (0, 1, 0, 1)^T & t \in (Q, T] \end{cases}$$
$$\text{s.t.} \quad r_1^*(T) = r_2^*(T) \tag{51}$$

type II
$$\text{if } \underline{\alpha} < \rho_1(0, T) - \rho_2(0, T) \le \bar{\alpha} \tag{52}$$
then
$$U^*(t) = (u_1^*(t), h_1^*(t), 0, 1)^T \quad t \in [0, T] \tag{53}$$
$$\text{s.t.} \quad \text{constraints (5)–(8) and (29).}$$

When $\rho_1(0, T) - \rho_2(0, T) > \bar{\alpha}$, as discussed earlier, there is no feasible solution satisfying the constraint $r_1(T) = r_2(T)$ and we resort to Theorem 2. Note that the solution of type II corresponds to a situation where the initial energy difference of the two batteries, expressed by $\rho_1(0, T) - \rho_2(0, T)$, is so large that it is optimal for battery 2 to recharge at full rate all the time while only battery 1 is utilized; this is similar to the solution determined by Theorem 2. From a practical standpoint, type I solutions are of greater interest, since they provide an insight as to how multiple batteries cooperate to serve a common load.

Obviously, the optimal solution is nonunique in both solution types shown in (51)–(53). We provide two numerical examples in Fig. 2(a) and (b) to verify this feature, where the parameters give rise to a type I solution. The solution in Fig. 2(a) was obtained using the generic numerical solver Tomlab/PROPT [31], where $U^*(t) = (1, 0, 0, 1)^T$ and $(0, 1, 1, 0)^T$ alternate in some arbitrary fashion over $[0, Q]$ and switch to $U^*(t) = (0, 1, 0, 1)^T$ over $(Q, T]$. Alternating

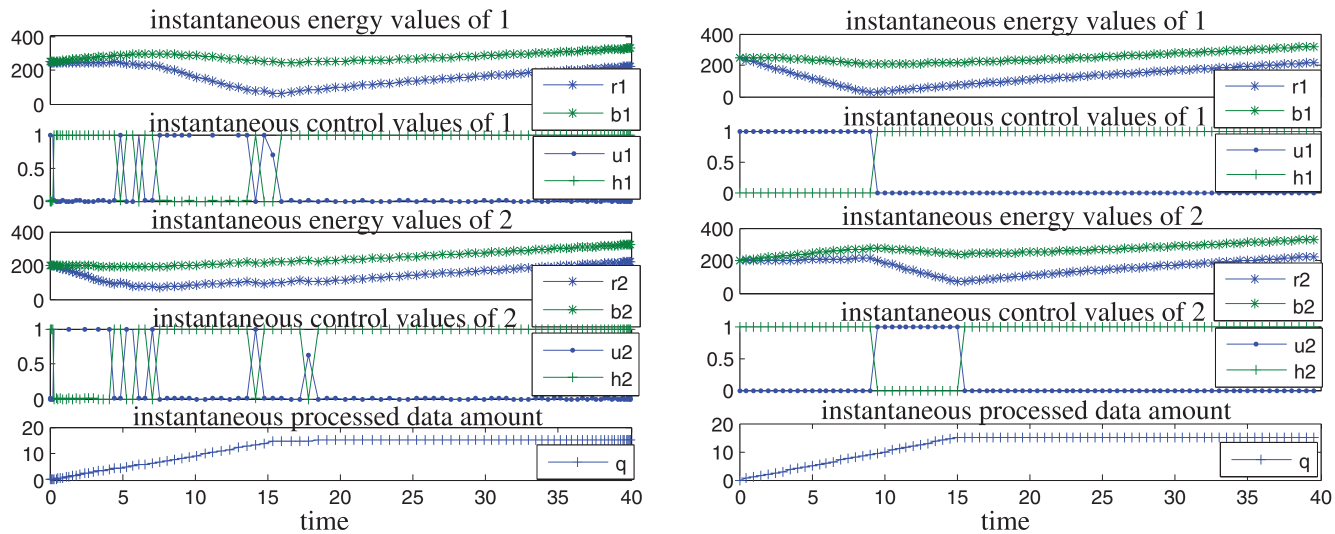


Fig. 2. (a) and (b) Optimal solution under $r_1(0) = 250$, $b_1(0) = 250$, $r_2(0) = 200$, $b_2(0) = 200$, $c_1 = 30$, $c_2 = 10$, $k = 0.05$, $T = 40$, and $Q = 15$.

between $(1, 0, 0, 1)^T$ and $(0, 1, 1, 0)^T$ ensures the constraint (29) is satisfied, which renders $r_1^*(T) = r_2^*(T) = 221.4762$ in this example. In Fig. 2(b), we present another optimal solution, in which, based on the knowledge that any $U^*(t)$ of the form (51) is optimal, the way of alternating between $(1, 0, 0, 1)^T$ and $(0, 1, 1, 0)^T$ over $[0, Q]$ is chosen to be much simpler, i.e., $U^*(t) = (1, 0, 0, 1)^T$ over $[0, t_1]$ and $U^*(t) = (0, 1, 1, 0)^T$ over $(t_1, Q]$ while $U^*(t)$ over $(Q, T]$ is still $(0, 1, 0, 1)^T$. The value of t_1 is 8.7804, leading to $r_1^*(T) = r_2^*(T) = 221.7324$, almost identical to the first (and different) solution, i.e., 221.4762. Note that the nonuniqueness of solutions is due to the assumption that batteries are identical. Thus, it is possible to obtain unique optimal solutions when c_1, c_2 in (2) and (3) are replaced by distinct parameters for each battery $c_{i1} \neq c_{j1}$ and $c_{i2} \neq c_{j2}$ for $i \neq j$.

