Perturbation Analysis and Control of Two-Class Stochastic Fluid Models for Communication Networks

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***Abstract*—This paper uses stochastic fluid models (SFMs) for control and optimization (rather than performance analysis) of communication network nodes processing two classes of traffic: one is uncontrolled and the other is subject to threshold-based buffer control. We derive gradient estimators for packet loss and workload related performance metrics with respect to threshold parameters. These estimators are shown to be unbiased and directly observable from a sample path without any knowledge of underlying stochastic characteristics of the traffic processes. This renders them computable in online environments and easily implementable for network management and control. We further demonstrate their use in buffer control problems where our SFM-based estimators are evaluated based on data from an actual system.**

***Index Terms*—Infinitesimal Perturbation Analysis, nonlinear optimization, stochastic fluid models (SFMs).**

I. INTRODUCTION

IN THIS paper, we use the framework of stochastic fluid models (SFMs) to capture key aspects of the behavior of complex stochastic discrete event systems and hence develop means for on-line control and performance optimization. Our motivation and ultimate goal is to use this approach for controlling communication networks with multiple classes of traffic. Although queueing models have long been used to capture the discrete event nature of packet-based operations in networks, huge traffic volumes have rendered such models highly impractical. In addition, the bursty nature of realistic traffic requires more sophisticated stochastic processes than those queueing theory can deal with, as well as the need to explicitly model buffer overflow phenomena which typically defy tractable analytical derivations. The SFM paradigm allows the aggregation of multiple events, associated with the movement of individual packets over a time period of a constant flow rate, into a single event associated with a rate change. It foregoes the identity and dynamics of individual packets and focuses instead on the aggregate flow rate. SFMs have recently been shown to be especially useful for analyzing various kinds of high-speed networks [1]–[8].

For the purpose of *performance analysis* with quality of service (QoS) requirements, the accuracy of SFMs depends on traffic conditions, the structure of the underlying system, and the nature of the performance metrics of interest. For the purpose of *control and optimization*, on the other hand, as long as a SFM captures the salient features of the underlying "real" system it is possible to obtain solutions to performance optimization problems even if we cannot estimate the corresponding performance with accuracy. In short, a SFM may be too "crude" for some performance analysis purposes, but able to capture sensitivity information for control purposes. This point of view is taken in [8], where a SFM is adopted for a single traffic class network node in which threshold-based buffer control is exercised. For the problem of determining a threshold (measured in packets or bytes) that minimizes a weighted sum of loss volume and buffer content, it is shown that a solution based on a SFM recovers or gives close approximations to the solution of the associated queueing model. Since solving such problems usually relies on gradient information, estimating the gradient of a given cost function with respect to key parameters, such as the aforementioned threshold, becomes an essential task. Perturbation analysis (PA) methods [9], [10] are therefore suitable, if appropriately adapted to a SFM viewed as a discrete-event system [11]. This approach has been used in [12], where incoming traffic rates were the parameters of interest, and in [8], where threshold parameters are optimized to solve buffer control problems. In [8], in particular, it was shown that infinitesimal perturbation analysis (IPA) yields remarkably simple *nonparametric* sensitivity estimators for packet loss and workload metrics with respect to threshold or buffer size parameters in a single-node SFM with a *single* incoming traffic stream. In addition, the estimators obtained are unbiased under very weak structural assumptions on the defining traffic processes.

In this paper, we consider a single node SFM with *two traffic streams*: one traffic stream is uncontrolled and the other is sub-

Manuscript received January 14, 2002; revised June 4, 2002 and January 16, 2003. The work of C. G. Cassandras, G. Sun, and C. G. Panayiotou was supported in part by the National Science Foundation under Grant ACI-98-73339 and Grant EEC-00-88073, by the Air Force Office of Scientific Research under Contract F49620-01-0056, by the Air Force Research Laboratory under Contract F30602-99-C-0057, by EPRI/ARO under Contract WO8333-03, by ARO under Grant DAAD19-01-0610, and by the NOKIA Research Center, Boston, MA. The work of Y. Wardi was supported in part by the National Science Foundation under Grant DMI-0085659 and by DARPA under Contract F30602-00-2-0556.

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Digital Object Identifier 10.1109/TAC.2003.811253

ject to threshold-based buffer control (see Fig. 1). Thus, we model a typical network node where the controlled stream represents a source of new traffic into the network at that node and the uncontrolled stream represents "interfering traffic," i.e., traffic originating at other nodes on its way to various destinations. This is an essential step toward the study of a complete network, which is the ultimate objective of this line of research. We assume that incoming traffic is not dependent on the threshold parameter being controlled, so that we limit ourselves to network settings operating with protocols such as user datagram protocol (UDP), but not transmission control protocol (TCP). However, traffic streams formed by multiplexing multiple TCP sources may be approximately independent of this threshold.

Interestingly, this model also captures the operation of the differentiated services (DS) protocol that has been proposed for supporting QoS requirements [13]–[15]. In a DS setting, packets arriving at a DS supporting domain are marked and aggregated into streams according to their classification. Subsequently, in all other nodes of the domain, all stream packets are treated according to that classification irrespective of the flow that they belong to. Thus, our model represents the handling of any one of the "assured forwarding" classes, where our two traffic classes correspond to different drop priorities; the uncontrolled stream corresponds to high priority (green) packets which are dropped only if the total buffer capacity is exceeded, while the controlled stream corresponds to low priority (yellow) packets which are dropped when the buffer exceeds a given threshold value θ. Otherwise, packets are treated alike. In this paper, we limit ourselves to a single controlled stream and assume an infinite capacity buffer. The natural extensions to a finite buffer model and to multiple controlled streams are possible and are the subject of ongoing work [16].

We point out that a central theme in the network management approach we propose is the fact that it is based on data directly available online, thus, requiring little or no information regarding the characteristics of traffic and service processes involved. Such measurement-based approaches have been proposed for network control (e.g., [17] and [18]), but the one we propose is a control strategy exclusively based on sensitivity analysis, capitalizing on the discovery that sample-path gradient information can be obtained *online* by extremely simple, often nonparametric, and unbiased estimators. On the other hand, the variance of gradient estimators may be high, leading to a tradeoff between fast and simple algorithms potentially yielding high-variance estimators. Our experience with sample path optimization reveals that fast convergence toward the optimal region of the minimum is often obtained by high-variance estimators based on few samples; however, this is an issue that requires further research.

The contributions of this paper are as follows. First, we derive IPA gradient estimators for performance metrics related to loss and workload levels (from the latter, fluid-based expected delay metrics can also be obtained; see [19]) with respect to the threshold parameter in a model with two traffic streams, one controlled and one uncontrolled. Compared to the nonparametric estimators derived for the single-stream SFM in [8], the estimators in the two-class case generally depend on traffic rate information, but not on the stochastic characteristics of the arrival and service processes involved. In addition, the estimators can be evaluated *based on data observed on a sample path of the actual (discrete-event) system*. Thus, we may use the SFM to obtain a gradient estimator whose form only is used, while the associated value at any operating point is obtained on line from real system data. Readers familiar with IPA will also notice that the form of the estimators derived for this type of system is quite different from the "classical" ones (e.g., see [9] and [10]). These estimators are also shown to be unbiased, a substantially more challenging task than in the single traffic class model studied in [8]. Finally, we use these estimators to illustrate how to solve buffer control problems in a two-class network setting.

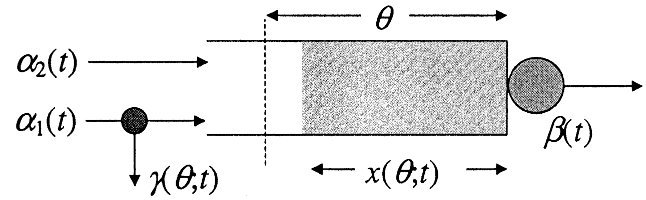

Fig. 1. SFM with two traffic classes.

The paper is organized as follows. First, in Section II, we present our model and define performance metrics and parameters of interest. In Section III, we derive an IPA estimator for the sensitivity of the expected loss rate with respect to the controlled traffic stream's threshold parameter. In Section IV, we repeat this process for the workload as a performance metric. In Section V, we establish the unbiasedness of the estimators derived. In Section VI, we show how the SFM-based derivative estimates can be used on line using data from the *actual* system (not the SFM) in order to solve buffer control problems. Finally, in Section VII we outline future research directions motivated by this work.

II. STOCHASTIC FLUID MODEL (SFM) SETTING

The SFM studied in this paper is based on the model described in [8] where a single node and single traffic stream was considered. In our case, as shown in Fig. 1, there are two "classes" of traffic: controlled (class 1) and uncontrolled (class 2). A threshold θ is associated with class 1 traffic, which has a time-varying arrival rate $\alpha_1(t)$. Uncontrolled traffic has a time-varying arrival rate $\alpha_2(t)$. A control policy is exercised so that when the total buffer content reaches a threshold θ, class 1 traffic is rejected, while class 2 traffic is not affected. The two traffic streams share a common FIFO buffer assumed of infinite size. The service rate is also time-varying and denoted by $\beta(t)$. In addition, let $\gamma(\theta;t)$ denote the loss rate when the buffer content exceeds the designated threshold level θ, and let $x(\theta;t)$ denote the buffer content at time t. The notational dependence on θ indicates that we will analyze performance metrics as functions of the given θ.

We are interested in studying sample paths of the SFM over a time interval $[0,T]$ for a given fixed $0 < T < \infty$. We assume that the processes $\{\alpha_1(t)\}$, $\{\alpha_2(t)\}$, and $\{\beta(t)\}$ are independent of θ (thus, we consider network settings operating with protocols such as UDP, but not TCP) and they are right-continuous piecewise continuously differentiable w.p. 1. Note that a typical sample path can be decomposed into two kinds of alternating intervals: *empty periods* and *buffering periods*. Empty periods (EPs) are intervals during which the buffer is empty, while buffering periods (BPs) are intervals during which the buffer is nonempty. Observe that during an EP, the system is not necessarily idle since the server may be active, processing traffic supplied to it at a rate that does not exceed $\beta(t)$, i.e., $\alpha_1(t)+\alpha_2(t)\leq\beta(t)$.

