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A Hierarchical Decomposition Method for Optimal Control of Hybrid Systems*

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Abstract

We consider optimal control problems for hybrid systems with a separable cost structure allowing us to decompose them into two components: a lower-level component with *time-driven* dynamics (describing the physical state of the system) interacting with a higher-level component with *event-driven* dynamics (describing the changes in the operating modes of the system). We develop a hybrid controller which aims at jointly optimizing the performance of both hierarchical components. We demonstrate this approach on two problems: a linear system switching from one operating mode to another and a multistage manufacturing system. In the first problem, the main difficulty is due to the coupling of the physical states across modes, whereas in the second it is due to the nondifferentiable event-driven dynamics.

1 Introduction

The term "hybrid" is used to characterize systems that combine *time-driven* and *event-driven* dynamics. The former are represented by differential (or difference) equations, while the latter may be described through various frameworks used for Discrete Event Systems (DES), such as timed automata, max-plus equations, or Petri nets (see [2]). Broadly speaking, two categories of modeling frameworks have been proposed to study hybrid systems: Those that extend event-driven models to include time-driven dynamics; and those that extend the traditional time-driven models to include event-driven dynamics; for an overview, see [1].

The hybrid system modeling framework we consider in this paper is motivated by the fact that it is often natural to hierarchically decompose systems into a lower-level component representing physical processes characterized by time-driven dynamics and a higher-level component representing discrete events related to these physical processes (e.g., switching from one mode of operation to another, as in shifting gears in an automotive system). Our objective is to formulate and solve optimal control problems associated with trade-offs between the operation of physical processes and timing issues related to the overall performance of the system. For a class of such optimal control problems, a hierarchical decomposition method was introduced in [6]. This method enables us to design a *hybrid controller* which has the task of communicating with both components and jointly solving coupled optimization problems, one for each component. In [6], this approach was used in the context of a single-stage manufacturing system where discrete entities (referred to as *jobs*) are processed to change their physical characteristics according to certain specifications. Associated with each job is a *temporal* state and a *physical* state. The temporal state of a job evolves according to event-driven dynamics and includes information such as the waiting time or departure time of the job at various work-

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centers. The physical state evolves according to time-driven dynamics modeled through differential (or difference) equations which, depending on the particular problem being studied, describe changes in such quantities as the temperature, size, weight, chemical composition, or some other measure of the "quality" of the job. The interaction of time-driven with event-driven dynamics leads to a natural trade-off between temporal requirements on job completion times and physical requirements on the quality of the completed jobs.

A feature of the manufacturing system considered in [6] is that the physical states of jobs are independent, i.e., the initial physical state of the ith job is decoupled from the final physical state of the $(i-1)$th job. In this paper, we extend the method to allow for physical state coupling and illustrate it through an optimal control problem for a simple linear system switching between two operating modes. Similar optimal control problems are tackled in [7] and [8] using dynamic programming techniques. Moreover, we extend the work in [6] to optimal control problems for multistage manufacturing systems. In the single-stage problem it was possible to overcome the difficulty due to nondifferentiabilities in the event-driven state dynamics by decomposing the optimal state trajectory into segments over which smaller and simpler constrained convex problems can be solved (see [4]). Identifying these segments is a combinatorially explosive task. However, it was recently shown [5] that it is possible to reduce this complexity from 2^{N-1} to at most N such convex problems (where N is the number of jobs processed). Since it seems unlikely that the same can be accomplished in multistage models, we commit instead to a Bezier approximation (as in [3]) to construct surrogate event-driven dynamics that avoid the nondifferentiability issue, so that a standard gradient-based procedure for solving a Two Point Boundary Value Problem (TPBVP) can be applied.