2) Constrained Case, $r_i(t) \geq 0$: Similar to the analysis presented in [29], when we incorporate the state constraint $r_i(t) \geq 0$ into the unconstrained case, chattering may occur depending on the values of the parameters $\rho_i(0, T)$. First, let us consider the two types of solutions (51)–(53) in the unconstrained case. In the type II solution, battery 1 processes all the workload Q while battery 2 recharges at full rate throughout $[0, T]$. Thus, $r_2^*(t) > 0$ over $(0, T]$. Regarding $r_1^*(t)$, since $\rho_2(0, T) > 0$, by (53)

$$\rho_1(0, T) > \int_0^Q \frac{1}{2}\left[(c_1 + c_2) + (c_1 - c_2)e^{2k(r-T)}\right] dr. \quad (54)$$

Denote the set of type II solutions by Π_2^*. Then, in view of (11) with $\int_0^T u_1^*(t)dt = Q$, we can achieve the lowest value of $r_1^*(t)$ over $t \in [0, T]$ among all $U^*(t) \in \Pi_2^*$ by taking $U^*(t)$ as (47) where the lowest value is $r_1^*(Q)$ under (47)

$$\min_{t\in[0,T],\, U^*\in\Pi_2^*} r_1^*(t) = \rho_1(0, T) - \int_0^Q c_1 \frac{1 + e^{2k(r-T)}}{2} dr$$

which is $> \int_0^Q c_2[(1 - e^{2k(r-T)})/2]dr > 0$ by (54). Therefore, the constraint $r_i(t) \geq 0$ is not active in a type II solution. As for a type I solution, note that $U^*(t) = (1, 0, 0, 1)^T$ or $(0, 1, 1, 0)^T$ during $[0, Q]$, which implies that if $r_1^*(t)$ reaches 0 we can switch the control to $(0, 1, 1, 0)^T$ and correspondingly increase $r_1^*(t)$ but decrease $r_2^*(t)$. A similar scheme applies when $r_2^*(t)$ reaches 0. However, when $r_1^*(t) = r_2^*(t) = 0$ at some time $t_c \in [0, Q)$, it is impossible for $U^*(t)$ to keep either $(1, 0, 0, 1)^T$ or $(0, 1, 1, 0)^T$ over $[t_1, Q]$ without violating $r_i(t) \geq 0$. At this time, referring to our analysis of this case in [29], chattering will also occur due to the same state dynamics (2)–(3) and constraints (5)–(6). This is further complicated by the presence of the constraint (7). Therefore, in practice it is desirable to avoid chattering and several approaches to achieving this goal are discussed in [29]. In addition, we can determine whether chattering will occur depending on the value of Q as follows. Note that in (51), $\sum_i u_i^*(t) = \sum_i h_i^*(t) = 1$ over $[0, Q)$. Moreover, despite the freedom of switching $U^*(t)$ between $(1, 0, 0, 1)^T$ and $(0, 1, 1, 0)^T$, chattering is still inevitable when $r_1^*(t) = r_2^*(t) = 0$ at $t_c \in [0, Q)$, i.e., the starting time of chattering. Therefore, by solving the following differential equations:

$$\sum_i \dot{r}_i(t) = -c_1 \sum_i u_i(t) + k\left[\sum_i b_i(t) - \sum_i r_i(t)\right] \quad (55)$$

$$\sum_i \dot{b}_i(t) = c_2 \sum_i h_i(t) - k\left[\sum_i b_i(t) - \sum_i r_i(t)\right] \quad (56)$$

with $\sum_i u_i^*(t) = \sum_i h_i^*(t) = 1$ over $[0, t_c]$, the given initial conditions $r_i(0), b_i(0)$ and the boundary conditions $r_1^*(t_c) = r_2^*(t_c) = 0$, we can determine t_c through

$$-\frac{1}{2}\left[\sum_i b_i(0) - \sum_i r_i(0) - \frac{c_1 + c_2}{2k}\right](e^{-2kt_c} - 1) + \sum_i r_i(0) - \frac{c_1 - c_2}{2} t_c = 0. \quad (57)$$

Thus, if $Q \leq t_c$, the optimal solution is the one obtained for the unconstrained problem. If $Q > t_c$, chattering occurs in the optimal trajectory. Therefore, if it is possible to select Q such that $Q \leq t_c$, where t_c can be calculated through (57), we can avoid chattering.

3) Constrained Case, $b_i(t) \leq B$: Next, we assume that the constraint $b_i(t) \leq B$ is active on the optimal trajectory, but do not impose the constraint $r_i(t) \geq 0$. We can employ the indirect adjoining approach to explicitly solve the state-constrained optimal control problem as in [29] for the single battery problem, where the optimal control when $b(t) = B$ turns out to be a boundary control $u^*(t) = 0, h^*(t) = [(k(B - r^*(t)))/c_2]$ throughout the remaining time interval. Thus, when the optimal control switches to the recharging mode, the battery can no longer discharge and remains in recharging mode. The analysis is similar in our multibattery problem and leads to the following solution.