Viewed as a discrete-event system, an *event* in a sample path of the above SFM may be either *exogenous* or *endogenous*. Of particular interest, as we will see, is any event that causes the difference function $[\alpha_1(t)+\alpha_2(t)-\beta(t)]$ or $[\alpha_2(t)-\beta(t)]$ to change sign. For our purposes, we identify two exogenous events: (e_1) an event where the buffer ceases to be empty, and (e_2) an event where the buffer content leaves the value $x(\theta;t)=\theta$ after it has maintained it for some finite length of time. An endogenous event is defined to occur whenever: (e_3) the buffer becomes empty, (e_4) the buffer content reaches the value $x(\theta;t)=\theta$ and then maintains it for some finite length of time, and (e_5) the buffer content crosses the value $x(\theta;t)=\theta$ from either below or above.

We will assume that the real-valued parameter θ is confined to a closed and bounded (compact) interval Θ; to avoid unnecessary technical complications, we assume that $\theta>0$ for all $\theta\in\Theta$. Let $\mathcal{L}(\theta):\Theta\rightarrow\mathbb{R}$ be a random function defined over the underlying probability space $(\Omega,\mathcal{F},\mathcal{P})$. Strictly speaking, we write $\mathcal{L}(\theta,\omega)$ to indicate that this sample function depends on the sample point $\omega\in\Omega$, but will suppress ω unless it is necessary to stress this fact. In what follows, we will consider two performance metrics, the *Loss Volume* $L_T(\theta)$ and the *Cumulative Workload* (or just *Work*) $Q_T(\theta)$, both defined on the interval $[0,T]$ as follows:

$$L_T(\theta)=\int_0^T\gamma(\theta;t)dt \tag{1}$$

$$Q_T(\theta)=\int_0^T x(\theta;t)dt \tag{2}$$

where, for simplicity, we assume that $x(\theta;0)=0$. Observe that $(1/T)\mathbb{E}[L_T(\theta)]$ is the *Expected Loss Rate* over the interval $[0,T]$, a common performance metric of interest (from which related metrics such as *Loss Probability* can also be derived). Similarly, $(1/T)\mathbb{E}[Q_T(\theta)]$ is the *Expected Buffer Content* over $[0,T]$. We may then formulate optimization problems such as the determination of θ^* that minimizes a cost function of the form

$$\begin{aligned}J_T(\theta)&=\frac{1}{T}\mathbb{E}\left[Q_T(\theta)\right]+\frac{R}{T}\mathbb{E}\left[L_T(\theta)\right]\\&\equiv\frac{1}{T}J_Q(\theta)+\frac{R}{T}J_L(\theta)\end{aligned} \tag{3}$$

where R represents a rejection cost due to class 1 loss (other cost functions are also possible, depending on network control objectives, e.g., workload minimization subject to some predefined loss rate constraint). In order to accomplish this task, we use stochastic approximation techniques [20] and rely on estimates of $dJ_L(\theta)/d\theta$ and $dJ_Q(\theta)/d\theta$ provided by the sample derivatives $dL_T(\theta)/d\theta$ and $dQ_T(\theta)/d\theta$ for use with these techniques. Accordingly, our objective is the estimation of the derivatives of $J_L(\theta)$ and $J_Q(\theta)$, which we will pursue through IPA techniques [9], [10]. Henceforth, we shall use the "prime" notation to denote derivatives with respect to θ, and will proceed to estimate the derivatives $J_L'(\theta)$ and $J_Q'(\theta)$. The corresponding sample derivatives are denoted by $L_T'(\theta)$ and $Q_T'(\theta)$, respectively.

III. IPA for Loss Volume With Respect to Threshold

Our objective here is to estimate the derivative $J_L'(\theta)=\mathbb{E}[L_T'(\theta)]$ through the sample derivative $L_T'(\theta)$ which is commonly referred to as the IPA estimator; comprehensive discussions of IPA and its applications can be found in [9] and [10]. The IPA derivative-estimation technique computes $L_T'(\theta)$ along an observed sample path ω. An IPA-based estimate $L_T'(\theta)$ of a performance metric derivative $d\mathbb{E}[L_T(\theta)]/d\theta$ is *unbiased* if $d\mathbb{E}[L_T(\theta)]/d\theta=\mathbb{E}[L_T'(\theta)]$. Unbiasedness is the principal condition for making the application of IPA useful in practice, since it enables the use of the sample (IPA) derivative in control and optimization methods that employ stochastic gradient-based techniques.

We will proceed by studying a sample path of the SFM over $[0,T]$. For a fixed $\theta\in\Theta$, the interval $[0,T]$ is divided into alternating EPs and BPs. Suppose that a sample path consists of K buffering periods denoted by $\mathcal{B}_k$, $k=1,\ldots,K$, in increasing order. Thus, given a BP $\mathcal{B}_k$, its starting point is one where the buffer ceases to be empty, i.e., there is a change in sign of the difference function $[\alpha_1(t)+\alpha_2(t)-\beta(t)]$ from nonpositive (hence, the buffer was empty) to positive; this corresponds to the exogenous event e_1 defined earlier. Since this function is locally independent of θ, the starting point of $\mathcal{B}_k$ is locally independent of θ. The ending point of $\mathcal{B}_k$ generally depends on θ. Denoting these points by ξ_k and $\eta_k(\theta)$, respectively, we express $\mathcal{B}_k$ as $\mathcal{B}_k=[\xi_k,\eta_k(\theta))$, $k=1,\ldots,K$, for some random integer K which is also locally independent of θ. Then, by (1), we may write

$$L_T(\theta)=\sum_{k=1}^{K}\int_{\xi_k}^{\eta_k(\theta)}\gamma(\theta;t)dt \tag{4}$$

and by differentiating with respect to θ we obtain

$$L_T'(\theta)=\sum_{k=1}^{K}\frac{d}{d\theta}\int_{\xi_k}^{\eta_k(\theta)}\gamma(\theta;t)dt \tag{5}$$

assuming that this sample derivative exists (we return to this issue later in this section). Let us now focus on a typical $\mathcal{B}_k$ and drop the index k in order to simplify notation. Thus, the BP in question is denoted by $\mathcal{B}=[\xi,\eta(\theta))$. Define the function $\lambda(\theta)$ as

$$\lambda(\theta)=\int_{\xi}^{\eta(\theta)}\gamma(\theta;t)dt \tag{6}$$

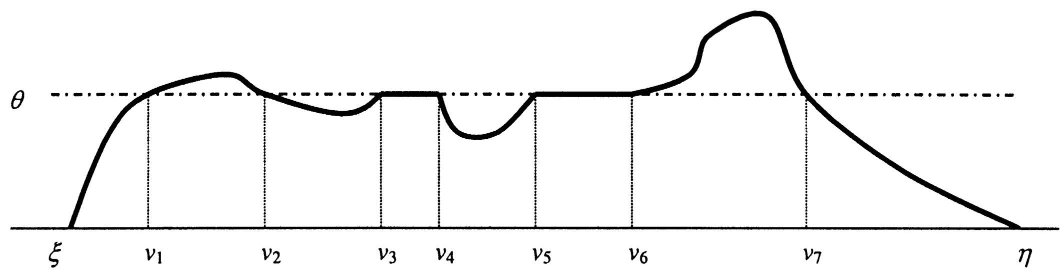

Fig. 2. Typical BP.

and we shall concentrate on evaluating $\lambda'(\theta)$.

For notational convenience, we define the instantaneous net inflow rates

$$A(t) = \alpha_1(t) + \alpha_2(t) - \beta(t) \tag{7}$$

$$B(t) = \alpha_2(t) - \beta(t). \tag{8}$$

Let v_i, $i = 0, \ldots, S$, be the event times of all exogenous and endogenous events ($e_1, \ldots, e_5$ as previously defined) in the BP. Note that $v_0 = \xi$ and $v_S = \eta(\theta)$. Fig. 2 shows a typical BP in a sample path of our SFM. According to the different levels of buffer content, we can divide the BP into periods (intervals)

$$p_i = [v_i(\theta), v_{i+1}(\theta)), \qquad i = 0, \ldots, S-1 \tag{9}$$

so that each belongs to one of the following three sets. To simplify the notation, we also define the open intervals $p_i^o = (v_i(\theta), v_{i+1}(\theta))$, $i = 0, \ldots, S-1$.

A. Partial Loss Period Set $U(\theta)$

During such periods, the buffer content is $x(t;\theta) = \theta$ and class 1 traffic experiences partial loss. In particular

$$\frac{dx(t)}{dt^+} = 0 \tag{10}$$

$$A(t) > 0 \quad \text{and} \quad B(t) < 0 \tag{11}$$

where (11) indicates that the total incoming traffic exceeds the processing capacity while the uncontrolled traffic rate is by itself below processing capacity. Therefore, the loss rate of class 1 is

$$\gamma(\theta; t) = A(t). \tag{12}$$

Formally, we define $U(\theta)$ as follows:

$$U(\theta) := \{p_i : x(t) = \theta,\ t \in p_i\} \tag{13}$$

where the end point v_{i+1} of each period is locally independent of θ, since the time when the buffer content leaves θ depends only on a change in sign of the net inflow function $A(t) = [\alpha_1(t) + \alpha_2(t) - \beta(t)]$ or $B(t) = [\alpha_2(t) - \beta(t)]$, as seen in (11); this corresponds to the exogenous event e_2 defined earlier. In Fig. 2, $p_3 = [v_3, v_4)$ and $p_5 = [v_5, v_6)$ are examples of partial loss periods within a BP.