2 Problem Formulation

In the hybrid system model that we consider, the state of the system consists of temporal and physical components. The temporal components keep track of the time information for system events that may cause switches in the operating mode of the system. Let $i = 1, 2, \ldots$ index these events. We denote the ith *physical state* of the system by $z_i(t)$ with dynamics:

$$\dot{z}_i = g_i(z_i, u_i, t), \quad z_i(x_{i-1}) = z_i^0 \tag{1}$$

where $u_i(t)$ is the control applied over an interval $[x_{i-1}, x_i)$ defined by two event occurrences at times x_{i-1} and x_i. In the case of a single event process in the system, the event-driven dynamics characterizing the *temporal states* x_i are given by

$$x_i = x_{i-1} + \gamma_i(z_i, u_i) \tag{2}$$

for $i = 1, 2, \ldots$, where $\gamma_i(z_i, u_i)$ represents the amount of time between switches which generally depends on the physical state and control. In the case of *multiple* asynchronous event processes in the system indexed by $j = 1, \ldots, M$, the event-driven dynamics are of the general form

$$x_i = x_{i-1} + f_i(y_{i,1}, \cdots, y_{i,M}, \gamma_i(z_i, u_i), t) \tag{3}$$

where $y_{i,1}, \cdots, y_{i,M}$ are the event clocks of a timed automaton which determines which of the M events in the system triggers the next switch and at what precise time (for details see [2]). The exact way in which the timed automaton works is not essential to the presentation that follows. We note, however, that $f_i(\cdot)$ above usually involves max and/or min operations (as illustrated in the model of Section 4), introducing nondifferentiabilities which can significantly complicate our problem. Looking at (1) and (3), note also that the choice of control u_i affects both the physical state z_i and the temporal state x_i. Thus, the switches at times $x_1, x_2, \ldots$ are generally *not* exogenous events that dictate changes in the state dynamics, but rather temporal states intrically connected to the control of the system; this is one of the crucial elements of a "hybrid" system.

The optimal control problem we consider has the general form

$$\min_{\mathbf{u}} J = \sum_{i=1}^{N} [\phi_i(z_i, u_i, x_i, x_{i-1}) + \psi_i(x_i)]$$

Here, $\phi_i(z_i, u_i, x_i, x_{i-1})$ is the cost of operating the system with control $u_i(t)$ resulting with the physical state $z_i(t)$ over the interval $[x_{i-1}, x_i)$, and $\psi_i(x_i)$ is the cost associated with the occurrence time x_i of the ith event. As an example,

$$\phi_i(z_i, u_i, x_i, x_{i-1}) = \frac{1}{2}h\left\|z_i^f - z_i^d\right\|^2 + \frac{1}{2}\int_{x_{i-1}}^{x_i} r\left\|u_i(t)\right\|^2 + q\left\|z_i(t)\right\|^2 dt \quad (4)$$

is a quadratic cost imposed on the deviation of the final state $z_i^f = z_i(x_i)$ from the desired value z_i^d and on the control and state variables over a processing interval $[x_{i-1}, x_i)$. Similarly,

$$\psi_i(x_i) = (x_i - x_i^d)^2$$

is a quadratic cost associated with the deviation of the ith event time from a desired target x_i^d.

Assuming stationarity of the cost $\phi_i(\cdot)$ in the sense that $\phi_i(z_i, u_i, x_i, x_{i-1}) = \phi_i(z_i, u_i, s_i)$ where $s_i = x_i - x_{i-1} \geq 0$, $i = 1, 2, \ldots$, the optimal control problem we consider can be written as

$$\min_{\mathbf{u}} J = \sum_{i=1}^{N} [\phi_i(z_i, u_i, s_i) + \psi_i(x_i)] \quad (5)$$

subject to (1) and (3), where $\mathbf{u} = \{u_1, \ldots, u_N\}$.

2.1 Hierarchical Decomposition

Since $u_i(t)$, $t \in [x_{i-1}, x_i)$, is a function of $z_i(t)$ and s_i (constrained to be non-negative), we can rewrite (5) as

$$\min_{\mathbf{z},\mathbf{s}} \sum_{i=1}^{N} \left[\min_{u_i(z_i,s_i)} \phi_i(z_i, u_i, s_i) + \psi_i(x_i)\right]$$