First, if a type I solution (51) applies, the nonuniqueness and the freedom to switch the control during $[0, Q]$ allows us to avoid $b_i(t) = B$ over $[0, Q)$. For the interval $[Q, T]$, the optimal control is fixed at $(0, 1, 0, 1)^T$, i.e., both batteries recharge at full rate; hence $b_i(t) \leq B$ is active over $(Q, T]$. As in [29], when $b_i^*(t) = B$ at some time $t_i \in (Q, T)$, then the optimal control turns to be $u_i^*(t) = 0, h_i^*(t) = [(k(B - r_i^*(t))/c_2]$ over $[t_i, T]$ for $i = 1, 2$.

On the other hand, if a type II solution (53) applies, then $u_2^*(t) = 0, h_2^*(t) = 1$ is fixed over $[0, T]$. Therefore, when $b_2^*(t) = B$ at some time $t_2 \in [0, T]$, we turn to a boundary control $u_2^*(t) = 0, h_2^*(t) = [(k(B - r_2^*(t)))/c_2]$ to continue recharging battery 2. As for battery 1, since we can select any feasible $(u_1^*(t), h_1^*(t))$ satisfying (53), it is possible to keep $b_1^*(t) \leq B$ inactive throughout $[0, T]$.

4) Constrained Case, $r_i(t) \geq 0$, $b_i(t) \leq B$: Finally, we allow both constraints $r_i(t) \geq 0$, $b_i(t) \leq B$ to become active on an optimal trajectory. If that does not happen, then the optimal solution reduces to one of the above two constrained cases or the unconstrained case. Similar to the analysis in [29], when the state constraints are both active, the solution is simply a combination of the solutions to the above two constrained cases.

An example of a type I solution where both constraints become active at some points over $[0, T]$ is shown in Fig. 3. From the trajectory of $q^*(t)$, we can see that $u_1^*(t) + u_2^*(t) = 1$ over $[0, t_c]$, where $t_c = 8.5260$ calculated through (57) and $r_1^*(t_c) = r_2^*(t_c) = 0$. Since the required workload $Q = 10$ is not achieved yet, the batteries start to chatter over the boundary arc $r_i(t) = 0$ until a point $t = 13$ when the full load requirement is met. After this point, both batteries recharge at full rate until some time when the boundary control $u_i^*(t) = 0, h_i^*(t) = [(k(B - r_i^*(t)))/c_2]$ continues recharging and terminates with $r_1^*(T) = r_2^*(T)$.

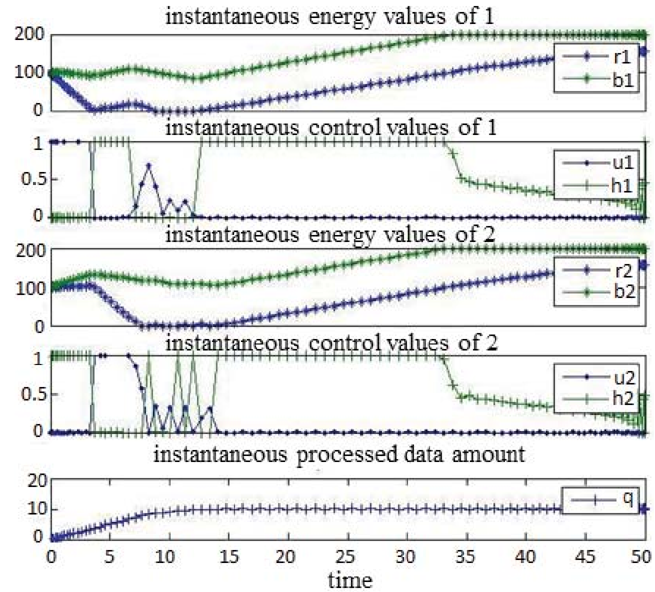


Fig. 3. Optimal solution under $r_1(0) = 100$, $b_1(0) = 100$, $r_2(0) = 100$, $b_2(0) = 100$, $c_1 = 30$, $c_2 = 10$, $k = 0.05$, $T = 50$, $Q = 10$, and $B = 200$, with $r(t) \geq 0$ and $b(t) \leq B$.

B. Solution of the $N > 2$ Case

We begin by setting $U(t) = (u_1(t), h_1(t), \ldots, u_N(t), h_N(t))^T$ and consider the unconstrained case in which (4) is relaxed. Since the state vector $\mathbf{x}(t) = (r_1(t), b_1(t), \ldots, r_N(t), b_N(t), q(t))^T$ has $2N + 1$ components in total, we let $\lambda(t) = (\lambda_{11}(t), \lambda_{12}(t), \ldots, \lambda_{N1}(t), \lambda_{N2}(t), \lambda_{2N+1}(t))^T$ be the costate vector and the Hamiltonian becomes

$$H(\mathbf{x}, \lambda, U) = \sum_{i=1}^{N} \Big([-c_1\lambda_{i1}(t) + \lambda_{2N+1}(t)]u_i(t) + c_2\lambda_{i2}(t) h_i(t) + k[\lambda_{i1}(t) - \lambda_{i2}(t)][b_i(t) - r_i(t)] \Big).$$

