Robust Identification of Uncertain Systems *

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Abstract

The problem of identification of uncertain systems arises whenever we must choose a model from a model class that only approximately describes the underlying system. Identification schemes here need to overcome the distortions in the data due to undermodeling error in addition to random stochastic noise. We introduce the notion of robust consistency, which requires estimating the optimal approximation from corrupted data. We then provide fundamental limits for when consistency is achievable for a wide variety of contexts in both unstructured as well structured uncertain environments. Structured uncertainty pertains to problems where additional prior information such as frequency weighting or specific component-level uncertainties are known. We show the optimality of well-known instrument-variable techniques in several cases and characterize the instruments and inputs that lead to robust identification. We show that robust consistency is achievable if and only if there is an instrument that can uniformly de-correlate both the input and noise. The final part of the paper deals with the question of design of input sequences that satisfy these necessary and sufficient conditions. In this regard, we show that any sequence having a bounded strictly positive spectral density suffices. The rest of the paper then deals with the design of optimal deterministic sequences for robust identification. We develop fast-sine-sweeps in this context and by appealing to elementary continued fraction theory show that these have polynomially and uniformly decaying auto-correlations properties.

Keywords: uncertain systems, robust identification, LFT, input design, instrumental variables

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### 1 Introduction

Robust identification of uncertain systems deals with approximate modelling of an unknown system directly from data. This issue arises whenever the underlying system is not adequately described by the chosen model class. Practically, these issues arise in a number of both stationary and non-stationary situations. For instance, acoustic cancellation systems, designed for automobile cabins, must deal with time-varying and complex acoustic dynamics. These result from a combination of large-scale multi-path reflections coupled with time-variation imposed both by passenger movement and environment [26]. Significant time variation relative to the delay spread of acoustic dynamics necessitates approximate modelling and tracking of the underlying time-varying system. Similar situations are encountered in the problem of channel identification in air-to-ground communication scenarios [26]. Such non-stationary situations are but stationary contexts viewed over finite data. There are also purely stationary situations such as distributed systems where approximate lumped parameter models can be sought because of limited spatial data. This situation is but similar to the non-stationary one viewed from a spatial context. The motivation for this paper stems from such situations where the objective is to approximately model an uncertain system directly from data. We focus on robust identification of uncertain LTI systems in this paper. The focus stems from the fact that LTI contexts are still poorly understood and that time-varying contexts are well modelled as LTI systems viewed over finite data.

The question of as to how to deal with situations where the underlying system does not belong to the model class has been dealt with extensively. In traditional statistical identification it has long been recognized [13] that the identified model has two sources of error: bias error that results from undermodeling and variance error that results from noise. The effect of measurement noise which is typically uncorrelated with the input is well understood. Recently, several researchers [8, 14] have focused attention on evaluation the so called “model error” (bias) in the identified model within a statistical paradigm. In parallel—inspired by the information-based-complexity (IBC) [24] framework—several researchers starting with [19, 20] (also see references therein) have extensively dealt with the so called uniform-bounded-noise (UBN) as a means towards incorporating unmodeled dynamics. Recognizing the fact that unlike measurement noise—where the statistics are usually known—the level of unmodeled dynamics are usually unknown, several researchers within this community have recently suggested a separation of unmodeled dynamics and noise [6, 28]. These questions have recently motivated [5] to deal with model error evaluation in the UBN setting. Again parallel efforts have been directed towards evaluating the level of unmodeled dynamics. This separation of unmodeled dynamics and noise was first explicitly introduced in [12, 32]. In [17] a wider and more extensive view of the exact nature of uncertainty and how it should be modelled is taken.

Nevertheless, there has been limited effort towards understanding the impact of undermodeling on parametric error. Part of the reason is that, once unmodeled dynamics is introduced, it is no longer clear as to how to further classify the identification error. This is because there are, in general, infinitely many decompositions of the underlying system into a model belonging to the chosen model class and the residual dynamics. Nevertheless, there are several situations where this distinction is natural. From a robust control perspective, parametric uncertainty has significantly less impact relative to the unstructured uncertainty. This implies that we should seek models that minimize unstructured uncertainty. Parametric error suffered during identification is then simply the difference between the identified parametric model and the optimal model approximation. This distinction is meaningful in other cases as well. As pointed out earlier there are several cases such as time-varying systems, where undermodeling is unavoidable. In these cases, the objective of estimating the best approximation is natural. Robust identification viewed in this sense seeks to minimize the error between the
estimated model and the best approximation. The term robust refers to the fact that identification schemes need
to overcome distortions in the data that result from undermodeling as well as measurement noise.

This objective of undermodeling a complex system and seeking to identify the best approximation was
formulated by the author in [27]. The novelty of this work pertained to the question of as to how to “model”
distortions due to unmodeled dynamics in the data. This overcame the inherent conservatism of the UBN
model in accounting for unmodeled dynamics(see [4,21,25]) where it was shown that parametric error linearly
scales with the level of unmodeled dynamics. On the other hand other more detailed noise models such as
the deterministic white noise models have been shown to be too stringent in being able to capture unmodeled
errors [28]. Based on these considerations in [27] structural constraints on the unmodeled errors were imposed.
These structural constraints incorporated a natural decomposition between the optimal approximation and
unstructured uncertainty resulting in a model component and an “orthogonal” unmodeled component. These
structural assumptions on the unmodeled dynamics showed that the parametric errors approach zero irrespective
of the level of unmodeled dynamics.

In this paper we continue and extend this theme along several fundamental directions. First, we address
the fundamental conditions required to achieve asymptotic consistency of the estimates. These conditions are
shown to be necessary not only for different norms but also for different model classes. These conditions imply
that robust consistency holds if and only if a specific instrumental variable method exists that annihilates both
unmodeled error as well as noise. Furthermore, these conditions fundamentally quantify as to how “model-
error” impacts estimating optimal approximations. These conditions also show that, although, consistency of
parameters does not hold in the bounded but arbitrary noise setting, it does hold almost surely except on a
set of Lebesgue measure zero. Second, these conditions help establish robust consistency for several structured
uncertain systems described by linear fractional transformations. In this light we deal with the question of as to
how to shape inputs to reflect prior frequency information about both noise spectrum and unmodeled dynamics.
Next, we demonstrate robust consistency for systems within a feedback loop and identification of an uncertain
component which forms a part of a series connection of other “known” uncertain systems and so on. The third
part of our paper deals with the design of input sequences that meet the necessary and sufficient conditions
derived earlier. It is well-known that inputs play a central role in system identification as they provide the
fundamental mechanism to shape the data to reveal useful information. Input design for identification of models
with statistically uncorrelated noise has had a long history [7,9,18,33]. Design of optimal sequences for bounded
but arbitrary noise models have been presented in [4,10,12,16,21,25,28]. Herein we design input sequences for
identification of uncertain systems that ensures robust consistency. We show that inputs must satisfy “infinite”
informativity, which amounts to requiring input sequences to have uniform decay of auto correlations. We
develop fast-sine-sweeps and show that they have polynomially decaying auto-correlations. The proof requires
application of concepts from continued fraction theory for which background material is provided for the sake of
completion.

The organization of the paper is as follows. In section 2 we present necessary and sufficient conditions
for robust consistency. Based on these results in section 3 we develop instrumental variables techniques for
identification of uncertain systems described by LFT structures. Finally, section 4 is devoted to the design of
input sequences for robust identification.
2 Asymptotic Consistency with Unstructured Uncertainty

In this section we deal with the problem of asymptotic consistency of systems with unstructured uncertainty. Unstructured uncertainty pertains to the situation where the only information available is that the underlying system does not belong to the chosen model class. We provide fundamental conditions required to achieve asymptotic convergence of the model estimates to their optimal approximations in the model class. We also quantify as to how “model-error/bias” impacts estimating optimal approximations. These conditions are shown to be necessary not only for different norms but for different model classes as well. These conditions imply that robust consistency holds if and only if a specific instrumental variable method (IV) exists that de-correlates both input and noise.