Moreover, given $u_i(t)$ along with the initial and final physical states, $z_i^0 = z_i(x_{i-1})$ and $z_i^f = z_i(x_i)$ and the ith time interval duration s_i, the physical state $z_i(t)$, $t \in [x_{i-1}, x_i)$, is specified through (1). Therefore, we can simplify the optimal control problem as follows:

$$\min_{\mathbf{z}^0,\mathbf{z}^f,\mathbf{s}} \sum_{i=1}^{N} \left[\min_{u_i(z_i^0, z_i^f, s_i)} \phi_i(z_i, u_i, s_i) + \psi_i(x_i)\right]$$

and impose a decomposition into a collection of inner minimization problems subject to (1), and an outer minimization problem subject to (3). At the lower level of this decomposition we first seek to determine the cost

$$\theta_i(z_i^0, z_i^f, s_i) \equiv \min_{u_i} \phi_i(z_i, u_i, s_i) \quad (6)$$

subject to (1) for all $i = 1, 2, \ldots$, which we view as the minimal cost for a *given* time interval s_i and boundary conditions z_i^0 and z_i^f for the physical state. Accordingly, the optimal control is

$$u_i^*(z_i^0, z_i^f, s_i) \equiv \arg\min_{u_i} \phi_i(z_i, u_i, s_i) \quad (7)$$

Note that $u_i^*(z_i^0, z_i^f, s_i)$ is in general time-varying over $[x_{i-1}, x_{i-1} + s_i)$. Once $u_i^*(z_i^0, z_i^f, s_i)$ and $\theta_i(z_i^0, z_i^f, s_i)$ are determined, we can proceed with the higher-level optimization problem:

$$\min_{\mathbf{z}^0,\mathbf{z}^f,\mathbf{s}} \sum_{i=1}^{N} [\theta_i(z_i^0, z_i^f, s_i) + \psi_i(x_i)] \quad (8)$$

subject to (3) where we try to determine the optimal event times and physical states at these times. Once these are known, the relationship (7) is used to determine the optimal controls for the N time intervals involved in the operation of the system.

The hybrid controller for coordinating the two problems above operates as follows (see Figure 1): (i) System Identification: the physical dynamics, $\mathbf{g}$, the costs associated with the physical dynamics, $\boldsymbol{\phi}$, the temporal dynamics, $\mathbf{f}$, and the costs associated with the temporal dynamics, $\boldsymbol{\psi}$, are all input to the controller. (ii) The lower level controller solves (6) to determine $\theta_i(z_i^0, z_i^f, s_i)$ and $u_i^*(z_i^0, z_i^f, s_i)$ for all $i = 1, \ldots, N$. (iii) The higher level controller solves (8) to determine the optimal values s_i^*, $(z_i^0)^*$, and $(z_i^f)^*$ for all $i = 1, \ldots, N$. (iv) The lower level controller evaluates $u_i^* = u_i^*((z_i^0)^*, (z_i^f)^*, s_i^*)$ for all $i = 1, \ldots, N$.

3 Two-mode Linear System Problem

Consider the linear system

$$\begin{aligned} \dot{z}_1 &= u_1, & z_1(x_0) &= z_0 \\ \dot{z}_2 &= \alpha z_2 + u_2, & z_2(x_1) &= z_1(x_1) \end{aligned}$$

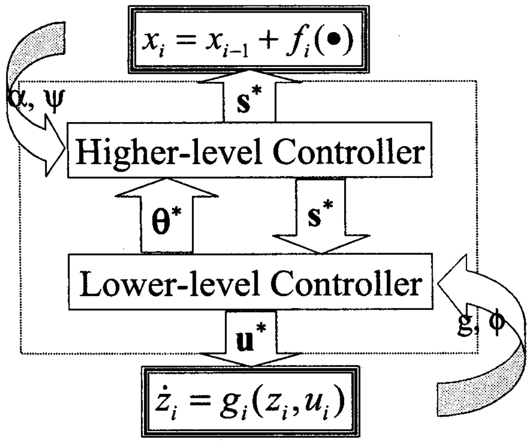

Figure 1: Hybrid Controller Operation

where the event-driven dynamics have the simple form given in (2):

$$\begin{aligned} x_1 &= x_0 + s_1(z_1, u_1) \\ x_2 &= x_1 + s_2(z_2, u_2) \end{aligned}$$

The cost to minimize is

$$J = \phi_1(z_1, s_1, u_1) + \phi_2(z_2, s_2, u_2) + \psi_2(x_2)$$

where

$$\begin{aligned} \phi_1(z_1, u_1, s_1) &= \int_0^{s_1} \frac{1}{2} r_1 u_1^2(t) dt \\ \phi_2(z_2, u_2, s_2) &= \frac{1}{2} h (z_2^f - z_d^f)^2 \\ &\quad + \int_0^{s_2} \frac{1}{2} r_2 u_2^2(t) dt \\ \psi_2(x_2) &= \beta x_2^2 \end{aligned}$$