The costate equations $\dot{\lambda} = -(\partial H/\partial \mathbf{x})$ are now

$$\dot{\lambda}_{i1}(t) = k(\lambda_{i1}(t) - \lambda_{i2}(t)), \quad \dot{\lambda}_{i2}(t) = -k(\lambda_{i1}(t) - \lambda_{i2}(t))$$
$$\dot{\lambda}_{2N+1}(t) = 0, \qquad i = 1, \ldots, N. \tag{58}$$

We can proceed with the terminal costate $\lambda(T) = [(\partial \Phi(\mathbf{x}(T)))/\partial \mathbf{x}]$ and then obtain $\lambda(t)$ over $[0, T]$ by solving (58). However, besides $q(T) = Q$, the terminal state constraints (29) result in $(1/2)N(N-1)$ (i.e., the combinatorial coefficient C_N^2) conditions, which gives rise to $(1/2)N(N - 1) + 1$ unknown multipliers ν_i in $\Phi(\mathbf{x}(T))$. Thus, unlike the $N = 2$ case, it is intractable to analyze all possible values for each ν_i. We proceed, as described next, by constructing aggregate states and formulating an equivalent problem.

1) Solution Using Aggregate States: First sum up (2) and (3) over 1 to N and set

$$\sum_{i=1}^{N} r_i(t) = R(t), \quad \sum_{i=1}^{N} b_i(t) = B(t)$$
$$\sum_{i=1}^{N} u_i(t) = X(t), \quad \sum_{i=1}^{N} h_i(t) = Y(t) \tag{59}$$

such that $\dot{R}(t) = -c_1 X(t) + k(B(t) - R(t))$, and $\dot{B}(t) = c_2 Y(t) - k(B(t) - R(t))$. Accordingly, the objective (30) can be transformed into $\max_{X(t), Y(t)} R(T)$. Moreover, by (5)–(7), we have $0 \leq X(t) \leq 1$ and $0 \leq Y(t) \leq N - n(S, t)$, where $n(S, t)$ denotes the number of $i \in S$ such that $u_i(t) > 0$ at time t. Along with the substitution of (59) into (8), we obtain an equivalent formulation of our N-battery optimal control problem based on these aggregate states

$$\max_{X(t),Y(t)} R(T) \tag{60}$$

$$\begin{aligned} \text{s.t.} \quad \dot{R}(t) &= -c_1X(t) + k(B(t) - R(t)) \\ \dot{B}(t) &= c_2Y(t) - k(B(t) - R(t)) \\ & 0 \le X(t) \le 1,\ 0 \le Y(t) \le N - n(S,t) \\ & \int_0^T X(t)dt = Q \\ r_i(T) &= r_j(T) \quad \forall i, j \in S. \end{aligned}$$

Note that if we relax the terminal conditions $r_i(T) = r_j(T)$, $\forall\, i, j \in S$ for the time being, then (60) can be viewed as a simple single-battery optimal control problem without the control constraints (5). We can employ the Euler-Lagrange approach where, similar to (31), we add a state $q(t)$ such that $\dot{q}(t) = X(t)$. Then, the constraint $\int_0^T X(t)dt = Q$ becomes $q(T) = Q$. Henceforth, we set $\mathbf{x}(t) = (R(t), B(t), q(t))^T$ and $\lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t))^T$ as the state and costate vector respectively. Then, the Hamiltonian for (60) is

$$\begin{aligned} H &= k(\lambda_1(t) - \lambda_2(t))[B(t) - R(t)] \\ &\quad + (-c_1\lambda_1(t) + \lambda_3(t))X(t) + c_2\lambda_2(t)Y(t) \end{aligned} \tag{61}$$

and the costate equations $\dot{\lambda} = -(\partial H/\partial \mathbf{x})$ give

$$\begin{aligned} &\dot{\lambda}_1(t) = k(\lambda_1(t) - \lambda_2(t)), \quad \dot{\lambda}_2(t) = -k(\lambda_1(t) - \lambda_2(t)) \\ &\dot{\lambda}_3(t) = 0. \end{aligned}$$

Also, given the state boundary equation $q(T) = Q$, we must satisfy $\lambda(T) = [(\partial\Phi(\mathbf{x}(T)))/\partial\mathbf{x}]$ where $\Phi(\mathbf{x}(T)) = \nu(q(T) - Q) - r(T)$ and ν is an unknown multiplier, so that $\lambda_1(T) = -1,\ \lambda_2(T) = 0,\ \lambda_3(T) = \nu$. Solving the costate equations, we get

$$\lambda_1(t) = -\frac{1 + e^{2k(t-T)}}{2},\ \lambda_2(t) = -\frac{1 - e^{2k(t-T)}}{2},\ \lambda_3(t) = \nu.$$