The main issue herein is that every identified model in the model class has a bias. Nevertheless, the choice of an identification scheme can impact the level of bias in the identified model. Furthermore, bias has two components: parametric error and non-parametric error. The parametric error is the error suffered within the model class, i.e., aspects of the data that can be accounted for by varying the parameters. The non-parametric component pertains to aspects of data that cannot be accounted for within the model class. One of the difficulties here is that it is possible to have infinitely many combinations of parametric and non-parametric components that result in the same system. Therefore, this raises the question of as to how to attribute the identification errors between these two components. A natural way to avoid this problem is presented in our earlier work [27]. The idea is that once a norm is defined on the space of systems, there is a “unique” model in the model class that minimizes the approximation error as shown schematically in Figure 1. The objective of identification then is to identify this approximate model. The parametric error is then given by the distance between the identified model and this approximate model. We illustrate the concept here for finite impulse response models for the sake of exposition. Suppose, the input output data is given by:

\[ y(k) = Hu(k) + w(k); \quad H \in \ell_1 \]

with \( w(0), w(1), \ldots \) is a sequence of i.i.d. random variables that are uniformly distributed (generalizations will be considered later) on the interval \( W = [-\sigma, \sigma] \). The inputs are bounded, i.e., \( |u(k)| \leq 1, \ k \in \mathbb{Z}^+ \). The input-output data is observed over a time interval starting from \( t = 0 \) to \( t = N \). Now, it is impossible to identify \( H \) since it is an arbitrary element in \( \ell_1 \). Indeed, even if \( H \) were bounded, say \( \|H\|_1 \leq \gamma \) with known bound, \( \gamma \), and data length, \( N \)—it is still not possible to estimate, \( H \), to a resolution better than \( \gamma \) in the worst-case. Therefore, rather than identify \( H \), we focus on identifying an approximation for \( H \) within a chosen model class as shown in Figure 1. One such class is the FIR model class \( G_{FIR} \) of models of order \( m \), which we focus here for the sake of exposition.

The best model, \( G^* = G(H) \), is one that is closest to the underlying uncertain system, \( H \), in some metric norm as this model has the smallest unmodeled dynamics associated with it. For the \( \ell_1 \) norm (other normed topologies are considered later) the best model, \( G(H) \in \mathcal{G} \), is:

\[ G(H) = \text{argmin}_{G \in \mathcal{G}} \|H - G\|_1 \]

For the FIR model class the best model is given by

\[ G(H) = \{h(k)\}_{k=0}^{m-1} \]

where \( h(k) \) is the impulse response sequence of the uncertain system. It follows that the input-output relation
can be rewritten as:

\[
y(t) = \sum_{k=0}^{m-1} h(k)u(t-k) + \sum_{k=m}^{\infty} h(k)u(t-k) + w(t), \ t = 0, 1, \ldots, N
\] (3)

Our objective reduces to estimating the first \(m\) impulse response coefficients from the input-output data. If we define the estimates by \(\hat{h}_N(k), k = 0, 1, \ldots, m-1\), the parametric error is given by:

\[
J(H, N) = \|P_m(h - \hat{h}_N)\|_1 = \sum_{k=0}^{m-1} |h(k) - \hat{h}_N(k)|
\]

We would like to find conditions to guarantee \textit{robust consistency} as defined below.

**Definition 1.** Let \(S(m, \gamma)\) denote the following subset of \(\ell_1\):

\[
S(m, \gamma) = \{H \in \ell_1 \mid H = G + \Delta, \ G \in \mathcal{G}_{FIR}, \ \Delta \in \ell_1, \ ||\Delta||_1 \leq \gamma\} \tag{4}
\]

An algorithm \(\hat{h}_N\) is said to be robustly consistent if,

\[
E \left\{ \sup_{H \in S(m, \gamma)} J(H, N) \right\} \xrightarrow{N \to \infty} 0 \quad \forall \ \gamma > 0
\]

where the expectation is taken with respect to the noise distribution.

**Remark:** Note that the issue of measurability of \(\sup_{H \in S(m, \gamma)} J(H, N)\) could be a cause for concern. However, if the set, \(S(m, \gamma)\) is a separable space, measurability is preserved under a supremum [1].

**Remark:** We note that this definition of consistency is more stringent relative to the conventional notions, where the estimation error is typically defined as the worst case (over parameters, \(H\)) expected error, which prohibits the dependence of the parameters on the realization of noise. However, for linear examples with continuous random variables it turns out that both notions usually yield similar results [31]. Nevertheless, the main advantage of the new definition, as it turns out, is that it yields tractable conditions for robust consistency.

In the following theorem we derive necessary and sufficient conditions for robust consistency. We first need the following definitions to state the theorem:
Definition 2. An n-window autocorrelation function (ACF) of a real/complex valued sequence, \( \{x(k)\}_{k \in \mathbb{Z}^+} \), is defined as:

\[
r^N_x(\tau) = \frac{1}{n} \sum_{k=0}^{N-\tau} u(\tau + k)u^*(k), \quad 0 \leq \tau \leq N
\]

where the asterisk denotes the complex conjugate. The n-window cross-correlation function between two real/complex valued sequences, \( x(\cdot), y(\cdot) \) is given by:

\[
r^N_{xy}(\tau) = \frac{1}{N} \sum_{k=0}^{N-\tau} x(\tau + k)y^*(k), \quad 0 \leq \tau \leq N
\]

We are now ready to state the necessary and sufficient conditions for robust consistency. Some parts of the proof have been presented in our recent paper and we provide the complete proof here for the sake of exposition.

Theorem 1. A robustly consistent algorithm exists if and only if there is an indexed family of instrument variables \( q^N(\cdot) \) which uniformly de-correlates the input and noise. In other words, there exists a number \( N(\rho) \) for every \( \rho > 0 \) such that for all \( N > N(\rho) \):

\[
\begin{align*}
|r^N_u(\tau)| & \leq \rho |r^N_{q^N u}(0)|, \quad \tau = 1, 2, \ldots, N \quad (5) \\
E\{|r^N_{q^N u}(0)|\} & \xrightarrow{N \to \infty} 0 \quad (6)
\end{align*}
\]

Proof. \((\Rightarrow)\) The proof of necessity follows from three steps as outlined below:

**Step A** We observe that all norms are equivalent on finite dimensional spaces \([15]\). Consequently, robust consistency in the \( \ell_1 \) norm implies robust consistency in the \( \ell_\infty \) norm. In particular, \( \ell_1 \) norm can be approximated by means of \( \ell_\infty \) norm. Thus for any estimator \( \hat{h}^N \) we have,

\[
\|P_m(h - \hat{h}^N)\|_\infty \leq \|P_m(h - \hat{h}^N)\|_1 \downarrow \\
E\left\{ \sup_{H \in \mathcal{S}_{(m, \gamma)}} \|P_m(h - \hat{h}^N)\|_\infty \right\} \leq E\left\{ \sup_{H \in \mathcal{S}_{(m, \gamma)}} \|P_m(h - \hat{h}^N)\|_1 \right\} \downarrow \\
E\left\{ \sup_{H \in \mathcal{S}_{(m, \gamma)}} |h(0) - \hat{h}^N(0)| \right\} \leq E\left\{ \sup_{H \in \mathcal{S}_{(m, \gamma)}} \|P_m(h - \hat{h}^N)\|_1 \right\} \downarrow \\
E\left\{ \sup_{H \in \mathcal{S}_{(1, \gamma)}} |h(0) - \hat{h}^N(0)| \right\} \leq E\left\{ \sup_{H \in \mathcal{S}_{(1, \gamma)}} \|P_m(h - \hat{h}^N)\|_1 \right\}
\]

We are now left with estimating the error of a 1-dimensional FIR with unmodeled error as seen by the LHS of the last inequality.

**Step B** Our next observation is that convergence in expectation implies convergence in probability,

\[
E\left\{ \sup_{H \in \mathcal{S}_{(1, \gamma)}} |h(0) - \hat{h}^N(0)| \right\} \xrightarrow{N \to \infty} 0 \downarrow \\
Prob\left\{ \sup_{H \in \mathcal{S}_{(1, \gamma)}} |h(0) - \hat{h}^N(0)| \geq \epsilon \right\} \xrightarrow{N \to \infty} 0
\]
This implies that for every $\epsilon, \xi > 0$ there exists an integer $N(\epsilon, \xi) \in \mathbb{Z}^+$ such that for all $N \geq N(\epsilon, \xi)$ we have,

$$\text{Prob} \left\{ \sup_{H \in \mathcal{S}(1, \gamma)} |h(0) - \hat{h}^N(0)| \geq \epsilon \right\} \leq \xi$$

This is equivalent to existence of a set $\mathcal{A} \subset \mathcal{W}^N$ with $\text{Prob}[\mathcal{A}] \leq \xi$ such that,

$$\sup_{w \in \mathcal{W}^N - \mathcal{A}} \sup_{H \in \mathcal{S}(1, \gamma)} |h(0) - \hat{h}^N(0)| \leq \epsilon$$

(Step C) Next we establish that the set $\mathcal{W}^N - \mathcal{A}$ can be chosen to be convex without impacting the above inequality. We state this in the form of a proposition below and the details can be found in the appendix.