B. Full Loss Period Set $V(\theta)$

In a full loss period, the buffer content is $x(t;\theta) > \theta$ (excluding the starting point $v_i(\theta)$) and *all* class 1 traffic is lost

$$V(\theta) := \{p_i : x(v_i(\theta)) = \theta \text{ and } x(t) > \theta,\ t \in p_i^o\} \tag{14}$$

and we have

$$\frac{dx(t)}{dt^+} = B(t) \tag{15}$$

$$\gamma(\theta; t) = \alpha_1(t). \tag{16}$$

Examples of full loss periods are $p_1 = [v_1, v_2)$ and $p_6 = [v_6, v_7)$ in Fig. 2. Note that in the former the full loss period starts upon crossing θ, whereas in the latter it follows a partial loss period and v_6 is locally independent of θ.

C. No Loss Period Set $W(\theta)$

During such periods the buffer content is $x(t;\theta) < \theta$ (excluding the starting point $v_i(\theta)$) and no loss occurs:

$$W(\theta) := \{p_0\} \cup \{p_i : x(v_i(\theta)) = \theta \text{ and } x(t) < \theta,\ t \in p_i^o\} \tag{17}$$

and we have

$$\frac{dx(t)}{dt^+} = A(t) \tag{18}$$

$$\gamma(\theta; t) = 0. \tag{19}$$

Examples of such periods are $p_0 = [\xi, v_1)$, $p_2 = [v_2, v_3)$, $p_4 = [v_4, v_5)$, and $p_7 = [v_7, \eta)$ in Fig. 2.

Then, returning to (6), we can rewrite $\lambda(\theta)$ as

$$\lambda(\theta) = \sum_{i=0}^{S-1} \int_{v_i(\theta)}^{v_{i+1}(\theta)} \gamma(\theta; t) \mathbf{1}[p_i \in U(\theta) \cup V(\theta)]\, dt \tag{20}$$

where $\mathbf{1}[\cdot]$ is the usual indicator function, and

$$\lambda'(\theta) = \sum_{i=0}^{S-1} \frac{d}{d\theta} \int_{v_i(\theta)}^{v_{i+1}(\theta)} \gamma(\theta; t) \mathbf{1}[p_i \in U(\theta) \cup V(\theta)]\, dt. \tag{21}$$

Since we are concerned with the sample derivative $L_T'(\theta)$ we have to identify conditions under which it exists (and, therefore, $\lambda'(\theta)$ also exists). Observe that any exogenous event time (corresponding to e_1 and e_2 as defined in the previous section) is locally independent of θ, whereas any endogenous event time (corresponding to e_3, e_4, and e_5), is generally a function of θ. The derivative $v_i'(\theta)$ exists as long as $v_i(\theta)$ is not a jump point of the net inflow function $A(t) = [\alpha_1(t) + \alpha_2(t) - \beta(t)]$ or $B(t) = [\alpha_2(t) - \beta(t)]$. Excluding the possibility of the simultaneous occurrence of two (exogenous or endogenous) events, the only situation preventing the existence of $v_i'(\theta)$, hence $\lambda'(\theta)$ and the sample derivative $L_T'(\theta)$, involves some t such that $A(t) = 0$

or $B(t)=0$ while $x(t)=\theta$; in such cases, the one-sided derivative of $L_T(\theta)$ still exists and can be obtained through a finite difference analysis (as in [8]). However, to keep the analysis simple, we focus only on the differentiable case by proceeding under the following technical conditions.

Assumption 1:

a) $\alpha_1(t)<\infty$, $\alpha_2(t)<\infty$ and $\beta(t)<\infty$ for all $t\in[0,T]$.
b) For every $\theta\in\Theta$, w.p. 1, no two events may occur at the same time.
c) W.p. 1, there exists no interval $(v_i(\theta), v_i(\theta)+\tau)$, $\tau>0$, such that $x(t)=\theta$ for all $t\in(v_i(\theta), v_i(\theta)+\tau)$, and either $A(t)=0$ or $B(t)=0$.

All three parts of **Assumption 1** are mild technical conditions. Regarding part c), as already pointed out, one-sided derivatives may still be used if a sample path happens to contain a partial loss period in which $A(t)=0$ or $B(t)=0$.

In order to proceed with the detailed derivation of $\lambda'(\theta)$ in (21), we need to study the derivatives $v_i'(\theta)$, $i=1,\ldots,S$. Let us consider an interval $p_{i-1}(\theta)=[v_{i-1}(\theta), v_i(\theta))$, $i=1,\ldots,S$, depending on whether it belongs to the set $U(\theta)$, $V(\theta)$, or $W(\theta)$.

1) $p_{i-1}(\theta)\in U(\theta)$: As already mentioned, v_i is independent of θ, so recalling (12)

$$\int_{v_{i-1}(\theta)}^{v_i(\theta)} \gamma(\theta;t)dt = \int_{v_{i-1}(\theta)}^{v_i} A(t)dt$$

and, upon taking derivatives w.r.t θ, we have

$$\frac{d}{d\theta}\int_{v_{i-1}(\theta)}^{v_i} A(t)dt = -A\left(v_{i-1}(\theta)\right)v_{i-1}'(\theta). \tag{22}$$

2) $p_{i-1}(\theta)\in V(\theta)$: In this case, we have $\gamma(\theta;t)dt=\alpha_1(t)$, so that

$$\frac{d}{d\theta}\int_{v_{i-1}(\theta)}^{v_i(\theta)} \alpha_1(t)dt = \alpha_1\left(v_i(\theta)\right)v_i'(\theta) - \alpha_1\left(v_{i-1}(\theta)\right)v_{i-1}'(\theta). \tag{23}$$

In addition, in a full loss period we have $x(v_{i-1}(\theta)) = x(v_i(\theta))=\theta$, and it follows from (15) that

$$\int_{v_{i-1}(\theta)}^{v_i(\theta)} B(t)dt = 0$$

so that, taking derivatives, we obtain

$$B\left(v_i(\theta)\right)v_i'(\theta) - B\left(v_{i-1}(\theta)\right)v_{i-1}'(\theta) = 0. \tag{24}$$

By adding the left-hand side of (24) to the right-hand side of (23), we get

$$\frac{d}{d\theta}\int_{v_{i-1}(\theta)}^{v_i(\theta)} \alpha_1(t)dt = A\left(v_i(\theta)\right)v_i'(\theta) - A\left(v_{i-1}(\theta)\right)v_{i-1}'(\theta). \tag{25}$$

3) $p_{i-1}(\theta)\in W(\theta)$: Here, we need to consider three cases. First, if $i=1$, then the buffer content evolves from $x(\xi)=0$ to $x(v_1(\theta))=\theta$ and, using (18), we have

$$\int_{\xi}^{v_1(\theta)} A(t)dt = \theta.$$

Upon taking derivatives, we get

$$A\left(v_1(\theta)\right)v_1'(\theta) = 1. \tag{26}$$

Second, if $1<i<S$, we have $x(v_{i-1}(\theta))=x(v_i(\theta))=\theta$, therefore, by (18)

$$\int_{v_{i-1}(\theta)}^{v_i(\theta)} A(t)dt = 0$$

and, upon differentiating

$$A\left(v_i(\theta)\right)v_i'(\theta) - A\left(v_{i-1}(\theta)\right)v_{i-1}'(\theta) = 0. \tag{27}$$

Finally, if $i=S$, then $x(v_{S-1}(\theta))=\theta$ and $x(v_S(\theta))=0$, so that

$$\int_{v_{S-1}(\theta)}^{v_S(\theta)} A(t)dt = -\theta$$

and, taking derivatives

$$A\left(v_S(\theta)\right)v_S'(\theta) - A\left(v_{S-1}(\theta)\right)v_{S-1}'(\theta) = -1. \tag{28}$$

Next, returning to (21), note that only terms in $U(\theta)$ and $V(\theta)$ contribute to $\lambda'(\theta)$, i.e., we need to evaluate the derivatives in (22) and (25). Observe that these depend only on the traffic rates at the end points v_{i-1} and v_i of the corresponding intervals where one of the events $e_1,\ldots,e_5$ takes place. Moreover, (24), (26), and (27) provide relationships between these quantities. Therefore, we only need to focus on these particular event points and ignore all system activity in between them. This also explains why IPA in this case is not dependent on the stochastic characteristics of the arrival and service processes. Another useful observation is that $\mathcal{B}=(\xi,\eta(\theta))$ always starts and ends with a period in the no loss set $W(\theta)$, while in between the sequence of periods $p_i(\theta)$, $i=1,\ldots,S-2$, can be arbitrary as long as no adjacent periods are from the same set.

We may now proceed by seeking a solution to the set of equations (24), (26), and (27), allowing us to obtain all $v_{i-1}'(\theta)$ and $v_i'(\theta)$ in (22) and (25), leading to an evaluation of $\lambda'(\theta)$ in (21).

As we shall see, if a BP contains at least one partial loss period, the associated IPA estimator is simply -1 (independent of all model parameters), which is the exact same result obtained in [8] for the case of a single controlled traffic class. In contrast, if a BP consists only of periods in the sets $V(\theta)$ and $W(\theta)$, then the IPA estimator does depend on some traffic rate values; we shall show, however, that its value is always limited to $[-1, 0]$.

Let us begin by simplifying notation even further through the introduction of the following, defined for $i=1,\ldots,S$:

$$A_i \equiv A\left(v_i(\theta)\right) = \alpha_1\left(v_i(\theta)\right)+\alpha_2\left(v_i(\theta)\right)-\beta\left(v_i(\theta)\right) \tag{29}$$

$$B_i \equiv B\left(v_i(\theta)\right) = \alpha_2\left(v_i(\theta)\right)-\beta\left(v_i(\theta)\right). \tag{30}$$

The following lemma shows that all event time derivatives of interest $v_i'(\theta)$ are expressed in terms of A_i and B_i; by convention, we shall set $A_0 \equiv 1$. Moreover, we establish the fact that after a partial loss period occurs, all ensuing event time derivatives are $v_i'(\theta) = 0$.

Lemma III.1: Suppose that $[v_m(\theta), v_{m+1})$, $1 \le m < S-1$ is the first partial loss period in a BP. Then, the following hold.