In (61), the optimal controls $X^*(t)$ and $Y^*(t)$ have associated switching functions $-c_1\lambda_1(t) + \lambda_3(t)$ and $c_2\lambda_2(t)$ respectively. This is similar to the single-battery case analyzed in [29] for all possible values of ν and it immediately leads to the optimal control

$$\begin{aligned} X^*(t) &= \begin{cases} 1 & t \in [0, t_s] \\ 0 & t \in (t_s, T] \end{cases} \\ Y^*(t) &= N - n^*(S,t), \quad t \in [0, T] \end{aligned} \tag{62}$$

where t_s is a switching time given by $t_s = Q$ in view of the constraint $\int_0^T X(t)dt = Q$. Obviously, (62) indicates that in order to maximize $R(T)$ in (60), the control $Y(t)$ should be maximized (see also (10), which implies $n^*(S,t) = \min_{u_i,h_i} n(S,t)$. Now given the definition of $n(S,t)$ and $X(t)$, to satisfy $X^*(t)$ in (62) and minimize $n(S,t)$ over $[0,T]$, we can select a single arbitrary $i \in S$ such that $u_i^*(t) = 1$, $t \in [0, Q]$, and $u_i^*(t) = 0$ for all $i \in S$ over $(Q, T]$, which also satisfies the control constraints (5). Thus, the optimal control can be summarized as

$$\begin{cases} u_i^*(t) = 1, \quad h_i^*(t) = 0, \quad \text{any arbitrary } i \in S \\ u_j^*(t) = 0, \quad h_j^*(t) = 1 \quad \forall\, j \in S/\{i\}, \quad t \in [0, Q\,] \\ u_i^*(t) = 0, \quad h_i^*(t) = 1 \quad \forall\, i \in S, \quad t \in (Q, T]. \end{cases} \tag{63}$$

As a result

$$n^*(S,t) = 1, \quad t \in [0, Q]; \quad n^*(S,t) = 0, \quad t \in (Q, T]. \tag{64}$$

As a last step, in order to make (63) feasible in problem (60), we need to satisfy the terminal conditions $r_i(T) = r_j(T)$, $\forall\, i, j \in S$. This constrains the arbitrary selection of $i \in S$ such that $u_i^*(t) = 1, h_i^*(t) = 0$ at any $t \in [0, Q]$ in (63). However, satisfying the terminal conditions also depends on the initial parameters $R(0), B(0)$, i.e., $\rho_i(0,T)$ in the original optimal control problem. Consequently, there are two cases: 1) the solution (63) is able to satisfy $r_i(T) = r_j(T)$, $\forall\, i, j \in S$ under the initial condition values $\rho_i(0,T)$, $i \in S$ and 2) the constraint $r_i(T) = r_j(T)$, $\forall\, i, j \in S$ cannot be met for all the solutions of (63) under the initial condition values $\rho_i(0,T)$, $i \in S$.

For case 1), the optimal solution to problem (60) is simply (63) subject to $r_i(T) = r_j(T)$, $\forall\, i, j \in S$ and is referred to as a type I solution, expressed as

$$\begin{aligned} &U^*(t) = \\ &\begin{pmatrix} u_k^*(t) = 1, & h_k^*(t) = 0, & \text{any } k \in S \\ u_i^*(t) = 0, & h_i^*(t) = 1, & \text{all } i \in S/\{k\} \end{pmatrix} \quad t \in [0, Q] \\ &\qquad u_i^*(t) = 0, \quad h_i^*(t) = 1 \qquad \forall\, i \in S \qquad t \in (Q, T] \\ &\text{s.t.} \quad r_i^*(T) = r_j^*(T) \qquad \forall i, j \in S. \end{aligned} \tag{65}$$

The condition under which this is indeed an optimal solution depends on the values of $\rho_i(0,T)$, $i \in S$. The following theorem provides a necessary and sufficient condition for (65) to be optimal.

Theorem 3: A necessary and sufficient condition for the optimality of a type I solution (65) to the N-battery optimal control problem is

$$\sum_{i=1}^{N} \rho_i(0,T) \le \underline{\alpha}_N \tag{66}$$

$$\begin{aligned} \underline{\alpha}_N &= \int_0^Q \left[c_1 \frac{1 + e^{2k(r-T)}}{2} + c_2 \frac{1 - e^{2k(r-T)}}{2} \right] dr \\ &\quad + N\rho_m(0,T), \quad m = \arg\min_{i \in S}\{\rho_i(0,T)\}. \end{aligned} \tag{67}$$

Proof: To establish necessity, note that if type I solution is optimal, then all type I solutions are feasible. To ensure feasibility, $\sum_{i\in S} r_i^*(T)$ subject to (29) must be guaranteed not to exceed $\sum_{i\in S} \rho_i(0,T)$. Unlike the $N = 2$ case, we have N initial conditions $\rho_i(0,T)$ to consider. However, due to (29), the largest value of $\sum_{i\in S} r_i^*(T)$ depends only on the largest value of $r_m^*(T)$ under all type I solutions, which can be obtained by letting battery m recharge at full rate all the time, i.e., $u_m^*(t) = 0,\ h_m^*(t) = 1$over $[0,T]$. Let Π_1^* denote the set of all optimal controls of type I. Then, based on (11)

$$\max_{U\in\Pi} r_m^*(T) = \rho_m(0,T) + \int_0^T c_2 \frac{1 - e^{2k(r-T)}}{2} dr \tag{68}$$

where Π is the feasible control set defined in Theorem 1. Accordingly

$$\max_{U^*\in\Pi_1^*}\sum_{i\in S} r_i^*(T) < N\left(\rho_m(0,T)+\int_0^T c_2\frac{1-e^{2k(r-T)}}{2}dr\right) \tag{69}$$

where the left-hand side cannot equal the right-hand side because in (65) there is one arbitrary $k \in S$ such that $u_k^*(t) = 1$, $h_k^*(t) = 0$. Furthermore, note that in (65) we have $\sum_{i\in S} u_i^*(t) = 1$, $\sum_{i\in S} h_i^*(t) = N-1$ over $[0, Q]$ and $\sum_{i\in S} u_i^*(t) = 0$, $\sum_{i\in S} h_i^*(t) = N$ over $(Q, T]$. Then, in view of (11), for a type I solution we have

$$\sum_{i\in S} r_i^*(T) = \sum_{i\in S}\rho_i(0,T) + N\int_0^T c_2\frac{1-e^{2k(r-T)}}{2}dr - \int_0^Q \left[c_1\frac{1+e^{2k(r-T)}}{2} + c_2\frac{1-e^{2k(r-T)}}{2}\right]dr.$$