Note that the cost is separable so the decomposition approach previously described may be applied. The Hamiltonian for the first stage $[x_0, x_0 + s_1)$ is

$$H_1(t) = \frac{1}{2} r_1 u_1^2(t) + p_1(t) u_1(t)$$

where $p_1(t)$ is the costate for the first stage, hence the optimality conditions for the first stage are

$$\begin{aligned} \dot{z}_1(t) &= u_1(t) \\ \dot{p}_1(t) &= 0 \\ 0 &= r_1 u_1(t) + p_1(t) \end{aligned}$$

Therefore, the optimal control in the first stage is constant and we get

$$u_1(t) = u_1 = \frac{z_1(x_1) - z_1(x_0)}{s_1}$$

and

$$\begin{aligned} \theta_1(s_1, z_1) &= \min_{u_1} \phi_1(z_1, s_1, u_1) = \frac{1}{2} r_1 u_1^2 s_1 \\ &= \frac{1}{2} \frac{r_1}{s_1} \left[z_1(x_1) - z_1(x_0) \right]^2 \end{aligned}$$

Similarly for the second stage the Hamiltonian is

$$H_2(t) = \frac{1}{2} r_2 u_2^2(t) + p_2(t) u_2(t) + \alpha p_2(t) z_2(t)$$

where $p_2(t)$ is the costate for the second stage, hence the optimality conditions for the second stage are

$$\begin{aligned} \dot{z}_2(t) &= \alpha z_2(t) + u_2(t) \\ \dot{p}_2(t) &= -\alpha p_2(t) \\ 0 &= r_2 u_2(t) + p_2(t) \end{aligned}$$

Therefore,

$$u_2(t) = -\frac{2\alpha \left(z_2^f - z_2^0 e^{\alpha s_2} \right)}{(e^{-\alpha s_2} - e^{\alpha s_2})} e^{-\alpha t}$$

and

$$\begin{aligned} \theta_2(s_2, z_2) &= \min_{u_2} \phi_2(z_2, s_2, u_2) \\ &= \frac{1}{2} h (z_2^f - z_d^f)^2 \\ &\quad + \frac{r_2 \alpha \left(z_2^f - z_2^0 e^{\alpha s_2} \right)^2}{(e^{\alpha s_2} - e^{-\alpha s_2})} e^{-\alpha s_2} \end{aligned}$$

The higher-level optimization problem hence becomes

$$\min_{\substack{s_1, s_2, z_1^0, \\ z_2^0, z_1^f, z_2^f}} \left[\begin{array}{c} \frac{1}{2} \frac{r_1}{s_1} \left(z_1^f - z_1^0 \right)^2 + \frac{1}{2} h \left(z_2^f - z_d^f \right)^2 \\ + \frac{\left(z_2^f - z_2^0 e^{\alpha s_2} \right)^2 \alpha r_2}{(e^{2\alpha s_2} - 1)} + \beta x_2^2 \end{array} \right]$$

subject to $z_1^0 = z_0$, $z_2^0 = z_1^f$, $s_1, s_2 \geq 0$, and $x_2 = s_1 + s_2$. The simplified optimization problem is

$$\min_{\substack{s_1, s_2, \\ z_1^f, z_2^f}} \left[\begin{array}{c} \frac{1}{2} \frac{r_1}{s_1} \left(z_1^f - z_0 \right)^2 + \frac{1}{2} h \left(z_2^f - z_d^f \right)^2 \\ + \frac{\left(z_2^f - z_1^f e^{\alpha s_2} \right)^2 \alpha r_2}{(e^{2\alpha s_2} - 1)} + \beta (s_1 + s_2)^2 \end{array} \right]$$

where $s_1, s_2 \geq 0$.