**Proposition 1.** If Equation 7 is satisfied then it follows that there exists a set, $\mathcal{A}_0$, with $\text{Prob}[\mathcal{A}_0] \leq \xi$ with the complementary set, $\mathcal{W}^N_0 = \mathcal{W}^N - \mathcal{A}_0$, being convex such that

$$\sup_{w \in \mathcal{W}^N_0} \sup_{H \in \mathcal{S}(1, \gamma)} |h(0) - \hat{h}^N(0)| \leq \epsilon$$

(Step D) The final step is based on an extension of Smolyak’s theorem [23, 24], which states that affine algorithms achieve smallest worst-case diameter of uncertainty for linear problems defined over convex but possibly unbalanced sets. Consequently, for every $\epsilon, \xi > 0$, it follows that there are a sequence of coefficients $\{q^N(k)\}_{k=1}^n$ not all zero such that,

$$\sup_{w \in \mathcal{W}^N} \sup_{H \in \mathcal{S}(1, \gamma)} \left| q^N(-1) + \sum_{k=0}^N q^N(k)w(k) - h(0) \right| \leq \epsilon, \quad \forall \ N \geq N(\epsilon, \xi)$$

Upon substitution of Equation 1 it follows that,

$$\left| q^N(-1) + \sum_{k=0}^N q^N(k)w(k) + \sum_{k=0}^N (q^N(k)u(k) - 1)h(0) + \sum_{j=1}^{N-1} \sum_{k=0}^N q^N(k + j)u(k)h(j) \right| \xrightarrow{N \to \infty} 0$$

Next, we observe that since, $\xi$, the probability of the set $\mathcal{A}_0$, can be chosen to be arbitrarily small, it follows that $0 \notin \mathcal{A}_0$. Indeed, $0 \notin \mathcal{A}_0$ together with convexity of the complementary set $\mathcal{W}^N_0$ implies that there must be a hyperplane passing through zero which separates $\mathcal{W}^N_0$ from zero. The size of such a set can at most be $1/2$ due to the symmetry of the uniform distribution. By setting $h(k) = 0$, $w(k) = 0$, as possible elements, it follows,

$$q^N(-1) \overset{N \to \infty}{\to} 0$$

Next, by noting that $\Delta$ is arbitrary with $\ell_1$ norm less than $\gamma$ and $w(\cdot)$ is an arbitrary element in the convex set $\mathcal{W}^N_0$, which also contains the zero element we have,

$$\sum_{k=0}^N q^N(k)u(k) = 1; \quad \max_{1 < j < N} \left| \sum_{k=0}^{N-j} q^N(k + j)u(k) \right| \leq \epsilon; \quad \left| \sum_{k=0}^N q^N(k)w(k) \right| \leq \epsilon, \; w \in \mathcal{W}^N_0$$

The proof now follows by observing that from the last inequality we have,

$$\text{Prob} \left\{ \left| \sum_{k=0}^N q^N(k)w(k) \right| \geq \epsilon \right\} \leq \xi$$

$$\implies \quad E \left\{ \sum_{k=0}^N q^N(k)w(k) \right\} \leq (1 - \xi) \text{Prob} \left\{ \left| \sum_{k=0}^N q^N(k)w(k) \right| \geq \epsilon \right\} + \|w\|_\infty \xi \xrightarrow{N \to \infty} 0$$
Corollary 1. To prove sufficiency we construct an IV technique that is robustly consistent. Basically we let, $q^N$ and its delayed copies up to $m$-delays serve as instruments. In particular, for length $N$ of data, we let, $Q_N$ be the matrix composed of the instrument $q^N$ and its $m$ delays, $U_N$ a corresponding matrix of input sequence with its $m$ delays and $R_N$ the residual delays of the input sequence.

$Q_N = \begin{bmatrix} q^N(0) & 0 & \ldots & 0 \\ q^N(1) & q^N(0) & \ddots & \vdots \\ \vdots & \vdots & \ddots & q^N(0) \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$; $U_N = \begin{bmatrix} u(0) & 0 & \ldots & 0 \\ u(1) & u(0) & \ddots & \vdots \\ \vdots & \vdots & \ddots & u(0) \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$; $R_N = \begin{bmatrix} 0 & 0 & \ldots \\ \vdots & \vdots & \vdots \\ u(0) & 0 & \ldots \\ \vdots & \vdots & \vdots \end{bmatrix}$ (11)

Suppose, $g$ is the column vector composed of the first $m$ impulse response coefficients, i.e., $g = P_m H$ and $\delta$ is the column vector of impulse response of the unmodeled dynamics, i.e., $\delta = (P_n - P_m)H$. Let $y$, $w$ be the output and noise signal vectors of length $N$ respectively. Then,

$$y = U_N g + R_N \delta + w \implies Q_N^T y = Q_N^T U_N g + Q_N^T R_N \delta + Q_N^T w$$ (12)

It follows from hypothesis that $Q_N^T U_N$ is an invertible matrix with bounded inverse for large enough $N$ since it converges to a lower triangular matrix with identical elements on the diagonal. Again from our hypothesis together with the fact that that $\|\Delta\| \leq \gamma$, it follows that, $Q_N^T R_N \delta$ converges to zero. Finally, $Q_N^T w$ also converges to zero in expectation by hypothesis (and noting that $(w(0), w(1), \ldots, w(n))$ are i.i.d. random variables.

We next consider several general remarks, which are a direct consequence of the result.

**Bounded but unknown noise:** It is well known that replacing stochastic noise with bounded but unknown noise in Equation 1 would lead to inconsistent estimates even in the case when the system is an FIR of known order. The proof of our theorem points to the fact that if we disallow a subset of vanishing lebesgue measure in Equation 1 would lead to inconsistent estimates even in the case when the system is an FIR of known order. The generalization to filtered (but bounded) i.i.d. noise can be established with minor modifications. For the unbounded stochastic noise case, the necessity falls short of the requirement of Equation 6. This can also be seen from Equation 10. Nevertheless, from Equation 10 it follows that for i.i.d. gaussian noise the necessary condition is given by convergence of $\sum q^N(k)w(k)$ to zero in probability as opposed to in expectation.

The necessary conditions derived in Theorem 1 assumes that the system belongs to a subset of $\ell_1$ as given by Equation 4. Nevertheless, these conditions hold more generally with $\ell_2$ and $H_\infty$ normed topologies. This follows from the fact that both these topologies are coarser than $\ell_1$ and hence a unit $\ell_1$ ball is contained in a unit $\ell_2$ and $H_\infty$ balls of real rational functions. We state this as a corollary next.

**Corollary 1.** The conditions given by Equations 5, 6 are necessary for robust consistency for estimating FIR models from input-output data given by Equation 1 with $H$ belonging to $S_2(m, \gamma)$ or $S_\infty(m, \gamma)$

$$S_2(m, \gamma) = \{H = G + \Delta, G \in \mathcal{G}_{FIR}, \Delta \in \mathbb{R}H_2, \|\Delta\|_2 \leq \gamma\}$$

$$S_\infty(m, \gamma) = \{H = G + \Delta, G \in \mathcal{G}_{FIR}, \Delta \in \mathbb{R}H_\infty, \|\Delta\|_{H_\infty} \leq \gamma\}$$
Proof. The proof follows from the observation that,

\[
E \left\{ \sup_{H \in \mathcal{S}_2(m,\gamma)} \| P_m(h - \hat{h}^N) \|_\infty \right\} \leq E \left\{ \sup_{H \in \mathcal{S}_2(m,\gamma)} \| P_m(h - \hat{h}^N) \|_2 \right\}
\]

\[
\downarrow
\]

\[
E \left\{ \sup_{H \in \mathcal{S}_2(1,\gamma)} | h(0) - \hat{h}^N(0) | \right\} \leq E \left\{ \sup_{H \in \mathcal{S}_2(1,\gamma)} | h(0) - \hat{h}^N(0) | \right\} \leq E \left\{ \sup_{H \in \mathcal{S}_2(m,\gamma)} \| P_m(h - \hat{h}^N) \|_2 \right\}
\]

Remark: Sufficient conditions are not mentioned as part of the corollary. Indeed for $\mathbb{R}H_\infty$ it turns out that robust consistency is not achievable [29].