Combining this with (69), we obtain $\sum_{i=1}^N \rho_i(0,T) \le \underline{\alpha}_N$ and prove necessity.

To establish sufficiency, we have already shown that the condition $\sum_{i=1}^N \rho_i(0,T) \le \underline{\alpha}_N$ ensures the feasibility of all type I solutions. On the other hand, since a type I solution is optimal as long as it is feasible, then sufficiency immediately follows. ■

We now turn our attention to case 2), which applies when the constraint $r_i(T) = r_j(T)$, $\forall i, j \in S$ cannot be met in (63). Extrapolating from the type II solution of the $N = 2$ case in (53), we consider

$$U^*(t) = \begin{pmatrix} u_m^*(t) = 0, \quad h_m^*(t) = 1 \\ m = \operatorname*{argmin}_{i\in S}\{\rho_i(0,T)\} \\ u_i^*(t), \quad h_i^*(t),\ i \in S/\{m\} \end{pmatrix}, \quad t\in[0,T] \tag{70}$$
$$\text{s.t.}\quad \text{constraints (5)–(8) and (29)}$$

where, as in (53), the solution is is nonunique. To verify the optimality of (70), observe that $u_m(t) = 0, h_m(t) = 1$ over $[0, T]$, which implies that $r_i(T) = \max_{U\in\Pi} r_m(T)$ for all $i \in S$, thus maximizing $\sum_{i\in S} r_i(T)$ subject to (29). The following theorem provides a necessary and sufficient condition for (70) to be optimal.

Theorem 4: A necessary and sufficient condition for the optimality of a type II solution (70) to the N-battery optimal control problem is

$$\underline{\alpha}_N < \sum_{i=1}^N \rho_i(0,T) \le \bar{\alpha}_N \tag{71}$$
$$\bar{\alpha}_N = N\rho_m(0,T) + (N-1)\int_0^T c_2\frac{1-e^{2k(r-T)}}{2}dr + \int_{T-Q}^T c_1\frac{1+e^{2k(r-T)}}{2}dr.$$

Proof: To establish necessity, first note that when $\sum_{i=1}^N \rho_i(0,T) > \underline{\alpha}_N$ in (67) then, obviously, a type I solution can no longer satisfy (29). Subject to (29) and (68), the following inequality still applies to a type II solution:

$$\sum_{i\in S} r_i^*(T) \le N\max_{U\in\Pi} r_m^*(T) = N\left(\rho_m(0,T) + \int_0^T c_2\frac{1-e^{2k(r-T)}}{2}dr\right). \tag{72}$$

Moreover, in view of (11), we have

$$\sum_{i\in S} r_i^*(T) = \sum_{i\in S}\rho_i(0,T) + \int_0^T\left(c_2\sum_{i\in S} h_i^*(r)\frac{1-e^{2k(r-T)}}{2} - c_1\sum_{i\in S} u_i^*(r)\frac{1+e^{2k(r-T)}}{2}\right)dr. \tag{73}$$

Let Π_2^* denote the set of all optimal controls of type II. Using (70) and given the increasing monotonicity of the exponential term in the integrand above, we get

$$\min_{U^*\in\Pi_2^*}\int_0^T\left(c_2\sum_{i\in S} h_i^*(r)\frac{1-e^{2k(r-T)}}{2} - c_1\sum_{i\in S} u_i^*(r)\frac{1+e^{2k(r-T)}}{2}\right)dr = \int_0^T c_2\frac{1-e^{2k(r-T)}}{2}dr - \int_{T-Q}^T c_1\frac{1+e^{2k(r-T)}}{2}dr. \tag{74}$$

Thus, along with (72) and (73), under solution type II we have

$$\sum_{i\in S}\rho_i(0,T) \le N\rho_m(0,T) + (N-1)\int_0^T c_2\frac{1-e^{2k(r-T)}}{2}dr + \int_{T-Q}^T c_1\frac{1+e^{2k(r-T)}}{2}dr$$

which proves necessity.