The optimality conditions are given in terms of four algebraic equations which can be solved to yield s_1, s_2, z_1^f, and z_2^f:

$$\begin{aligned}\frac{r_1}{s_1}(z_1^f - z_0) &= \frac{2\left(z_2^f - z_1^f e^{\alpha s_2}\right)\alpha r_2}{(e^{\alpha s_2} - e^{-\alpha s_2})} \\ h(z_2^f - z_d^f) &= -\frac{2\left(z_2^f - z_1^f e^{\alpha s_2}\right)\alpha r_2}{(e^{2\alpha s_2} - 1)}\end{aligned}$$

For $s_1 > 0$ and $s_2 > 0$, we get:

$$\begin{aligned}r_1(z_1^f - z_0)^2 &= 4\beta s_1^2(s_1 + s_2) \\ \beta(s_1 + s_2) &= \frac{\alpha^2 r_2\left(z_2^f - z_1^f e^{\alpha s_2}\right)(z_1^f e^{\alpha s_2})}{(e^{2\alpha s_2} - 1)} \\ &+ \frac{\alpha^2 r_2 e^{2\alpha s_2}\left(z_2^f - z_1^f e^{\alpha s_2}\right)^2}{(e^{2\alpha s_2} - 1)^2}\end{aligned}$$

For $s_1 = 0$ and $s_2 = 0$, we get $z_1^f = z_0$ and $z_2^f = z_1^f$, respectively. For a numerical example, setting $r_1 = 2$, $r_2 = 10$, $h = 10$, $z_d = 10$, $z_0 = 0$, $\alpha = 1$ and $\beta = 10$, yields the following solution: it is optimal to start operating in the first mode with constant control $u_1(t) = 5.72$ and switch to the second mode at time $x_1 = 0.4$ when $z_1^f = 2.29$. The system operates in the second mode with control $u_2(t) = 1.66e^{-t}$ until time $x_2 = 1.64$ when $z_2^f = 9.67$.

4 Multistage Manufacturing System Problem

Consider an M-stage hybrid manufacturing system which incurs the cost

$$J(\mathbf{u}, \mathbf{s}, \mathbf{x}) = \sum_{i=1}^{N}\sum_{j=1}^{M}[\phi_{i,j}(z_{i,j}, u_{i,j}, s_{i,j}) + \psi_{i,j}(x_{i,j})]$$

while processing N jobs. In this case, it is convenient to replace (3), which is based on a global event counter i, by M equations describing the well-known event-driven dynamics at each stage in terms of a local event counter. We will assume that, for $i = 1, ..., N$ and $j = 1, ..., M$, the temporal state $x_{i,j}$ for the ith job at the jth stage of the process evolves according to the well-known event-driven dynamics

$$\begin{aligned}x_{i,j} &= \max(x_{i-1,j}, x_{i,j-1}) + s_{i,j}(u_{i,j}) \quad (9) \\ x_{i,0} &= \alpha_i \ , \ x_{0,j} = -\infty\end{aligned}$$

where α_i is the ith job arrival time. The cost $\psi_{i,j}(x_{i,j})$ associated with the temporal state $x_{i,j}$ is

$$\psi_{i,j}(x_{i,j}) = \begin{cases} \beta(x_{i,j} - \alpha_i)^2 & j = M \\ 0 & \text{otherwise}\end{cases}$$

which penalizes the *total system time* of the ith job. The physical state $z_{i,j}$, on the other hand, evolves according to the time-driven dynamics

$$\begin{aligned}\dot{z}_{i,j} &= u_{i,j} \ , \\ z_{i,j}(x_{i,j-1}) &= 0, \quad z_{i,j}(x_{i,j}) = z_j^d\end{aligned}$$

where the initial and the final states are all fixed. For simplicity, we assume that $u_{i,j} = 0$ during the waiting time, i.e., the state $z_{i,j}$ changes only during the process and not while being queued at any of the stages. The cost of processing the ith job at the jth stage is

$$\phi_{i,j}(z_{i,j}, u_{i,j}, s_{i,j}) = \int_0^{s_{i,j}} \frac{1}{2} r_j u_{i,j}^2$$

Following the four basic steps of the hybrid controller described in Section 2, first the dynamics and the cost information above are provided to the controller. Since the initial and the final states are provided the higher level optimization task is simplified.