Our next step is to extend these results to more general model spaces. The basic structure used in the proof (as seen from going from Equation 8 to Equation 9) is that the convex set $\mathcal{S}(m,\gamma)$ leads to a convex decomposition of its components consisting of unmodeled dynamics and the model subspace. In the $\ell_1$ setting this is true only for FIR model subspaces. One possible means to preserve this convexity is to consider Hilbert space topologies, where convex sets leads to convex decompositions resulting in the optimality of IV techniques for such spaces.

Theorem 2. Consider, the subspace, $\mathcal{G}$ and the set of uncertain systems, $\mathcal{S}(m,\gamma)$ given by:

\[
\mathcal{G} = \{ G = \sum_{k=1}^{m} \theta_k G_k, \ G_k \in \mathbb{R}H_2, \ \theta_k \in \mathbb{R} \}
\]

\[
\mathcal{S}(m,\gamma) = \{ H = G + \Delta, \ G \in \mathcal{G}, \ \| \Delta \|_1 \leq \gamma \} \subset \mathbb{R}H_2
\]

It follows that the conditions given by Equations 5, 6 is necessary and sufficient for robust-consistency. Furthermore, the instruments, $\{ \tilde{q}^N(\cdot) \}_{i=1}^{m}$ for the instrument variable technique that achieves robust consistency is given by:

\[
\tilde{q}^N_i(k) = (F_i \tilde{q}^N)(k)
\]

where, the linear filter, $F_i$, with impulse response, $f_i(\cdot)$, is aligned with the model $G_i$ and annihilates the unmodeled error $\Delta$:

\[
\langle G_i, F_i \rangle = \sum_{k=0}^{\infty} g_i(k) f_i(k) = 1; \quad \langle \Delta, F_i \rangle = \sum_{k=0}^{\infty} \delta(k) f_i(k) = 0
\]

Proof. We present the proof for $\mathcal{H}_2$ case for simplicity. The proof follows along the lines of Theorem 1 up until Step C. In application of Sukharev’s theorem [23], robust consistency would imply the existence of an instrument such that,

\[
\left| q^N(-1) + \sum_{k=0}^{N} q^N(k) w(k) + \sum_{j=0}^{N} \sum_{k=0}^{N-j} q^N(k+j) u(k) h(j) - \theta_1 \right| \xrightarrow{N \to \infty} 0, \ \forall \ \theta_1 \in \mathbb{R}
\]

Again following the arguments of Theorem 1, the fact that the first two terms must converge to zero can be established. We are now left with an instrument that must satisfy,

\[
\left| \sum_{j=0}^{N} \sum_{k=0}^{N-j} q^N(k+j) u(k) h(j) - \theta_1 \right| = \left| \sum_{j=0}^{N} \sum_{k=0}^{N-j} q^N(k+j) u(k) (\theta_1 g_1(k) + \delta(k)) - \theta_1 \right| \leq \epsilon
\]

where, $\Delta$ and $G_1$ are orthogonal by virtue of the decomposition of $H$. By arbitrarily varying $\Delta$ we see that,

\[
\sum_{j=0}^{N} r_q u(j) g_1(j) = 1; \sup_{\Delta} \left| \sum_{j=0}^{N} r_q u(j) \delta(j) \right| \xrightarrow{N \to \infty} 0
\]
We can decompose $r_{q^{N}u}$ (over finite time) into two components: one aligned with the finite sequence $\{g_1(k)\}_{k \leq N}$ and the other perpendicular to it.

$$r_{q^{N}u}(\cdot) = \frac{1}{\|P_N g\|_2} P_N g_1 + \frac{\epsilon}{\|(P_N g)^\perp\|_2^2} (P_N g)^\perp$$

(17)

This shows that the $\ell_2$ norm of the cross-correlation sequence is uniformly bounded and converges to the real rational transfer function $g_1 \in \mathbb{R} \mathbb{H}_2$. The denominator, $B_1(z^{-1})$ of $G_1(z^{-1}) = A_1(z^{-1})/B_1(z^{-1})$ is a stable polynomial that is causally invertible. Therefore, we let,

$$q^N(\cdot) \mapsto \tilde{q}^N(\cdot) = B_1(z^{-1}) q^N(\cdot)$$

which is an invertible transformation. Consequently, Equation 17 is equivalent to

$$r_{\tilde{q}^{N}u}(\cdot) \rightarrow a_1(\cdot)$$

where, $a_1(\cdot)$ are the impulse response coefficients of the numerator, which by definition are only finite many. We pick the last non-zero coefficient, say, $j$. It follows that, by choosing $\tilde{q}^N(k) = \tilde{q}(k + j)$, we obtain the desired instrument modulo a constant scaling. The orthogonal conditions follow by noting that $1/B_1(z^{-1})$ is aligned with $G_1$ and perpendicular to $\Delta$.

Remark: It can be shown that the conditions of Equation 5, 6 are also sufficient for achieving robust consistency for the general model space in the $\ell_1$ setting (see [27] for further details).

We have so far discussed the question of robust consistency but have not provided bounds for identification error for finite data. However, these are quite straightforward to establish since the instruments to be used for the IV techniques have already been characterized. These are provided in the following theorem.

**Theorem 3.** Let $G$ be an m-dimensional model subspace and $S(m, \gamma)$ characterize the set of uncertain systems given by Equations 13, 14 respectively. Let $q^N(\cdot)$ be a set of instruments characterized by Equations 5, 6. It follows that,

$$E \left\{ \sup_{H \in S(m, \gamma)} \| \theta^n - \theta \|_2^2 \right\} \leq C_0 \sum_{k=1}^{m} \left( \sum_{j=0}^{N} r_{q^{N}u}(j) (f_k(-\tau) * \delta(\tau))(j) \right)^2 + \sum_{k=1}^{m} E \left\{ \sum_{\tau=1}^{N} r_{q^{N}w}(\tau) f_k(\tau) \right\}^2$$

where, $*$ is the usual convolution operation.

**Remark:** It can be shown that the upper bound provided here is close to being optimal for Hilbert spaces. This follows from arguments similar to that provided in [24], where it is shown that linear algorithms are optimal when the set $S(m, \gamma)$ can be characterized as the inverse image of some closed unit ball on a Hilbert space.

In summary we have characterized conditions for asymptotic consistency and optimality (for Hilbert spaces) for systems with unstructured uncertainty. The IV technique provides a framework for extension of these results to structured uncertainty. Furthermore, the IV characterization is fundamentally a characterization of satisfactory inputs. We further develop this characterization in the following section.

### 2.1 Input Characterization

In this section we provide a simpler characterization in terms of persistently exciting signals for instruments and inputs that meet the conditions for robust consistency. Broadly, a persistent input is an $\ell_\infty$ sequence with unbounded energy, i.e.,
**Definition 3.** An $\ell_\infty$ bounded sequence, $(x(0), x(1), \ldots, x(k), \ldots)$, is said to be persistent if

$$\lim\inf_n \|P_n x\|_2 \longrightarrow \infty$$

Although, we did not assume a persistently exciting input in our derivation of Theorem 1, this is indirectly implied from the necessary conditions. The claim follows through contradiction: Suppose, the input is bounded in $\ell_2$ norm, and robust consistency is achievable. It follows from Equation 5 that the instrument will be of the form $q^n(\cdot) = u/\|u\|^2 + u^\perp$ for large enough $N$ (since $u$ decays to zero). This implies that $q^n$ does not asymptotically approach zero and Equation 6 cannot be satisfied—i.e. cannot completely average out the noise—violating the hypothesis. This motivates the following definition:

**Definition 4.** The ordered input-instrument pair of persistently exciting bounded signals, $(u, v)$, respectively is said to have UDC property if

$$\max_{0 < k \leq n} \frac{|r^n_{vu}(k)|}{|r^n_{vu}(0)|} \xrightarrow{n \to \infty} 0$$

It is easy to see that the Theorems 1, 2, 3 presented earlier can be re-derived by only requiring the UDC condition with little modification. The UDC property ensures that the instrument is uncorrelated from the past inputs. The definition captures the fact that since the unmodeled dynamics has infinite memory, the corresponding output is correlated with all the past inputs. Robust consistency requires that we find an instrument that is uncorrelated with all the past inputs, and yet is aligned with the current input. Note that because of the ordering the definition does not impose that current instrument is uncorrelated from future unmodeled outputs.