The proof of sufficiency is similar to that of Theorem 3. ■

Based on Theorems 3 and 4, we can now summarize the solution to problem (60) as follows:

type I

$$\text{if } \sum_{i=1}^N \rho_i(0,T) \le \underline{\alpha}_N, \text{ then}$$
$$U^*(t) = \begin{cases} \begin{pmatrix} u_k^*(t) = 1, \quad h_k^*(t) = 0, \text{ any } k\in S \\ u_i^*(t) = 0, \quad h_i^*(t) = 1, \text{ all } i\in S/\{k\} \end{pmatrix}, & t\in[0,Q] \\ u_i^*(t) = 0, \quad h_i^*(t) = 1 \quad \forall i \in S, & t\in(Q,T] \end{cases}$$
$$\text{s.t.}\quad r_i^*(T) = r_j^*(T) \quad \forall i, j\in S \tag{75}$$

type II

$$\text{if } \underline{\alpha}_N < \sum_{i=1}^N \rho_i(0,T) \le \bar{\alpha}_N, \text{ then}$$
$$U^*(t) = \begin{pmatrix} u_m^*(t) = 0, \quad h_m^*(t) = 1 \\ m = \operatorname*{argmin}_{i\in S}\{\rho_i(0,T)\} \\ u_i^*(t), \quad h_i^*(t),\ i \in S/\{m\} \end{pmatrix}, t\in[0,T]$$
$$\text{s.t.}\quad \text{constraints (5)–(8) and (29).} \tag{76}$$

Furthermore, when $\sum_{i=1}^{N} \rho_i(0,T) > \bar{\alpha_N}$, it follows that $\sum_{i\in S} r_i(T) > N \max_{U\in\Pi} r_m(T)$, which cannot satisfy the constraint (29) for any $U \in \Pi$. Thus, there exists no feasible policy $\pi_0 \in \Pi$ such that $r_i(T) = r_j(T)$ for all $i, j \in S$, so that the determination of the optimal objective is obtained from Theorem 2. Note that setting $N = 2$ in conditions (66) and (71), it is easy to check that they reduce to conditions (50) and (52) respectively that were derived in our analysis of the $N = 2$ case.

Remark 4: Type I solutions are clearly the ones of most interest. in this case, the initial energy values $r_i(0)$, $b_i(0)$ are relatively balanced to satisfy (66), requiring all the batteries to serve the load cooperatively in a way prescribed by (75). In contrast, the remaining cases indicate that when $\rho_i(0,T)$ are so unbalanced as to satisfy (71), then battery m recharges at full rate all the time while the remaining $N-1$ batteries carry out the task of meeting the load requirement Q; this, however, is a more unusual situation. ■

V. Conclusion

We used a KBM to study the problem of optimally controlling the discharge and recharge processes of multiple nonideal batteries so as to maximize the minimum residual energy among all batteries at the end of a given time period $[0,T]$ while performing a prescribed amount of work Q over this period. Based on the use of a KBM for each battery, we showed that the optimal policy has the property that the residual energies of all batteries are equal at T as long as such a policy is feasible. This helps transform the original max-min optimization problem to a more standard optimal control problem with terminal state constraints. Moreover, through the analysis of the $N = 2$ case exploiting this property, we can characterize the optimal policy and show that it is generally not unique. We were also able to extend our analysis to the general $N \geq 2$ case through a state aggregation approach and obtain explicit expressions for two possible types of solutions characterized by associated necessary and sufficient conditions on the initial battery energy levels. Note that repeatedly discharging and recharging a battery in practice reduces its lifetime; therefore, a solution based on the structure (65)–(70) should be chosen so as to impose as few switches on the same battery as possible. This is easily achievable, as shown in using the solution of Fig. 2(b), rather that that of Fig. 2(a), without affecting performance.

Determining models for nonideal batteries such that they combine accuracy with computational efficiency remains a crucial open problem. Although we adopted the KBM for our analysis, it is possible to formulate similar optimal control problems for some of the more elaborate models mentioned in the Introduction, a research direction we plan to further pursue. Moreover, identifying the proper parameter values for any such model is a challenging process in itself. Additional future work will aim at extending the approach in this paper to problems where the batteries are not all shared at a single location, but rather distributed over a network of devices with one or more batteries placed on board and powering each device. This leads to resource allocation and network lifetime maximization problems where a nonideal battery model is employed. We also plan to extend our analysis to nonidentical batteries and to incorporate explicit switching costs between recharging and discharging any of the batteries.

References

[1] F. Yao, A. Demers, and S. Shenker, "A scheduling model for reduced CPU energy," in *Proc. IEEE 36th Annu. Found. Comput. Sci. Conf.*, Oct. 1995, pp. 374–382.

[2] V. Rao, G. Singhal, and A. Kumar, "Real time dynamic voltage scaling for embedded systems," in *Proc. 17th Int. Conf. VLSI Design*, Jan. 2004, pp. 650–653.

[3] J. Mao, C. G. Cassandras, and Q. Zhao, "Optimal dynamic voltage scaling in energy-limited nonpreemptive systems with real-time constraints," *IEEE Trans. Mobile Comput.*, vol. 6, no. 6, pp. 678–688, Jun. 2007.

[4] K. Kar, A. Krishnamurthy, and N. Jaggi, "Dynamic node activation in networks of rechargeable sensors," *IEEE/ACM Trans. Netw.*, vol. 14, no. 1, pp. 15–26, Feb. 2006.

[5] M. Doyle and J. S. Newman, "Analysis of capacity-rate data for lithium batteries using simplified models of the discharge process," *J. Appl. Electrochem.*, vol. 27, no. 7, pp. 846–856, 1997.

[6] T. L. Martin, "Balancing batteries, power, and performance: System issues in CPU speed-setting for mobile computing," Ph.D. dissertation, Dept. Electr. Comput. Eng., Carnegie Mellon Univ., Pittsburgh, PA, 1999.