1) For $v_i \le v_m$:

$$v_1'(\theta) = \frac{A_0}{A_1} \tag{31}$$

$$v_{2n}'(\theta) = \prod_{i=1}^{n} \frac{B_{2i-1}}{B_{2i}} \cdot \frac{A_{2i-2}}{A_{2i-1}} \tag{32}$$

where $1 \le n \le (m/2)$ if m is even, and $2 \le n \le ((m-1)/2)$ if m is odd, and $m > 1$

$$v_{2n+1}'(\theta) = \frac{A_{2n}}{A_{2n+1}} \cdot \prod_{i=1}^{n} \frac{B_{2i-1}}{B_{2i}} \cdot \frac{A_{2i-2}}{A_{2i-1}} \tag{33}$$

where $1 \le n \le ((m-2)/2)$ if m is even, and $1 \le n \le ((m-1)/2)$ if m is odd, and $m > 2$.

2) For all $v_i \ge v_{m+1}$

$$v_i'(\theta) = 0.$$

Proof: See the Appendix.

Remark: Readers familiar with IPA applied to event times will notice that the estimators (31)–(33) are quite different from the "classical" form encountered in standard queueing systems (e.g., see [9] and [10]). In particular, IPA derivatives for event times evaluated over a buffering period of a queueing system are *sums* of terms reflecting the effect of some parameter perturbation on the accumulated traffic processed during this buffering period. Here, however, we see in (32) and (33) expressions with a *multiplicative* effect of perturbations over specific crucial events, i.e., events that cause a buffer overflow.

The following lemma establishes a property of the ratios B_{2n+1}/A_{2n+1} and A_{2n}/B_{2n} which turn out to play a role in the eventual evaluation of $\lambda'(\theta)$ in Lemma III.3.

Lemma III.2: Suppose that a BP contains a partial loss period $[v_m(\theta), v_{m+1})$ with $1 \le m < S-1$. Then, for $v_i, \ldots, v_{m-1}$

$$0 < \frac{B_{2n+1}}{A_{2n+1}} \le 1, \qquad 0 \le n \le \frac{m-2}{2} \tag{34}$$

$$0 < \frac{A_{2n}}{B_{2n}} \le 1, \qquad 0 < n \le \frac{m-1}{2}. \tag{35}$$

Proof: See the Appendix.

Lemma III.3: For any BP $[\xi, \eta(\theta))$, if at least one partial loss period is present, then

$$\lambda'(\theta) = -1. \tag{36}$$

If no partial loss period is present, then

$$\lambda'(\theta) = -1 + \prod_{i=1}^{\frac{(S-1)}{2}} \frac{A_{2i}}{B_{2i}} \cdot \frac{B_{2i-1}}{A_{2i-1}} \tag{37}$$

and

$$-1 < \lambda'(\theta) \le 0. \tag{38}$$

Proof: See the Appendix.

Motivated by our analysis thus far, let $U_k(\theta)$, $V_k(\theta)$, and $W_k(\theta)$ be the partial loss, full loss, and no loss period sets, respectively, in the kth BP, $k = 1, \ldots, K$. Similarly, let $v_{k,i}(\theta)$ denote the ith event time in the kth BP, $i = 0, \ldots, S_k$, and $A_{k,i}$, $B_{k,i}$ be the obvious extensions of A_i, B_i in (29) and (30). Then, define

$$\Phi(\theta) = \{k \in \{1, \ldots, K\} \, : \, U_k(\theta) \ne \emptyset\} \tag{39}$$

to be the set of BPs containing *at least one partial loss period*, and set

$$\lambda_k'(\theta) = -1 + \prod_{i=1}^{\frac{(S_k-1)}{2}} \frac{A_{k,2i}}{B_{k,2i}} \cdot \frac{B_{k,2i-1}}{A_{k,2i-1}}. \tag{40}$$

Theorem III.1: The sample derivative $L_T'(\theta)$ is given by

$$L_T'(\theta) = -\sum_{k=1}^{K} \mathbf{1}\,[k \in \Phi(\theta)] + \sum_{k=1}^{K} \mathbf{1}\,[k \notin \Phi(\theta)]\, \lambda_k'(\theta) \tag{41}$$

where K is the (random) number of buffering periods contained in $[0, T]$, including a possibly incomplete last buffering period.

Proof: The result follows from Lemma III.3, using the definitions in (39) and (40). ∎

The expression in (41) provides the IPA estimator for the loss metric defined in (1). We shall prove the unbiasedness of this estimator in Section V. Note that $L_T'(\theta)$, shown previously, does not depend on any distributional information regarding the traffic arrival and service processes and involves only flow rates at event times $v_{k,i}(\theta)$ which may be estimated on line. From an implementation standpoint, (40) requires observing events e_2, e_4, and e_5 within a BP and the corresponding rates of α_1, α_2, and β at their occurrence times, so that we can evaluate $A_{k,i}$ and $B_{k,i}$. If BPs include at least one partial loss period, then the only implementation requirement is that such a period be detected and the contribution of this entire BP is simply -1.

IV. IPA for Work With Respect to Threshold

In this section, we derive the IPA estimator for the Cumulative Workload (or simply Work) defined in (2) by carrying out an analysis similar to that of the previous section under **Assumption 1**. First, note that we can write

$$Q_T(\theta) = \sum_{k=1}^{K} \int_{\xi_k}^{\eta_k(\theta)} x(\theta; t)dt \tag{42}$$

where, as before, we consider BPs $\mathcal{B}_k = [\xi_k, \eta_k(\theta))$, $k = 1, \ldots, K$. Differentiating with respect to θ, we obtain

$$Q_T'(\theta) = \sum_{k=1}^{K} \frac{d}{d\theta} \int_{\xi_k}^{\eta_k(\theta)} x(\theta; t)dt. \tag{43}$$

where the sample derivative exists under **Assumption 1**. Then, focusing on a particular $\mathcal{B}_k$ and dropping the index k, we define

$$q(\theta) = \int_{\xi}^{\eta(\theta)} x(\theta; t)dt. \tag{44}$$

Taking the derivative with respect to θ yields

$$q'(\theta) = \int_{\xi}^{\eta(\theta)} x'(\theta;t)dt + x(\theta;\eta(\theta))\eta'(\theta)$$
$$= \int_{\xi}^{\eta(\theta)} x'(\theta;t)dt \tag{45}$$

since the BP ends at $\eta(\theta)$, hence $x(\theta;\eta(\theta)) = 0$. To evaluate $x'(\theta;t)$, we consider all possible cases regarding the location of t in the BP.

Case 1) $t \in [\xi, v_1(\theta))$. In this case

$$x(\theta;t) = \int_{\xi}^{t} A(\tau)d\tau$$

and we see that $x(\theta,t)$ is independent of θ, therefore

$$x'(\theta;t) = 0. \tag{46}$$

Case 2) $t \in [v_{i-1}(\theta), v_i(\theta)) \in U(\theta)$, i.e., t belongs to a partial loss period. Therefore, $x(\theta,t) = \theta$ and

$$x'(\theta;t) = 1. \tag{47}$$

Case 3) $t \in [v_{i-1}(\theta), v_i(\theta)) \in V(\theta)$. In this case, t belongs to a full loss period, therefore $x(\theta; v_{i-1}(\theta)) = \theta$ and $x(\theta;t) > \theta, t \in (v_{i-1}(\theta), t)$. It follows that

$$x(\theta;t) = \theta + \int_{v_{i-1}(\theta)}^{t} B(\tau)d\tau$$

and, upon differentiating, we obtain

$$x'(\theta;t) = 1 - B(v_{i-1}(\theta))v'_{i-1}(\theta)$$
$$= 1 - B_{i-1}v'_{i-1}(\theta). \tag{48}$$

Case 4) $t \in [v_{i-1}(\theta), v_i(\theta)) \in W(\theta)$, $i > 2$. In this case, t belongs to a no loss period, therefore $x(\theta; v_{i-1}(\theta)) = \theta$ and $x(\theta;t) < \theta, t \in (v_{i-1}(\theta), t)$, so that

$$x(\theta;t) = \theta + \int_{v_{i-1}(\theta)}^{t} A(\tau)d\tau$$

and

$$x'(\theta;t) = 1 - A(v_{i-1}(\theta))v'_{i-1}(\theta)$$
$$= 1 - A_{i-1}v'_{i-1}(\theta) \tag{49}$$

where we have used the definition (29).

We can now see that (45) can be written as

$$q'(\theta) = \sum_{i=2}^{S} \int_{v_{i-1}(\theta)}^{v_i(\theta)} x'(\theta;t)dt \tag{50}$$

where $x'(\theta;t)$ is given by (46)–(49), depending on the type of period encountered in this BP. We can further evaluate the terms $B_{i-1}v'_{i-1}(\theta)$ and $A_{i-1}v'_{i-1}(\theta)$ appearing in (48) and (49) by making use of (31)–(33) in Lemma III.1, to obtain

$$B_1 v'_1(\theta) = \frac{B_1}{A_1} \tag{51}$$

$$A_{2n}v'_{2n}(\theta) = \prod_{i=1}^{n} \frac{A_{2i}}{B_{2i}} \cdot \frac{B_{2i-1}}{A_{2i-1}},\ n > 0 \tag{52}$$

$$B_{2n+1}v'_{2n+1}(\theta) = \frac{B_{2n+1}}{A_{2n+1}} \prod_{i=1}^{n} \frac{A_{2i}}{B_{2i}} \cdot \frac{B_{2i-1}}{A_{2i-1}},\ n > 0 \tag{53}$$

for all v_i that precede the first partial loss period (if one exists). Then, define

$$\phi_i = \begin{cases} 1 - B_i v'_i, & i \text{ odd} \\ 1 - A_i v'_i, & i \text{ even} \end{cases}. \tag{54}$$

Note that, by (51)–(53) and Lemma III.2, we can see that $0 \le \phi_i < 1$.