A related notion is that of a sequence that is uniformly de-correlating with respect to itself, such as white noise processes:

**Definition 5.** A persistently exciting bounded signal, $u$, is said to have uniformly decaying auto-correlation property UDAC if

$$\max_{0 < k \leq n} \frac{|r^n_{uu}(k)|}{|r^n_{uu}(0)|} \xrightarrow{n \to \infty} 0$$

With these definitions we are ready to state a lemma that provides a characterization of UDC preserving transformations.

**Lemma 1.** If $(u, v)$ form a UDC input-instrument pair, then so do $(z = Fu + w, v)$ for any BIBO stable $F$ with $f(0) \neq 0$ and $w$ uncorrelated w.r.t. $u$.

**Proof.** The cross correlation between, $v$ and $z$, satisfies (for $\tau > 0$):

$$|r^n_{vz}(\tau)| = \sum_{j=0}^{n-\tau} r^n_{vu}(\tau + j)f(j) \leq \|F\|_1 \max_{0 < \tau \leq n} |r^n_{vu}(\tau)|$$

while, for $\tau = 0$,

$$|r^n_{vz}(0)| = \sum_{j=0}^{n} r^n_{vu}(j)f_j \xrightarrow{n \to \infty} |f(0)||r^n_{vu}(0)|$$

where we have omitted the noise term as this follows by direct substitution. The proof now follows as $(u, v)$ form a UDC pair.

This lemma points to how new UDC pairs can be obtained from existing ones and will be used repeatedly in the following section to derive robust consistency results for structured uncertainties. In the next section we describe extensions of these results to uncertain systems with structured uncertainties and deal with input design in the subsequent section.
3 Robust Identification with Structured Uncertainties

In the previous section we obtained a characterization of input sequences that lead to robust consistency of the estimates. The procedure involved starting with a UDC pair and modifying the instrument by filtering it through the annihilating filter to obtain the appropriate instrument for general model spaces. In this section we deal with identification of systems with structured uncertainty. Structured uncertainty arises when there is additional information about the structure of uncertainty in addition to information that the underlying system does not belong to the chosen model class. The following section deals with input shaping in situations where frequency information about noise and unmodeled dynamics is available. In the following sections we deal with structured uncertainty in the form of linear fractional transformations [2].

3.1 Input Shaping

In this section we discuss input design for situations where we have prior knowledge on unmodeled dynamics and noise. A common case is to consider priors that are in the form of frequency weights, which provide relative levels of uncertainty as a function of frequency. The setup is schematically shown in Figure 2(a) and some examples of frequency weighting functions are shown in Figure 2(b). Mathematically, we can write:

\[ y(k) = Gu(k) + \Delta W_1 u(k) + W_2 v(k) \]

where, \( v(\cdot) \) is additive white gaussian noise, \( W_1 \) and \( W_2 \) are two LTI stable filters. Our task is to determine input sequences which not only guarantee robust consistency but also exploits prior information.

First note that the orthogonal decomposition described in Equation 16 is still valid. This is by virtue of the fact that all components of \( H \) aligned with \( G \) are still modelled as part of the model, \( G \). The only modification is the additional information that the unmodeled error belongs to the frequency band \( W_1 \) and the noise spectrum to the frequency band \( W_2 \). In the classical setting, the frequency weight \( W_2 \)—without loss of generality—is chosen to be minimum-phase stable filter [13]. This observation leads to the so called whitening of the observations/noise. In this scheme the input and the output is first pre-filtered by the whitening filter followed by least-squares estimation of the model parameters.
Our situation is slightly more complex. The weight $W_1$ characterizing the unmodeled error need not be minimum phase. Furthermore, pre-filtering together with least-squares can lead to significant bias even when both $W_1$ and $W_2$ are in the same frequency band. This follows from interpreting least-squares as an IV technique where both the instrument and input are identical. Now since unmodeled dynamics can have infinite memory and the pre-filtered input is in general colored, the instrument is correlated with unmodeled output for such situations and leads to significant bias. One possibility for optimization is provided by Lemma 1. We introduce a new optimization variable $L \in \ell_1$, which serves as input shaping function. The idea is that for every feasible input-instrument pair $(u, q)$ we have the flexibility of searching over all corresponding pairs $(Lu, q)$. Furthermore, if $W_1^\perp$ is an annihilator for $W_1$ it is possible to further enlarge our set of choices by considering all pairs of the form $(Lu, W_1^\perp q)$. Our task now reduces to choosing the optimal shaping function, $L$, to minimize the upper bound derived in Theorem 3.

The identification scheme proceeds as follows. We first choose a UDC pair, $(u, q)$. We pre-filter the input and output with the weight $W_2$ as in the classical setting:

$$W_2^{-1}y(k) = GW_2^{-1}u(k) + \Delta W_1 W_2^{-1}u(k) + v(k)$$

(18)

Next, we can employ an additional pre-filter $L_1$ for the input without violating the conditions for robust consistency. It follows that the input $\tilde{u}$ is given by:

$$\tilde{u} = Lu = W_2^{-1}L_1u, \quad \tilde{y} = \sum_{k=1}^{m} \theta_k G_k \tilde{u} + \Delta W_1 \tilde{u} + v$$

The instrument can also be modified by filtering it through $W_1^\perp$. Consequently, Equation 15 is modified to:

$$\hat{q}_k = W_1^\perp F_k q, \quad k = 1, 2, \ldots, m$$

We now form matrices, $\hat{Q}_N, \hat{U}_N, \hat{R}_N$, as in Equation 11 with the jth column of $\hat{Q}_N$ corresponding to the jth instrument sequence, $\hat{q}_j(\cdot)$; $\hat{U}_N$ is a block toeplitz matrix formed from the model outputs, i.e., $[G_1 \tilde{u}, G_2 \tilde{u}, \ldots, G_m \tilde{u}]$; $\hat{R}_N$ is a lower triangular toeplitz matrix formed with the sequence, $\tilde{u}$. With this setup, the estimate and the corresponding error is described by:

$$\theta^N - \theta = (\hat{Q}_N^T \hat{U}_N)^{-1} \left( \hat{Q}_N^T \hat{R}_N \delta + v_N \right)$$

where, $\delta, v_N$ are N-dimensional column vectors consisting of the noise sequence and the first N taps of the impulse response coefficients of unmodeled error. The error bound can now be obtained in terms of the size of the unmodeled error.

$$E \left\{ \sup_{\delta} \|\theta^N - \theta\| \right\} \leq \sup_{\delta} \left\| (\hat{Q}_N^T \hat{U}_N)^{-1} \hat{Q}_N^T \hat{R}_N \delta \right\| + E \left\{ \left\| (\hat{Q}_N^T \hat{U}_N)^{-1} v_N \right\| \right\} \leq \left\| (\hat{Q}_N^T \hat{U}_N)^{-1} \right\|_1 \left\| \hat{Q}_N^T \hat{R}_N \right\|_\infty \|\delta\|_1 + E \left\{ \left\| (\hat{Q}_N^T \hat{U}_N)^{-1} v_N \right\| \right\} \leq \left\| (\hat{Q}_N^T \hat{U}_N)^{-1} \right\|_1 \left\| \hat{Q}_N^T \hat{R}_N \right\|_\infty \|\delta\|_1 + C_0 \left\| (\hat{Q}_N^T \hat{U}_N)^{-1} \right\|^2_2$$

where the last inequality is obtained by observing that that term inside the expectation is a gaussian random vector and scales with its variance. Consequently, the problem reduces to minimization of the last term over all stable filters $L$. Although, $L$ enters $\hat{U}_N, \hat{R}_N$ linearly, the problem is non-convex and has to be solved numerically. However, the problem can be simplified for FIR model class when the extra annihilating filter $W_1^\perp$ is not used.
In this case it can be verified that the leading term $\| (\hat{Q}_N^T \hat{U}_N)^{-1} \|$ converges to $w_2(0)l(0)$ in this case. For each value of $l(0)$ we are left to solve the following optimization problem:

$$\inf_{L \in \ell_1} \| \hat{Q}_N^T \hat{R}_N \|_{\infty} \approx \inf_{L \in \ell_1} \| \hat{Q}_N^T R_N W_2 L \|_{\infty}$$