Next, the lower-level controller evaluates $\theta_{i,j}(s_{i,j})$ and $u_{i,j}(s_{i,j})$ for all $i = 1, \ldots, N$ and $j = 1, ..., M$. Using calculus of variations as in the previous section, $u_{i,j}^*(t) = u_{i,j}^*$ which is a constant, and integrating the state equation $\dot{z}_{i,j} = u_{i,j}^*$ gives

$$u_{i,j}^* = \frac{z_j^d}{s_{i,j}}$$

The optimal control, therefore, will incur a cost

$$\theta(s_{i,j}) = \int_0^{s_{i,j}} \frac{1}{2} r_j u_{i,j}^{*2} dt = \frac{1}{2} r_j \frac{(z_j^d)^2}{s_{i,j}} = \frac{\gamma_j}{s_{i,j}}$$

where $\gamma_j = \frac{1}{2}r_j(z_j^d)^2$. The higher level optimization problem now becomes

$$\min_{\mathbf{s}} \sum_{i=1}^{N} \left[\sum_{j=1}^{M} \frac{\gamma_j}{s_{i,j}} \right] + \beta(x_{i,M} - \alpha_i)^2$$

subject to (9). Note the simplification due to fixed physical states at times $x_{0,j}, \ldots x_{N,j}$ for all $j = 1, \ldots, M$. The nondifferentiable dynamics due to the max operation in (9), on the other hand, introduce an additional difficulty common in the analysis of many discrete event systems.

Forming the augmented cost

$$\begin{aligned}\bar{J}(\mathbf{s}, \mathbf{x}, \boldsymbol{\lambda}) &= \sum_{i=1}^{N} \left[\begin{array}{c} \sum_{j=1}^{M} \frac{\gamma_j}{s_{i,j}} \\ +\lambda_{i,j}(\max(x_{i-1,j}, x_{i,j-1}) \\ +s_{i,j} - x_{i,j}) \end{array} \right] \\ &\quad +\beta(x_{i,M} - \alpha_i)^2\end{aligned}$$

the optimality equations for $i = 1, ..., N$ and $j = 1, ..., M$ are

$$\frac{\partial \bar{J}}{\partial s_{i,j}} = 0, \quad \frac{\partial \bar{J}}{\partial \lambda_{i,j}} = 0, \quad \frac{\partial \bar{J}}{\partial x_{i,j}} = 0$$

The first equation gives

$$\frac{\partial \bar{J}}{\partial s_{i,j}} = -\frac{\gamma_j}{s_{i,j}^2} + \lambda_{i,j} = 0$$

The second equation gives (9), while the third equation yields the following:

For $i < N$ and $j < M$

$$\begin{aligned}\lambda_{i,j} &= \lambda_{i,j+1} \frac{\partial \max(x_{i,j}, x_{i-1,j+1})}{\partial x_{i,j}} \\ &\quad +\lambda_{i+1,j} \frac{\partial \max(x_{i+1,j-1}, x_{i,j})}{\partial x_{i,j}}\end{aligned}$$

For $j < M$

$$\lambda_{N,j} = \lambda_{N,j+1} \frac{\partial \max(x_{N,j}, x_{N-1,j+1})}{\partial x_{N,j}}$$

For $i < N$

$$\lambda_{i,M} = \frac{\partial \psi_{i,M}}{\partial x_{i,M}} + \lambda_{i+1,M} \frac{\partial \max(x_{i+1,M-1}, x_{i,M})}{\partial x_{i,M}}$$

Finally, $\lambda_{N,M} = \partial \psi_{N,M} / \partial x_{N,M}$. Using the Bezier approximation approach described in [3], this TPBVP can be solved effectively to evaluate the optimal service time sequence $\{s_{i,j}^*\}$. The last remaining step is for the hybrid controller to evaluate $u_{i,j}^*$ for all $i = 1, \ldots, N$ and $j = 1, ..., M$. In particular, the lower-level controller evaluates $u_{i,j}^* = u_{i,j}(s_{i,j}^*) = z_{d,j}^f / s_{i,j}^*$ for all $i = 1, \ldots, N$ and $j = 1, ..., M$. The optimal control input $u_{i,j}^*$ is fed to the jth stage while processing the ith job (during the $[\max(x_{i-1,j}, x_{i,j-1}), x_{i,j})$ interval) which departs at time $x_{i,j}$.

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