Lemma IV.1: Suppose that $[v_m(\theta), v_{m+1})$, $1 \le m < S-1$, is the first partial loss period in a BP. Then,

$$q'(\theta) = \sum_{i=1}^{m-1} (v_{i+1} - v_i)\phi_i + (v_S - v_m). \tag{55}$$

Proof: See the Appendix.

Remark: It should be clear that if the BP does not contain a partial loss period, then $q'(\theta)$ is given by the sum in (55) evaluated over all $i = 1, \ldots, S-1$. Moreover, looking at the proof of the lemma, note that the loss derivative $\lambda'(\theta)$ in (37) is the same as the value of $-x'(\theta;t)$ at v_{S-1}. This implies that an IPA estimator implemented for the workload metric can simultaneously provide an estimate for the loss metric as well.

Let $v_{k,i}(\theta)$ denote the ith event time in the kth BP, $i = 0, \ldots, S_k$, and set

$$\phi_{k,i} = \begin{cases} 1 - B_{k,i}v'_{k,i}, & i \text{ odd} \\ 1 - A_{k,i}v'_{k,i}, & i \text{ even} \end{cases} \tag{56}$$

and

$$q'_k(\theta) = \sum_{i=1}^{m_k-1} (v_{k,i+1} - v_{k,i})\phi_{k,i} + (v_{k,S_k} - v_{k,m_k}) \tag{57}$$

with the understanding that if the kth BP contains no partial loss period, then $m_k = S_k$.

Theorem IV.1: The sample derivative $Q'_T(\theta)$ is given by

$$Q'_T(\theta) = \sum_{k=1}^{K} q'_k(\theta) \tag{58}$$

where K is the (random) number of buffering periods contained in $[0,T]$, including a possibly incomplete last buffering period.

Proof: The result follows from Lemma IV.1, using the definitions in (56) and (57). ■

The expression in (58) provides the IPA estimator for the work metric defined in (2). Its implementation requires the same information as that for the loss metric with the addition of timers to measure the duration of periods $[v_{k,i}, v_{k,i+1})$ within each BP observed in $[0,T]$, as well as $(v_{k,S_k} - v_{k,m_k})$ if one or more partial loss periods are included, with the first one starting at v_{k,m_k}.

V. IPA Estimator Unbiasedness

We now prove the unbiasedness (as defined in Section III) of the IPA derivatives $L'_T(\theta)$ and $Q'_T(\theta)$ obtained previously. Note that we do not concern ourselves with the issue of estimator consistency which involves letting $T \to \infty$, since it is hard to justify steady state in the setting we are considering; rather we concentrate on obtaining reliable shorter-term sensitivity information tracking the behavior of the network and seeking to continuously improve its performance.

In general, the unbiasedness of an IPA derivative $\mathcal{L}'(\theta)$ is ensured by the following two conditions (see [21, Lemma A2, p. 70]): i) For every $\theta \in \Theta$, the sample derivative $\mathcal{L}'(\theta)$ exists w.p. 1. ii) W.p. 1, the random function $\mathcal{L}(\theta)$ is Lipschitz continuous throughout Θ, and the (generally random) Lipschitz constant has a finite first moment. We have already discussed the mild technical conditions required to ensure the existence of $L'_T(\theta)$ and $Q'_T(\theta)$. Consequently, establishing the unbiasedness of $L'_T(\theta)$ and $Q'_T(\theta)$ as estimators of $d\mathbb{E}[L_T(\theta)]/d\theta$ and $d\mathbb{E}[Q_T(\theta)]/d\theta$, respectively, reduces to verifying the Lipschitz continuity of $L_T(\theta)$ and $Q_T(\theta)$ with appropriate Lipschitz constants. Let $N(T)$ be the random number of all events (exogenous and endogenous) in $[0,T]$. Then, under the assumption that $\mathbb{E}[N(T)] < \infty$, we shall establish next that $L'_T(\theta)$ and $Q'_T(\theta)$ are indeed unbiased estimators.

As stated in Section III, the buffer content $x(\theta;t)$ over any sample path can be classified as belonging to one of three sets, i.e., partial loss $U(\theta)$, full loss $V(\theta)$, and no loss $W(\theta)$. Let us denote by $x(\theta+\Delta\theta;t)$ the buffer content in a *perturbed* sample path resulting when θ is replaced by $\theta+\Delta\theta$. Then, this classification applied to both the nominal and perturbed sample paths is as follows:

$$U: x(\theta;t) = \theta \quad x(\theta+\Delta\theta;t) = \theta+\Delta\theta \tag{59}$$

$$V: x(\theta;t) \geq \theta \quad x(\theta+\Delta\theta;t) \geq \theta+\Delta\theta \tag{60}$$

$$W: x(\theta;t) \leq \theta \quad x(\theta+\Delta\theta;t) \leq \theta+\Delta\theta. \tag{61}$$

At this point it is worth recalling that $x(\theta;t)$ and $x(\theta+\Delta\theta;t)$ are continuous functions of t due to **Assumption 1 a)**. Next, we show in Lemma V.2 that $0 \leq \Delta x(t) \leq \Delta\theta$ for all $t \in [0,T]$, where $\Delta x(t) = x(\theta+\Delta\theta;t) - x(\theta;t)$ and $\Delta\theta > 0$ (the case $\Delta\theta < 0$ is similarly handled). To do so, we first show the following result.

Lemma V.1: Consider a BP $\mathcal{B}_k = [\xi_k, \eta_k(\theta))$, and assume that $0 \leq \Delta x(\xi_k) \leq \Delta\theta$. Then

$$0 \leq \Delta x(t) \leq \Delta\theta \quad \text{for all } t \in [\xi_k, \eta_k(\theta)). \tag{62}$$

Proof: See the Appendix.

Lemma V.2: For all $t \in [0,T]$

$$0 \leq \Delta x(t) \leq \Delta\theta.$$

Proof: See Appendix.

We are now ready to show the unbiasedness of $L'_T(\theta)$ and $Q'_T(\theta)$.

Theorem V.1: The IPA estimates $L'_T(\theta)$ (given in (41)) and $Q'_T(\theta)$ (given in (58)) are unbiased estimates of $d\mathbb{E}[L_T(\theta)]/d\theta$ and $d\mathbb{E}[Q_T(\theta)]/d\theta$, respectively. In other words

$$\mathbb{E}[L'_T(\theta)] = \frac{d\mathbb{E}[L_T(\theta)]}{d\theta} \quad \text{and} \quad \mathbb{E}[Q'_T(\theta)] = \frac{d\mathbb{E}[Q_T(\theta)]}{d\theta}.$$

Proof: We start with $L'_T(\theta)$ and recall (4). By partitioning $[0,T]$ into all buffering and empty periods of the nominal sample path and setting $\Delta\gamma(\tau) = \gamma(\theta+\Delta\theta;t) - \gamma(\theta;t)$, we get

$$\Delta L_T(\theta) = \sum_{k=1}^{K} \int_{\xi_k}^{\eta_k(\theta)} \Delta\gamma(\tau)d\tau + \sum_{k=1}^{K-1} \int_{\eta_k(\theta)}^{\xi_{k+1}} \Delta\gamma(\tau)d\tau = \sum_{k=1}^{K} \int_{\xi_k}^{\eta_k(\theta)} \Delta\gamma(\tau)d\tau \tag{63}$$

where the second term is zero since for all $t \in [\eta_k(\theta), \xi_{k+1})$ we have $x(\theta;t) = 0$ and from Lemma V.2 $x(\theta+\Delta\theta;t) \leq \Delta\theta < \theta+\Delta\theta$, which implies that no loss is possible in the perturbed sample path over $[\eta_k(\theta), \xi_{k+1})$. Next, we can write

$$x(\theta;\eta_k(\theta)) = x(\theta;\xi_k) + \int_{\xi_k}^{\eta_k(\theta)} A(\tau)d\tau - \int_{\xi_k}^{\eta_k(\theta)} \gamma(\theta;\tau)d\tau$$

$$x(\theta+\Delta\theta;\eta_k(\theta)) = x(\theta;\xi_k) + \Delta x(\xi_k) + \int_{\xi_k}^{\eta_k(\theta)} A(\tau)d\tau - \int_{\xi_k}^{\eta_k(\theta)} \gamma(\theta+\Delta\theta;\tau)d\tau$$

where we have used $\Delta x(\xi_k) = x(\theta+\Delta\theta;\xi_k) - x(\theta;\xi_k)$. Subtracting the first from the second equation, we get

$$\Delta x(\eta_k) = \Delta x(\xi_k) - \int_{\xi_k}^{\eta_k(\theta)} \Delta\gamma(\tau)d\tau.$$

Using $0 \leq \Delta x(t) \leq \Delta\theta$ in Lemma V.2, we get from the previous equation

$$\Delta\theta \geq \Delta x(\xi_k) - \int_{\xi_k}^{\eta_k(\theta)} \Delta\gamma(\tau)d\tau \geq -\int_{\xi_k}^{\eta_k(\theta)} \Delta\gamma(\tau)d\tau$$

and

$$0 \leq \Delta x(\xi_k) - \int_{\xi_k}^{\eta_k(\theta)} \Delta\gamma(\tau)d\tau \leq \Delta\theta - \int_{\xi_k}^{\eta_k(\theta)} \Delta\gamma(\tau)d\tau$$

and it follows that

$$\left| \int_{\xi_k}^{\eta_k(\theta)} \Delta\gamma(\tau)d\tau \right| \leq \Delta\theta.$$

Combining this inequality with (63), we get

$$|\Delta L_T(\theta)| \leq \sum_{k=1}^{K} \left| \int_{\xi_k}^{\eta_k(\theta)} \Delta\gamma(\tau)d\tau \right| \leq K|\Delta\theta|. \tag{64}$$

where $\mathbb{E}[K] < \infty$ since, by assumption, $\mathbb{E}[N(T)] < \infty$.