The latter problem is a convex optimization problem and can be reduced to solution of a linear program [3]. An example of the performance improvement of the squared norm error that can be realized is provided in the simulation in Figure 3. In the example, the problem is to choose an FIR of order 5. The unmodeled dynamics has an $\ell_1$ norm of unity. The noise spectra and unmodeled dynamics is confined to high frequency with $W_2 = 1.1 - 1.09z^{-1}$ and $W_1 = -0.9 + z^{-1}$ as shown in Figure 2(b). For the weighted LS approach we pre-filter the input and output by the stable filter $W_2^{-1}$ as in Equation 18 and apply least-squares with the pre-filtered data. Despite the fact that unmodeled dynamics is concentrated primarily at large frequencies and matches the noise spectrum, the simulations show that the LS algorithms result in a large bias. This follows from earlier arguments that the unmodeled dynamics is correlated with the input. The simulations also show the relatively large errors when no shaping filter is applied. The optimal shaping shows significant improvement in accuracy.

### 3.2 Uncertain LFT structures

In this section we point out several LFT structures for which robustly consistent algorithms can be derived. We consider first the multiplicative uncertainty as shown in Figure 4(a). The input-output equation is given by:

$$y(t) = G(1 + \Delta)u(t) + w(t) = Gu(t) + G\Delta u(t) + w(t); \ G \in \mathcal{G}$$

where, $\mathcal{G}$ is given by Equation 13. This is a situation similar to the case considered in Section 3.1, except that the prior weight $G$ is now unknown. Nevertheless, the orthogonal property (between unmodeled dynamics and model) still holds because as before $G$ models all the aspects that are accountable in the model class $\mathcal{G}$. It follows
that by starting with a UDC pair \( (u, q) \), modifying the instrument, \( q \mapsto \tilde{q} = Fq \), using the IV technique, and proceeding as in the previous section the parameter error is given by:

\[
\theta - \theta^N = (\tilde{Q}_N^T \tilde{U}_N)^{-1} \left( \tilde{Q}_N^T \tilde{R}_N \delta + v_N \right)
\]

The point is that \( \delta \) now depends on \( \theta \) and so the error bound also has a multiplicative structure, i.e.,

\[
\|\theta - \theta^N\| \leq C_0(N)\|\theta\| + C_1(N)
\]

where, \( C_0(N) \) and \( C_1(N) \) that functions of data length and approach zero. On account of the multiplicative nature we need a new definition of robust consistency. An algorithm is said to be relatively robustly consistent, if:

\[
E \left\{ \sup_{\delta} \|\theta - \theta^N\| \right\} \leq C(N)|\theta|; \ C(N) \xrightarrow{N \to \infty} 0
\]

This is to be expected since unbounded parameter enhances the unmodeled error.

Our next setup is provided in Figure 4(b). In this problem two uncertain systems, \( H_0, H_1 \) are connected in series. The system \( H_1 \) is approximately known, i.e., its best approximation \( G_1 \) (in the class \( G_1 \)) is known, and the system \( H_0 \) must be approximately identified in the model class, \( G_0 \) from data. The interpretation of the setup is that we are given a set of uncertain components connected in series, where it is not possible to isolate any single component for the purpose of calibration. The question arises as to when an approximate model for the uncertain component can be derived. We first write the input-output equation as follows:

\[
y(k) = (G_0 + \Delta_0)(G_1 + \Delta_1)u(k) + w(k)
\]

where, we have assumed \( W_0, W_1 \) and \( L \) to be equal to identity for simplicity. It is clear from the setup that the unmodeled dynamics enters the equations in a non-linear fashion. Nevertheless, fundamental conditions for robust consistency can still be established.

**Theorem 4.** For the above setup there exists a relatively robustly consistent algorithm for estimating \( G_0 \) if and only if the is a UDC pair \( (u, q) \) and a filter, \( F \) such that

\[
(F, G_0 G_1) = 1; \ (F, \Delta_1) = 0, \ \forall, \ k
\]

where \( \Delta_1 \) is the unmodeled error corresponding to the known uncertain system \( H_1 \).
Proof. (necessity) Upon expanding the Equation 19 we have,
\[ y(t) = G_0 G_1 u(t) + G_1 \Delta_0 u(t) + G_0 \Delta_1 u(t) + \Delta_0 \Delta_1 u(t) + w(t) \]

By picking \( \Delta_0 = 0 \) we get a lower bound for the identification error in Equation 19. Now from Theorem 2 it follows that identifiability requires an instrument that is aligned with \( G_0 G_1 \) and annihilates \( \Delta_1 \).

(sufficiency) Since \( G_0 \) is orthogonal to \( \Delta_0 \) by definition, we construct an instrument \( \tilde{q} = F_0 F q \). Together with the UDC pair \((u, q)\), it is seen by application of Theorem 2 that this instrument annihilates all three components of the unmodeled errors.

As an instantiation of the theorem consider the case when \( G_0 \) and \( G_1 \) are FIRs of order \( m \) and \( n \) respectively. Then no consistent algorithm exists when the order, \( m \), of \( G_0 \) is larger than \( n \). This is because the unmodeled error \( G_0 \Delta_1 \) and the class of FIR models of order \( m \) are no longer orthogonal.

3.3 Uncertain systems in feedback loop

Consider the feedback setup shown in Figure 5(a). The goal is to estimate the optimal approximation to \( H = G + \Delta \) in the model class \( G \in \mathcal{G} \) from input-output data. Our task is to design the input signal \( u \) to guarantee robust convergence to the optimal approximation.

Such problems arise in the context of design of adaptive acoustic echo-cancellation systems that operate in a feedback loop [30]. In general the high order acoustic dynamics coupled with significant time variation severely limits the choice of echo-canceller order. Moreover, the feedback filter, typically used to enhance the quality of speech, can impact the stability of the echo-cancellation system. In this light, a lower order echo-cancellation system is designed in the hope that robust performance can be ensured at the cost of sacrificing some performance. A dither input signal, \( u \), is employed as an input to the loudspeaker and the output, \( v \), of
the loudspeaker is recorded in addition to the microphone signals, \( y \). Thus the input output data consists of \((u, v, y)\) as illustrated in Figure 5(a).

We now focus on the solution to the identification problem. For simplicity of exposition we again consider a one dimensional parametrization, \( \mathcal{G} = \{ \theta(1 - az^{-1})^{-1}, \theta \in \mathbb{R} \} \). We first simplify the problem by decomposing the system in a \( \mathcal{H}_2 \) space into orthogonal subspaces \( \mathcal{G} = \begin{pmatrix} 1 \\ 1 - az^{-1} \end{pmatrix} \) and \( \Delta = (z^{-1} - .2)/(1 - az^{-1})\Delta_0 \) as follows:

\[
y(t) = Hv(t) + w(t) = \frac{\theta}{1 - az^{-1}}v(t) + \frac{z^{-1} - a}{1 - az^{-1}}\Delta_0 v(t) + w(t), \quad \theta \in \mathbb{R}
\]

where, \( w \) is AWGN. Next, by straightforward algebraic manipulations of the feedback loop we obtain that:

\[
v(t) = W_1 u(t) + W_2 w(t), \quad W_1 = (1 - HK)^{-1}, \quad W_2 = K W_1
\]

We point out that the system, \( W \), is generally unknown in this situation since it is a LFT of the controller with the unknown system. Nevertheless, we assume that the controller is stabilizing and consequently the system \( W_1, W_2 \) are stable transfer functions. The algorithm now proceeds as follows. We observe that the input \( u \) and the signal \( v \) are jointly UDAC as they satisfy the conditions of Lemma 1. Theorem 2 provides the characterization of the appropriate instrument required for robust consistency and Theorem 3 provides the corresponding parametric error bound over finite data respectively.