[7] M. R. Jongerden and B. R. Haverkort, "Battery modeling," Centre Telemat. Inf. Technol., Univ. Twente, Enschede, Netherlands, Tech. Rep. TR-CTIT-08-01, 2008.

[8] T. F. Fuller, M. Doyle, and J. S. Newman, "Modeling of galvanostatic charge and discharge of the lithium/polymer/insertion cell," *J. Electrochem. Soc.*, vol. 140, no. 6, pp. 1526–1533, 1993.

[9] J. Newman. (1998). *FORTRAN Programs for the Simulation of Electrochemical Systems* [Online]. Available: http://www.cchem.berkeley.edu/jsngrp/fortran.html

[10] S. C. Hageman, "Simple PSpice models let you simulate common battery types," *Electron. Design News*, vol. 38, pp. 117–129, Oct. 1993.

[11] M. Chen and G. Rincon-Mora, "Accurate electrical battery model capable of predicting runtime and I–V performance," *IEEE Trans. Energy Convers.*, vol. 21, no. 2, pp. 504–511, Jun. 2006.

[12] C. Chiasserini and R. Rao, "Pulsed battery discharge in communication devices," in *Proc. 5th Int. Conf. Mobile Comput. Netw.*, 1999, pp. 88–95.

[13] C. F. Chiasserini and R. R. Rao, "A model for battery pulsed discharge with recovery effect," in *Proc. Wireless Commun. Netw. Conf.*, 1999, pp. 636–639.

[14] C. F. Chiasserini and R. R. Rao, "Energy efficient battery management," *IEEE J. Sel. Areas Commun.*, vol. 19, no. 7, pp. 1235–1245, Jul. 2001.

[15] V. Rao, G. Singhal, A. Kumar, and N. Navet, "Battery model for embedded systems," in *Proc. 18th Int. Conf. VLSI Design Conf.*, Jan. 2005, pp. 105–110.

[16] D. Rakhmatov and S. Vrudhula, "An analytical high-level battery model for use in energy management of portable electronic systems," in *Proc. Int. Conf. Comput.-Aided Design*, 2001, pp. 488–493.

[17] S. Vrudhula and D. Rakhmatov, "Energy management for battery-powered embedded systems," *ACM Trans. Embedd. Comput. Syst.*, vol. 2, no. 3, pp. 277–324, 2003.

[18] J. Manwell and J. McGowan, "Lead acid battery storage model for hybrid energy systems," *Solar Energ.*, vol. 50, no. 5, pp. 399–405, 1993.

[19] F. Zhang and Z. Shi, "Optimal and adaptive battery discharge strategies for cyber-physical systems," in *Proc. 48th IEEE Conf. Decision Control*, Dec. 2009, pp. 6232–6237.

[20] O. Barbarisi, F. Vasca, and L. Glielmo, "State of charge Kalman filter estimator for automotive batteries," *Control Eng. Pract.*, vol. 14, no. 3, pp. 267–275, 2006.

[21] J. F. Manwell and J. G. McGowan, "Extension of the kinetic battery model for wind/hybrid power systems," in *Proc. EWEC*, 1994, pp. 294–289.

[22] X. Ning and C. G. Cassandras, "On maximum lifetime routing in wireless sensor networks," in *Proc. 48th IEEE Conf. Decision Control*, Dec. 2009, pp. 3757–3762.

[23] T. Wang and C. G. Cassandras, "Optimal control of batteries with fully and partially available rechargebility," *Automatica*, vol. 48, no. 8, pp. 1658–1666, 2012.

[24] M. Gatzianas, L. Georgiadis, and L. Tassiulas, "Control of wireless networks with rechargeable batteries," *IEEE Trans. Wireless Commun.*, vol. 9, no. 2, pp. 581–593, Feb. 2010.

[25] K. M. Chandy, S. H. Low, U. Topcu, and H. Xu, "A simple optimal power flow model with energy storage," in *Proc. 49th IEEE Conf. Decision Control*, Dec. 2010, pp. 1051–1057.

[26] R. Liu, P. Sinha, and C. Koksal, "Joint energy management and resource allocation in rechargeable sensor networks," in *Proc. IEEE INFOCOM*, Mar. 2010, pp. 1–9.

[27] S. Moura, J. Forman, S. Bashash, J. Stein, and H. Fathy, "Optimal control of film growth in lithium-ion battery packs via relay switches," *IEEE Trans. Ind. Electron.*, vol. 58, no. 8, pp. 3555–3566, Aug. 2011.

[28] J. M. Foster and M. C. Caramanis, "Energy reserves and clearing in stochastic power markets: The case of plug-in-hybrid electric vehicle battery charging," in *Proc. 49th IEEE Conf. Decision Control*, Dec. 2010, pp. 1037–1044.

[29] T. Wang and C. G. Cassandras, "Optimal discharge and recharge control of battery-powered energy-aware systems," in *Proc. 49th IEEE Conf. Decision Control*, Dec. 2010, pp. 7513–7518.

[30] J. Chang and L. Tassiulas, "Maximum lifetime routing in wireless sensor networks," *IEEE/ACM Trans. Netw.*, vol. 12, no. 4, pp. 609–619, Aug. 2004.

[31] *MATLAB Optimal Control Software*. (2012) [Online]. Available: http://tomdyn.com

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