Next, we consider $Q'_T(\theta)$ for which we can write

$$Q_T(\theta) = \int_0^T x(\theta;\tau)d\tau$$

therefore,

$$|\Delta Q_T(\theta)| = \left| \int_0^T \Delta x(\tau) d\tau \right| \leq T|\Delta\theta| \tag{65}$$

where the last inequality is again due to Lemma V.2. Equations (64) and (65) show that the performance measures of interest are Lipschitz continuous and the proof is complete. ■

Remark: For the more commonly used performance metrics $(1/T)\mathbb{E}[L_T(\theta)]$ (the Expected Loss Rate over $[0, T]$) and $(1/T)\mathbb{E}[Q_T(\theta)]$ (the expected buffer content over $[0, T]$), the Lipschitz constants become $N(T)/T$ and 1, respectively. As $T \to \infty$, the former quantity typically converges to the exogenous event rate.

VI. Optimal Buffer Control Using SFM-Based IPA Estimators

Let us return to the buffer control problem with cost function (3), illustrating one of several possible means to quantify network performance objectives by trading off the expected loss rate (with a rejection penalty R) of class 1 for the expected queue length. The optimal threshold parameter, θ^*, may be determined through a standard stochastic approximation algorithm

$$\theta_{n+1} = \theta_n - \nu_n H_n\left(\theta_n, \omega_n^{\text{SFM}}\right), \; n = 0, 1, \ldots \tag{66}$$

where $\{\nu_n\}$ is a step size sequence and $H_n(\theta_n, \omega_n^{\text{SFM}})$ is an estimate of $dJ_T/d\theta$ evaluated at $\theta = \theta_n$ and based on information obtained from a sample path of the SFM denoted by ω_n^{SFM}. In our case, the gradient estimator $H_n(\theta, \omega_n^{\text{SFM}})$ is the IPA estimator of $dJ_T/d\theta$ based on (41) and (58), evaluated over a simulated sample path ω_n^{SFM} of length T. The estimation is followed by a control update performed through (66) based on the value of $H_n(\theta, \omega_n^{\text{SFM}})$. Details on stochastic approximation algorithms, including conditions required for convergence to an optimum (generally local, unless the form of the cost functions ensures the existence of a single optimum) may be found, for instance, in [20]. However, as already mentioned in the previous section, recognizing the absence of steady state in networks, our main concern is with tracking network performance and seeking continuous improvements as operating conditions change.

The interesting observation here is that the same estimator may be used in the real system as follows. We can observe all events involved in the evaluation of $Q'_T(\theta)$ and $L'_T(\theta)$ in (58), (41), i.e., the starting and ending points of partial loss, full loss, and no loss periods on a sample path of the actual system (denoted by ω_n^{DES}). Assuming that at these event times the arrival rates of both class 1 and class 2 traffic, as well as the service rate, are known (otherwise, they have to be measured on line), then the threshold parameter is updated as follows:

$$\theta_{n+1} = \theta_n - \nu_n H_n\left(\theta_n, \omega_n^{\text{DES}}\right), \; n = 0, 1, \ldots \tag{67}$$

where the only difference from (66) is that data are obtained from ω_n^{DES} (a sample path of the "real" system) instead of ω_n^{SFM} (as sample path of the SFM which one can only simulate). In other words, the *form* of the IPA estimators is obtained by analyzing the system as a SFM, but the associated *values* are based on real data.

Recall that our analysis over an interval $[0, T]$ was based on the convention that $x(\theta; 0) = 0$. Thus, after a control update in (67) the state should be reset to 0. In the case of off-line control, this simply amounts to simulating the system after resetting its state to 0. In the more interesting case of online control, we can take advantage of the following simple observation. Looking at (40) and (57), note that both estimators depend *only* on the portion of a BP that starts at v_1, i.e., the first time the buffer reaches the value θ (if at all); all data since the BP starts and prior to this event are irrelevant to the gradient estimator $H_n(\theta_n, \omega_n^{\text{DES}})$. The implication is that *any initial value* $x(\theta_n; 0) < \theta_n$ may be used without affecting the IPA gradient estimates obtained over an interval $[0, T]$ at the nth iteration of (67). If the nth iteration ends at time τ_n and the state is $x(\theta_n; \tau_n) < \theta_{n+1}$, then the next iteration immediately starts. Otherwise, one may either wait until a value below θ_{n+1} is observed and initiate the next iteration or simply proceed right away, thus, incurring an initialization (transient) error in the estimate (which is negligible for large values of T as also seen in the numerical results obtained when this approach is taken).

Fig. 3 shows the results of the application of this scheme to a single-node SFM with two traffic classes (as in Fig. 1) where the service process is assumed deterministic with $\beta(t) = \beta$ remaining constant throughout the simulation and the service rate is 20 000 packets per second, which corresponds approximately to a 10 Mb/s link processing 512 b packets. The arrival rate process $\alpha_1(t)$ is piecewise constant; each interval over which $\alpha_1(t)$ remains constant is exponentially distributed with rate parameter 25 (i.e., a mean of 0.04 s) and the corresponding traffic rate value is uniformly distributed over [1000,14000] packets per second. Similarly, $\alpha_2(t)$ is piecewise constant and each interval over which $\alpha_2(t)$ remains constant is exponentially distributed with rate parameter 100 (i.e., a mean of 0.01 s) and the corresponding traffic rate value is uniformly distributed over [2000,22000] packets per second. Both class 1 and class 2 packet interarrival times are exponentially distributed. The rejection cost is $R = 1$, and the simulation length in between control updates in (67) is $T = 100$ s. For simplicity, the step size is kept constant with $\nu_n = 500$. This is consistent with our earlier point that our concern is with tracking network performance rather than seeking some optimal threshold value assuming a stationary setting; in the latter case, we would choose a decreasing sequence $\{\nu_n\}$ that satisfies standard conditions (e.g., see [20]), whereas by maintaining a fixed value we are able to respond to changes captured by varying cost sensitivities. In Fig. 3, 'J (DES)' denotes cost curves obtained by estimating $J_T(\theta)$ over different discrete values of $\theta = 1, 2, \ldots$, 'J (SFM)' denotes curves obtained by estimating $J_T(\theta)$ over different values of θ, and 'Opt.Algo.' represents the optimization process (67), where we maintain real-valued thresholds throughout. All cost curves are obtained by averaging 30 sample paths, while the 'Opt.Algo' curve is obtained by executing (67) only once. During a simulated sample path, packets are generated according to the characteristics described above so that arrival rates are known when their values are required in (41) and (58). In order to detect events that start or end a partial loss, full loss, or no loss period, we simply observe the state of the buffer. When the buffer content increases and

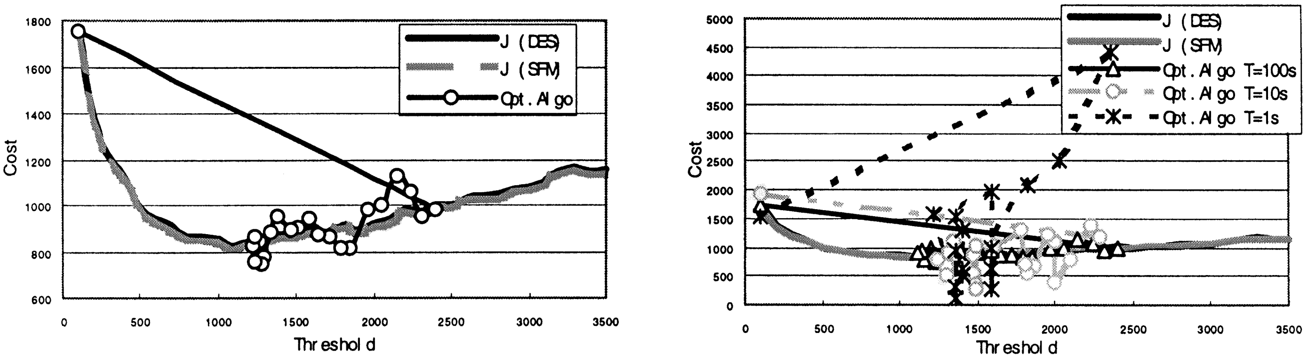

Fig. 3. Optimal threshold determination in an actual system using SFM-based gradient estimators.

reaches the value θ_n, we look at the values of α_1, α_2, and β at that time. If $\alpha_1 + \alpha_2 - \beta > 0$ and $\alpha_2 - \beta < 0$, we identify the start of a partial loss period; if $\alpha_2 - \beta > 0$, this implies the start of a full loss period, and so on. Based on this information, we can calculate the IPA estimator $H_n(\theta_n, \omega_n^{\mathrm{DES}})$ along the sample path until we reach the end of a control update interval $[0, T]$. At this point, the threshold is updated and the process repeats. As seen in Fig. 3, the threshold value obtained through (67) using the SFM-based gradient estimator based on (41), (58) either recovers the optimal threshold θ^* or is very close to it. Similar results have been obtained for many examples involving different arrival processes and traffic intensities.

Finally, in order to investigate the effect of the parameter T in the optimization process, we implemented (67) with $H_n(\theta_n, \omega_n^{\mathrm{DES}})$ estimated over shorter interval lengths $T = 10$ s and $T = 1$ s. The results are shown in the second plot of Fig. 3, compared with the original $T = 100$ s. As expected, the variance of the cost at each step increases with shorter estimation intervals and becomes very large for estimation intervals as short as $T = 1$ s. However, the sensitivity estimates are still sufficient to drive threshold adjustments toward the optimal cost (which we did not actually attempt to achieve). We should also point out that the class 1 loss probabilities on the 'J (DES)' cost curve for threshold values above 2000 (where the algorithm operates after the first few steps) are of the order of 10^{-3}.