We verify these results on an example and provide comparisons with the least squares technique. The model space is given by \( \mathcal{G} = \frac{\theta}{1 - az^{-1}} \). The system \( H \) and the controller \( K \) is chosen so that the optimal \( \theta = 1 \) and the worst-case closed loop sensitivity function, \( W \) has \( \ell_1 \) norm less than one. The IV technique developed above is applied. Figure 5(b) shows the parametric error plotted as a function of data length. The plot also shows the performance of the least squares approach (with \( u \) as the input and \( y \) as the output) in the absence of noise but with a persistently exciting input, \( r \). It is seen that the least squares approach leads to large bias. This is not surprising considering the fact that the unmodeled output is correlated with the model output.
4 Design of UDAC Input Sequences

In Section 2 we dealt with the problem of characterizing the conditions required to achieve robust consistency. The following section dealt with input design for structured uncertain systems. The results point to the fact that a fundamental property required for achieving robust consistency is the UDC property, which is the topic of this section. It is not difficult to see that the UDC property implies that infinite informativity—the ability to estimate models of arbitrary order when the output is corrupted by additive stochastic noise. This is stated in the following theorem:

**Theorem 5.** If an input-instrument pair \((u, q)\) is UDC then the input \(u\) is infinitely informative, i.e., it is effectively persistently exciting of arbitrary order.

**Proof.** From the proof of sufficiency of Theorem 1 we notice that \(Q_N U_N\) converges to a lower triangular matrix with ones on the main diagonal irrespective of the number of parameters \(m\). Hence, it has a bounded inverse. The proof now follows by definition.

We are now left with the problem of constructing UDC sequences. In the following theorem we show that any random process with bounded strictly positive power spectral density suffices.

**Theorem 6.** A wide-sense-stationary input sequence is part of a UDC pair if it has a bounded strictly positive spectrum.

**Proof.** The proof follows from the spectral factorization theorem and is omitted.

The theorem relies on the fact that a stationary input sequence with bounded positive real spectrum can be whitened by means of a causal-causally invertible stable transformation. The theorem points to the fact that underneath UDC sequences are essentially UDAC sequences, i.e., persistent sequences whose out-of-phase autocorrelation function decays uniformly to zero. Lemma 1 points to the fact that new UDC sequences are constructed through linear transformations. For this reason we next focus on the construction of deterministic UDAC sequences. To motivate this topic we point out several commonly used sequences that do not satisfy the UDAC property. First, UDAC sequences must have bounded amplitude and be persistent which rules out any sequence with finite energy such as impulses. UDAC sequences must have decaying autocorrelations, which rules out periodic sequences. Consequently, the widely used PRBS being periodic lacks UDAC property. It can be argued that for all practical purposes one could possibly take truncations of PRBS sequences with arbitrary long period. However, it turns out that truncations of PRBS sequences do not necessarily have small autocorrelations. This can also be seen in Figure 6 where the worst-case auto-correlations of truncated PRBS of period \(2^{15} - 1\) has large auto-correlations. Finally, it is perhaps surprising that the widely used sine-sweep, \(u(t) = \exp(i\alpha t^2), \alpha \in \mathbb{R}, t = 0, 1, 2, \ldots\), is not UDAC either. In [27], we show that,

\[
\limsup_{0 < \tau \leq n} \max_{0 \leq k \leq n} |r_u^n(\tau)| \geq 1/2
\]

The basic reason is that the auto-correlation function turns out to be equal to \(\sin(n\tau\alpha/2)/(n\sin(\tau\alpha))\) and subsequences \(\tau_j, n_j\) can be suitably chosen so that the limiting value of \(\tau_j\alpha \mod (2\pi)\) tends to zero as seen in Figure 6. These examples serve to illustrate the difficulty in constructing deterministic UDAC sequences.

On the other hand our optimism stems from considering bounded i.i.d. random processes. By means of straightforward arguments based on chernoff bounds [27] it is possible to show that:

\[
P\left\{ \max_{0 \leq \tau \leq n} |r_u^n(\tau)| \geq \epsilon \right\} \approx \exp(-n\epsilon^2)
\]
Figure 6: Comparison of Worst-case auto-correlation functions for various lengths of a $2^{15} - 1$ order PRBS, sine-sweeps and FSS.

The significance of the expression is threefold: 1) autocorrelation reduces as $O(n^{-1/2})$; 2) a substantial number of sample paths have small autocorrelation; 3) this level of reduction in auto-correlation is optimal, i.e., the set of sample paths with a larger decay of auto-correlation has measure zero. This latter fact can be established by means of large-deviations theory (see [1]). These facts lead us to the following points: a) we should be able to find a large number of robust sequences; b) the rate of decay cannot be expected to be higher than $O(1/\sqrt{n})$. For these reasons we wish to design sequences whose worst-case out-of-phase auto-correlation decays at a polynomial rate as defined below.

**Definition 6.** A persistent bounded sequence, $\{u\}$, is said to have a uniformly polynomial decay (PDACF) property if there is a $\gamma > 0$ such that:

$$\max_{0 < k \leq n} \left| \frac{r_n^u(k)}{r_n^u(0)} \right| \leq Cn^{-\gamma}, \ \forall \ n \in \mathbb{Z}^+$$

It turns out that fast-sine-sweeps (FSS) sequences that match the decay rate of the random i.i.d. inputs. In this section we will show that FSS sequences have the PDACF property, in that, there out-of-phase ACF decays to zero at a polynomial rate, which matches the rate obtained with i.i.d. sequences. The FSS sequences are a family of complex valued sequences that are persistent and bounded. They are parameterized by irrational numbers, $\alpha$, as in the following expression.

$$u(t) = \exp(i2\pi\alpha t^3), \ \alpha \in \mathbb{R} - \mathbb{Q}, \ t = 0, 1, 2, \ldots$$

In applications the real part of the complex sequence will actually be applied and to simplify our exposition we consider the complex valued signal here.

**Theorem 7.** The FSS sequence has the PDACF property for all algebraic numbers.

**Remark:** Recall that algebraic numbers are those numbers that are roots of any polynomial defined over a field of integers.
Remark: The PDACF property holds for almost all real numbers except on a set of Lebesgue measure zero and will be reported elsewhere.

The proof is broken down into several steps. In the first step we will upper bound the ACF function by means of simpler functions. Consider, the function, \( \phi(x) \) that maps any real number to the interval \((-1/2, 1/2] \) corresponding to the distance from the closest integer, i.e.,

\[ \phi(x) = x - [x] \]  

(20)

where, \([x]\) is the integer closest to \(x\) and for all numbers, \(x\) such that \(x = 0.5 \mod (1)\), we define \(\phi(x) = 0.5\).

The following elementary facts follows directly from the definition of \(\phi\).

**Proposition 2.** \(\phi(\cdot)\) satisfies the following properties

\[ \begin{align*}
\phi(-x) &= -\phi(x) \\
\phi(kx) &= \phi(k\phi(x)) \\
|\phi(kx)| &= |\phi(k|\phi(x)|)| \\
\sin(2\pi x) &\in \left[ \frac{\phi(x)}{2}, \phi(x) \right]
\end{align*} \]

Proof. The proof is provided in the appendix.

With these preliminaries we can reduce the expression for ACF in terms of the function \(\phi\).