VII. Conclusion and Future Work

Our ultimate goal in using IPA for SFMs is to develop an approach for on-line network-wide control that is efficient and does not require any node decomposition assumptions. Toward this goal, we have considered in this paper an SFM of a communication network node with two traffic classes, one uncontrolled and one subject to threshold-based buffer control. Our objective is to control the threshold parameter so as to optimize performance captured by combining loss and workload metrics. We have developed IPA estimators for these metrics with respect to the threshold and shown them to be unbiased. The simplicity of the estimators derived and the fact they are not dependent on knowledge of the traffic arrival or service processes makes them attractive for online control and optimization. For a class of buffer control problems, we have shown how to use an optimization scheme (and illustrated it through numerical examples) for a discrete-event model (viewed as a real, queueing-based single-node system) using the IPA gradient obtained from its SFM counterpart. It is worth pointing out that there is no IPA derivative for the discrete-event model, since its associated control parameter is discrete.

As in our earlier work [8], where we considered a SFM for a single node and single traffic class, we have found that SFMs provide means for determining optimal control parameter settings (rather than attempting to use them for performance analysis). The presence of multiple traffic classes complicates the analysis required, but still yields simple IPA estimators. The model we have considered here assumes infinite buffer capacity; this assumption can be relaxed and our results can be extended to account for a finite buffer, which also directly leads to an extension of the analysis to more than two traffic classes. This is the subject of ongoing work (see [16]).

As already mentioned, our ongoing research is geared toward the use of SFMs and IPA methods for network-wide control and optimization. This requires analyzing the effect of perturbation propagation across network nodes, each node modeled as shown in Fig. 1, with the ability to control incoming traffic while also accommodating interfering (uncontrolled) traffic that has originated elsewhere in the network. Thus, we envision an on-line network congestion control capability that does not require node decomposition and is general in the sense that it does not require knowledge of the traffic and service processes involved and only limited rate information. Toward the same goal, our ongoing work is also considering how to develop IPA methods that include network feedback effects (i.e., allowing arriving traffic processes to depend on the buffer content in different ways) and how to allow for the possibility of packet processing other than through the usual first-in–first-out discipline.

Appendix

Proof of Lemma III.1: Let us first consider periods prior to $[v_m(\theta), v_{m+1})$. We start with the observation that before the first partial loss period at $v_m(\theta)$, we must alternately observe no loss periods in $W(\theta)$ and full loss periods in $V(\theta)$. Moreover, the first period must belong to $W(\theta)$. It follows that if m is even, periods $[v_{2n}(\theta), v_{2n+1}(\theta))$, $n = 0, 1, \ldots, (m/2)$, belong to $W(\theta)$, while $[v_{2n+1}(\theta), v_{2n+2}(\theta))$, $n = 0, 1, \ldots, (m/2)$, belong to $V(\theta)$. If m is odd, then $[v_{2n}(\theta), v_{2n+1}(\theta))$, $n = 0, 1, \ldots, ((m-1)/2)$, belong to $W(\theta)$, and $[v_{2n+1}(\theta), v_{2n+2}(\theta))$, $n = 0, 1, \ldots, ((m-1)/2)$ belong to $V(\theta)$.

For the case $i = 1$, note that (31) immediately follows from (26) and the convention $A_0 \equiv 1$. For periods $[v_{2n}(\theta), v_{2n+1}(\theta))$, $n > 0$, that belong to $W(\theta)$, by (27) we have $v'_{2n+1}(\theta) = (A_{2n}/A_{2n+1})v'_{2n}(\theta)$, and for periods $[v_{2n}(\theta), v_{2n+1}(\theta))$, $n > 0$, in $V(\theta)$, by (24), we have $v'_{2n}(\theta) = (B_{2n-1}/B_{2n})v'_{2n-1}(\theta)$. Therefore, given (31), we can start with this equation setting $n = 1$ to obtain $v'_2(\theta)$ and, proceeding recursively, we obtain all $v'_i(\theta)$ in (33) and (32).

Finally, we consider $v'_i(\theta)$ for all periods following $[v_m(\theta), v_{m+1})$. Recall the fact that v_{m+1} is locally independent of θ, therefore $v'_{m+1} = 0$. Moreover, v_{m+1} is the starting point of either a full loss period or a no loss period. In the former case, it follows from (24) that $v'_{m+2} = 0$. In the latter case, it follows from (27) that $v'_{m+2} = 0$. This process repeats until the next partial loss period (if any) is encountered, and all subsequent event time derivatives maintain a zero value until the end of the BP. ■

Proof of Lemma III.2: Let us first consider event times of the general form v_{2n+1} with $n \geq 0$. Beginning with v_1, when the no loss period $[\xi, v_1)$ which starts the BP switches to a full loss period, note that all v_{2n+1} correspond to starts of full loss periods, therefore $B(v_{2n+1}) > 0$ and by (30) we have $B_{2n+1} > 0$. Then, using (29), we have $A_{2n+1} \geq B_{2n+1} > 0$, therefore, $0 < B_{2n+1}/A_{2n+1} \leq 1$. On the other hand, event times of the general form v_{2n} with $n > 0$ are instants when no loss periods start, implying that $A(v_{2n}) < 0$. Thus, by (29), $A_{2n} < 0$, and, recalling (30), we get $B_{2n} \leq A_{2n} < 0$, so that $0 < A_{2n}/B_{2n} \leq 1$ completing the proof. ■

Proof of Lemma III.3: We start with the case where a BP contains at least one partial loss period and let $[v_m(\theta), v_{m+1}) \in U(\theta)$, $m \geq 1$ be the first such period. Let us first consider the general case where $m > 1$; the special case where $m = 1$ will easily follow. Thus, if $m > 1$, then all periods preceding $[v_m(\theta), v_{m+1})$ belong to either $W(\theta)$ or $V(\theta)$ and appear alternately with the first one belonging to $W(\theta)$. From (21), it is clear that terms contributing to the loss derivative $\lambda'(\theta)$ prior to $v_m(\theta)$ are due only to full loss periods belonging to $V(\theta)$ and given by (25); in particular, every full loss period $[v_{i-1}(\theta), v_i(\theta))$ will contribute $-A_{i-1}v'_{i-1}(\theta) + A_i v'_i(\theta)$, $i = 2, 4, \ldots$. In addition, we know that the ensuing period $[v_i(\theta), v_{i+1}(\theta))$ is a no loss period and from (27) we have $A_i v'_i(\theta) = A_{i+1}v'_{i+1}(\theta)$, $i = 2, 4, \ldots$, and following is another full loss period $[v_{i+1}(\theta), v_{i+2}(\theta))$ contributing, by (25), $-A_{i+1}v'_{i+1}(\theta) + A_{i+2}v'_{i+2}(\theta)$, $i = 2, 4, \ldots$. We can see that the combination of the aforementioned results is $-A_{i-1}v'_{i-1}(\theta) + A_{i+2}v'_{i+2}(\theta)$, $i = 2, 4, \ldots$. Proceeding in this fashion for $i = 1, \ldots, m-1$, we get the contribution to $\lambda'(\theta)$ as follows: i) If $[v_{m-1}(\theta), v_m(\theta))$ is a no loss period, then the contribution from all full loss periods up to $v_m(\theta)$ is $-A_1 v'_1(\theta) + A_{m-1}v'_{m-1}(\theta)$; and ii) if $[v_{m-1}(\theta), v_m(\theta))$ is a full loss period, then the contribution from all full loss periods up to $v_m(\theta)$ is $-A_1 v'_1(\theta) + A_m v'_m(\theta)$. Next, look at the first partial loss period $[v_m(\theta), v_{m+1}(\theta))$, $m > 1$. According to (22) its contribution to $\lambda'(\theta)$ is $-A_m v'_m(\theta)$. If ii) holds, then adding the last two contributions we immediately see that the accumulated contribution at v_{m+1} is $-A_1 v'_1(\theta)$. If i) holds, then recall (27), implying that $A_{m-1}v'_{m-1}(\theta) = A_m v'_m(\theta)$. Thus, we again get the accumulated contribution at v_{m+1} as $-A_1 v'_1(\theta)$. Moreover, from (26), we know that $-A_1 v'_1(\theta) = -1$, which implies that the contribution to $\lambda'(\theta)$ up to this point is simply -1. It remains to show that following the first partial loss period $[v_m(\theta), v_{m+1}(\theta))$ all further contributions to the loss derivative $\lambda'(\theta)$ are 0. By Lemma III.1, $v'_i(\theta) = 0$ for all $i = m+1, \ldots, S-1$. Then, observing in (22) and (25) that all derivatives consist of terms with multiplicative factors in these $v'_i(\theta)$, it follows that all contributions to $\lambda'(\theta)$ are indeed 0 and we obtain (36).

In the case where $m = 1$, the only contribution to $\lambda'(\theta)$ comes from the partial loss period $[v_1(\theta), v_2)$, which, by (22), contributes $-A_1 v'_1(\theta)$. Since, from (26), we have $-A_1 v'_1(\theta) = -1$, (36) is immediately obtained.

Next, consider the case where the BP $[\xi, \eta(\theta))$ contains no partial loss period. This means the whole BP is a sequence of alternating no loss periods and full loss periods. Proceeding exactly as above when analyzing the periods preceding the first partial loss period, we obtain a similar result as ii): $-A_1 v'_1(\theta) + A_{S-1}v'_{S-1}(\theta)$. We already know $-A_1 v'_1(\theta) = -1$, so it remains to determine $A_{S-1}v'_{S-1}(\theta)$. Because the first and last periods, $[\xi, v_1(\theta))$ and $[v_{S-1}(\theta), \eta(\theta))$, respectively, are no loss periods, in between the number of periods must be odd, hence, $S-1$ is even. Using (32) in Lemma III.1, we obtain (37). Finally, by Lemma III.2, (38) immediately follows. ■

Proof of Lemma IV.1: From Lemma III.1, we know that $v'_i(\theta) = 0$ for all $v_i \geq v_{m+1}$. Therefore, looking at (47)–(49) we see that $x'(\theta; t) = 1$ for all $t \in [v_i(\theta), v_{i+1})$, $i \geq m$. Thus, from (50), we have

$$\sum_{i=m+1}^{S} \int_{v_{i-1}(\theta)}^{v_i(\theta)} x'(\theta; t)dt = \sum_{i=m+1}^{S} [v_i(\theta) - v_{i-1}(\theta)] = v_S - v_m. \tag{68}$$

Let us now consider periods $p_i(\theta)$ preceding $p_m(\theta)$. We know that p_{2n+1}, $n = 0, 1, \ldots$ must be full loss periods and p_{2n}, $n = 0, 1, \ldots$ must be no loss periods. Thus, for any p_{2n+1}, (48) applies and we have $x'(\theta; t) = 1 - B_{2n+1}v'_{2n+1}(\theta)$, and for p_{2n}, (49) applies and we have $x'(\theta; t) = 1 - A_{2n}v'_{2n}(\theta)$. Observe that these expressions for $x'(\theta; t)$ are independent of t. Thus, using (54), it follows from (50) that each period preceding p_m contributes a term $(v_{i+1} - v_i)\phi_i$ (except for the first period for which (46) applies). Combining this with (68) yields (55). ■

Proof of Lemma V.1: Consider the increasing sequence $t_0 < t_1 < \cdots < t_N < t_{N+1}$ such that $t_0 = \xi_k$, $t_{N+1} = \eta_k(\theta)$ and at each t_j, $j = 1, \ldots, N$, either the state $x(\theta; t)$ of the nominal sample path switches to a different set among $U(\theta)$, $V(\theta)$, and $W(\theta)$, or the state $x(\theta + \Delta\theta; t)$ of the perturbed sample path switches to a different set among $U(\theta + \Delta\theta)$, $V(\theta + \Delta\theta)$, and $W(\theta + \Delta\theta)$. Therefore, for all $t \in [t_j, t_{j+1})$, the state of each sample path is contained in one of the three sets $U(\cdot)$, $V(\cdot)$, or $W(\cdot)$. Let $I_j = [t_j, t_{j+1})$. Depending on which set $x(\theta; t)$ and $x(\theta + \Delta\theta; t)$ belong to over the interval I_j, there are nine cases to consider. In the following, we use the notation (S_1, S_2), $S_i \in \{U, V, W\}$ to denote the state set in the nominal and perturbed sample paths, respectively. The proof is by induction over all intervals I_j, $j = 0, \ldots, N$. For the first interval $I_0 = [\xi_k, t_1)$, the nominal and perturbed sample paths are in the no loss set

$W(\theta)$ and $W(\theta+\Delta\theta)$ respectively, therefore the change in the state of the two sample paths is identical, given by (18). As a result $\Delta x(t) = \Delta x(\xi_k) \leq \Delta\theta$ for all $t \in [\xi_k, t_1)$ where the last inequality is by assumption. Similarly, $\Delta x(t) \geq 0$, since $\Delta x(\xi_k) \geq 0$. Next, we assume that $0 \leq \Delta x(t) \leq \Delta\theta$ for all $t \in I_{j-1} = [t_{j-1}, t_j)$ and show that $0 \leq \Delta x(t) \leq \Delta\theta$ for all $t \in I_j = [t_j, t_{j+1})$ by considering all nine possible cases previously mentioned.

1) (U,U): Using (59), we immediately get $\Delta x(t) = \Delta\theta$.
2) (V,V): The change in the state of the two sample paths is identical given by (15), therefore, $\Delta x(t) = \Delta x(t_j) \leq \Delta\theta$ for $t \in I_j$, where the inequality is due to the continuity of $x(t)$ and the induction hypothesis. Similarly, $\Delta x(t) \geq 0$, since $\Delta x(t_j) \geq 0$.
3) (W,W): By a similar argument as in 2), $0 \leq \Delta x(t) \leq \Delta\theta$.
4) (U,V): Since $x(\theta;t) = \theta$ due to (59) and $x(\theta+\Delta\theta;t) \geq \theta+\Delta\theta$ due to (60), we get that $\Delta x(t_j) \geq \Delta\theta$. By the induction hypothesis and the continuity of $x(t)$, $\Delta x(t_j) \leq \Delta\theta$, therefore we must have $\Delta x(t_j) = \Delta\theta$. Since $x(\theta;t_j) = \theta$, it follows that $x(\theta+\Delta\theta;t_j) = \theta+\Delta\theta$. In order to have a full loss period in the perturbed sample path, we *must* have $x(\theta+\Delta\theta;t_j^+) > \theta+\Delta\theta$, i.e., $(dx(\theta+\Delta\theta;t_j)/dt^+) > 0$, therefore $\alpha_2(t_j^+) - \beta(t_j^+) > 0$ (see (15)). However, since the nominal sample path is in a partial loss period, $\alpha_2(t)-\beta(t) < 0$ for all $t \in (t_j, t_{j+1})$ [see (11)], yielding a contradiction. As a result, this case cannot possiblly occur.
5) (V,U): From (59), we have $x(\theta+\Delta\theta;t) = \theta+\Delta\theta$ and from (60), $x(\theta;t) \geq \theta$. Therefore, we immediately get $\Delta x(t) \leq \Delta\theta$. To prove that $\Delta x(t) \geq 0$, note that since the perturbed sample path is in a partial loss period, $\alpha_2(t) - \beta(t) < 0$ for all $t \in I_j$. Moreover, since the nominal sample path is in a full loss period, we must have $(dx(\theta;t)/dt^+) = \alpha_2(t)-\beta(t) < 0$ [see (15)]. Therefore, since $x(\theta+\Delta\theta;t) = \theta+\Delta\theta$ and $\Delta x(t_j) \geq 0$ by the induction hypothesis, it follows that $\Delta x(t) \geq \Delta x(t_j) \geq 0$ for all $t \in I_j$.
6) (W,V): Using an argument similar to 4), we can show this is also an impossible case.
7) (V,W): Using an argument similar to 5), we can show $0 \leq \Delta x(t) \leq \Delta\theta$ for all $t \in I_j$.
8) (W,U): Using an argument similar to 4), we can show this is also an impossible case.
9) (U,W): From (61) $x(\theta+\Delta\theta;t) \leq \theta+\Delta\theta$, and from (59), $x(\theta;t) = \theta$. It follows that $\Delta x(t) \leq \Delta\theta$. To prove that $\Delta x(t) \geq 0$, note that since the nominal sample path is in a partial loss period we must have $\alpha_1(t)+\alpha_2(t)-\beta(t) > 0$. Therefore, since the perturbed sample path is in a no loss period, we have $(dx(\theta+\Delta\theta;t)/dt^+) = \alpha_1(t)+\alpha_2(t)-\beta(t) > 0$ ([see (18)] for all $t \in I_j$. Since $x(\theta;t) = \theta$ and $\Delta x(t_j) \geq 0$ by the induction hypothesis, it follows that $\Delta x(t) \geq \Delta x(t_j) \geq 0$ for all $t \in I_j$. ■

Proof of Lemma V.2: Consider a BP $\mathcal{B}_k = [\xi_k, \eta_k(\theta))$ and observe that $x(\theta;\xi_k) = 0$ and $x(\theta;\eta_k) = 0$ since ξ_k and η_k are the beginning and end points of this BP, respectively. The proof is by induction over all buffering periods $k = 1, \ldots, K$. Since the nominal and perturbed sample paths start out at the same initial state, for the first buffering period we have $\Delta x(\xi_1) = 0$, therefore, using Lemma V.1, we get $0 \leq \Delta x(t) \leq \Delta\theta$ for all $t \in [\xi_1, \eta_1(\theta))$. Next we assume that $0 \leq \Delta x(t) \leq \Delta\theta$ for all $t \in [\xi_k, \eta_k(\theta))$ and show that $0 \leq \Delta x(t) \leq \Delta\theta$ for all $t \in [\xi_{k+1}, \eta_{k+1}(\theta))$. The interval $[\eta_k(\theta), \xi_{k+1})$ corresponds to an empty period in the nominal sample path. Therefore, $x(\theta;t) = 0$ and $A(t) \leq 0$ for all $t \in [\eta_k(\theta), \xi_{k+1})$, and, by continuity, $x(\theta;\xi_{k+1}) = 0$. It follows that

$$\begin{aligned}\Delta x(\xi_{k+1}) &= x(\theta+\Delta\theta;\xi_{k+1}) - 0\\ &= \max\left\{0, x\left(\theta+\Delta\theta;\eta_k(\theta)\right) + \int_{\eta_k(\theta)}^{\xi_{k+1}} A(\tau)d\tau\right\}\end{aligned} \tag{69}$$

since, in general, the perturbed state over the nominal sample path empty period $[\eta_k(\theta), \xi_{k+1})$ is such that $x(\theta+\Delta\theta;t) \geq 0$. The right-hand side is $\leq x(\theta+\Delta\theta;\eta_k(\theta)) = \Delta x(\eta_k(\theta))$, since $A(\tau) \leq 0$ and $x(\theta;\eta_k(\theta)) = 0$. Therefore, $\Delta x(\xi_{k+1}) \leq \Delta x(\eta_k(\theta))$. From the induction hypothesis and the continuity of $x(t)$, we know that $\Delta x(\eta_k(\theta)) \leq \Delta\theta$, therefore $\Delta x_{k+1}(\xi_{k+1}) \leq \Delta\theta$. Moreover, $\Delta x(\xi_{k+1}) \geq 0$ in view of (69) and

$$\max\left\{0,\ x\left(\theta+\Delta\theta;\eta_k(\theta)\right) + \int_{\eta_k}^{\xi_{k+1}} A(\tau)d\tau\right\} \geq 0.$$

Using Lemma V.1, the proof is complete. ■

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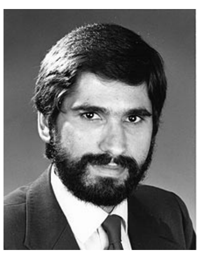

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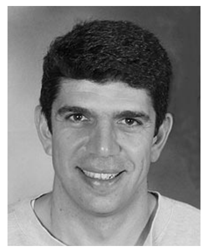

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