**Lemma 2.** The ACF function, \(|r_n^\alpha(\tau)|\), for the FSS satisfies the following inequality for any irrational number \(\alpha \in \mathbb{R}\).

\[ |r_n^\alpha(\tau)|^2 \leq \sum_{k=0}^n \frac{1}{1 + n|\phi(k\beta)|}, \beta = |\phi(\tau\alpha)| < 1/2 \]

Proof. First, the ACF function can be simplified by straightforward algebraic manipulations as follows:

\[ |r_n^\alpha(\tau)|^2 \leq \frac{1}{n^2} \sum_{k=1}^{n-\tau} \left| \frac{\sin(2\pi ak\tau(n - \tau))}{\sin(2\pi ak\tau)} \right| + C = \frac{C}{n} + \frac{1}{n^2} \sum_{k=1}^{L} \left| \frac{\sin(2\pi Lk\tau)}{\sin(2\pi k\tau)} \right|, \quad L = n - \tau, \quad x = \tau\alpha \]  

(21)

To simplify this expression we introduce the function, \(H_L(\cdot)\):

\[ H_L(x) = \frac{3L}{1 + L|\phi(x)|} \]

We next have the following proposition:

**Proposition 3.** The function \(H_L(x)\) is a monotonic function over \(L\), i.e., \(H_j(x) \leq H_k(x)\) for \(j < k\). Furthermore, it satisfies:

\[ \left| \frac{\sin(2\pi Lx)}{\sin(2\pi x)} \right| \leq H_L(x) \]

Proof. The proof is provided in the appendix.

Applying proposition 3 in Equation 21 leads to the following inequality:

\[ |r_n^\alpha(\tau)|^2 \leq \frac{1}{n^2} \sum_{k=1}^{n-\tau} H_{n-\tau}(k\tau) \leq \frac{1}{n^2} \sum_{k=1}^{n} H_n(k\tau) = \frac{3}{n} \sum_{k=1}^{n} \frac{1}{1 + n|\phi(k\tau\alpha)|} \]

Finally, for the last step we apply proposition 2 to obtain the result:

\[ \frac{3}{n} \sum_{k=1}^{n} \frac{1}{1 + n|\phi(k\tau\alpha)|} = \frac{3}{n} \sum_{k=1}^{n} \frac{1}{1 + n|\phi(k|\phi(\tau\alpha)|)|} = \frac{3}{n} \sum_{k=1}^{n} \frac{1}{1 + n|\phi(k\beta)|}, \beta = |\phi(\tau\alpha)| < 1/2 \]  

20
The main difficulty now in establishing the result is that \( \lim \inf_q \phi(q\alpha) = 0 \) for every real number, i.e., \( \phi(j\alpha) \) comes close to zero infinitely often. The proof therefore rests on the fact that very few terms in the sequence \( \phi(\tau\alpha), \phi(2\tau\alpha), \ldots, \phi(n\tau\alpha) \) are close to zero for any phase \( 0 < \tau \leq n \). A well known result in continued fraction expansion theory \[11\] states that rational approximations cannot approach an algebraic number faster than the size of the denominator, i.e.,

\[
\frac{C_\alpha}{j} \leq \phi(j\alpha), \; j \in \mathbb{Z}^+
\]

Unfortunately, this inequality alone is insufficient for deriving the upper bounds for worst-case ACF, as shown below:

\[
\max_{0 < \tau \leq n} \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{1 + n\phi(k(\tau\alpha))} \leq \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + \frac{nB(\alpha)}{k\max}} \not \to 0
\]

We summarize properties from elementary continued fraction theory for the purpose of completion next. These properties and the accompanying notation have been adopted from \[22\].

\[\text{(A) Basics:}\]

The continuous fraction expansion of any positive irrational number \( \alpha \) is given by:

\[
\alpha = \left[ a_0; a_1, a_2, \ldots, a_j, \ldots \right] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}, \; a_j \in \mathbb{Z}^+
\]

where for the sake of brevity the second expression is typically used to denote the expansion. By truncating the continuous fraction expansion we obtain a sequence of rational numbers called convergents, i.e.,

\[
\frac{A_k}{B_k} = [a_0; a_1, a_2, \ldots, a_k]
\]

The numerator and denominator of the convergents satisfy the following recursion:

\[
A_{k+1} = a_{k+1}A_k + A_{k-1}, \; A_{-1} = 1; \; A_0 = a_0, \; k \geq 0 \tag{22}
\]

\[
B_{k+1} = a_{k+1}B_k + B_{k-1}, \; B_{-1} = 0; \; B_0 = 1, \; k \geq 0 \tag{23}
\]

Since, \( a_k \)s are positive integers the numerator and denominator sequences \( B_k, A_k \)s form a strictly monotonically increasing sequence. It follows that:

\[
B_k = a_kB_{k-1} + B_{k-2} = (a_ka_{k-1} + 1)B_{k-2} + B_{k-3} \geq 2B_{k-2} \geq 2^{(k-1)/2} \tag{24}
\]

where in the penultimate equation we have used the fact that \( a_k \geq 1 \).

\[\text{(B) Optimal Approximation:}\]

The convergents are optimal approximants, in that, the \( k \)th convergent is the best approximation to the irrational number among all rationals which have a smaller denominator:

\[
|\phi(B_k\alpha)| \leq |\phi(j\alpha)|, \; \forall \; 0 < j \leq B_k, \; k \geq 1 \tag{25}
\]

For simplicity the value \( \phi(B_k\beta) \) is denoted by the symbol \( D_k \) and it turns out that there is a simple expression relating \( D_k \) to \( A_k, B_k \)s:

\[
D_k = \phi(B_k\alpha) = B_k\beta - A_k
\]

Moreover, the \( D_k \)s follows a simple a recursion identical to Equations 22, i.e.,

\[
D_k = a_kD_{k-1} + D_{k-2}, \; D_{-1} = -1 \tag{26}
\]
Figure 7: Illustration of how even and odd convergents approach an irrational number

It follows that the so called even convergents, \( \frac{A_{2k}}{B_{2k}} \), monotonically approach \( \beta \) from the right while the odd convergents, \( \frac{A_{2k+1}}{B_{2k+1}} \), approach \( \beta \) from the left as shown in Figure 7. It can also be shown that the value of \( D_k \) is bounded from above and below by the size of the denominators given by:

\[
\frac{1}{B_k + B_{k+1}} \leq |D_k| = \phi(B_k \beta) \leq \frac{1}{B_k}
\]  

(27)

Algebraic numbers have the special property that the \( a_k \)s are periodic. Consequently, from Equation 22 it follows that the ratio between any two subsequent convergent denominators, \( B_k, B_{k+1} \) is bounded. Consequently, the smallest rational approximation error for any algebraic number can be given by the denominator of the convergent.

(C) **Decomposition:**

In the so called Ostrowski’s representation, each integer, \( m \), is expressed as a linear combination of the \( B_k \)s. It turns out that this representation is unique in the following sense. Any integer \( m \in \mathbb{Z}^+ \) with \( B_p \leq m \leq B_{p+1} \) can be uniquely decomposed as:

\[
m = \sum_{j=0}^{p} c_{j+1} B_j
\]

(28)

with \( 0 \leq c_{j+1} \leq a_j+1 \) for \( j \geq 1 \) and \( 0 \leq c_1 < a_1 \). Moreover, \( c_j = 0 \) if \( c_{j+1} = a_{k+1} \). This decomposition is obtained by first dividing \( m \) with the largest denominator, \( B_p \), smaller than \( m \) and then decomposing the remainder with a corresponding largest denominator and so on.

(D) **Lower Bounds** The significance of ostrowski representation follows from upper and lower bounds. To do this we need to define the class of an integer \( m \). Let, \( m = \sum_{j=0}^{p} c_{j+1} B_j \), be the ostrowski representation for \( m \). The class \( \gamma(m) \) of an integer \( m \) is an integer defined as:

\[
\gamma(m) = \min \{ j \mid c_{j+1} \neq 0 \}
\]

The type of the integer, \( q \), signifies how small \( \phi(q\beta) \) can get. A well-known result in continued fraction theory states that for any irrational number \( \beta < 1/2 \) with \( \gamma(q) \geq 1 \), the value of \( \phi(q\beta) \) is given by the linear combination of \( \phi(B_k\beta) \)s, weighted by the ostrowski coefficients, i.e.,

\[
q = \sum_{j=\gamma(q)}^{p} c_{j+1} B_j \implies \phi(q\beta) = \sum_{j=\gamma(q)}^{p} c_{j+1} D_j
\]

(29)

\[
|\phi(m\beta)| \geq \langle c_{\gamma(m)+1} - 1 \rangle D_{\gamma(m)} + D_{\gamma(m)+1}
\]

(30)

We will need a stronger result for our proof, which we will derive in the course of the paper. But for now notice that the type, \( \gamma(m) \), of a number, \( m \), effectively controls the value of \( \phi(m\alpha) \).
We are now ready to prove the main theorem by a combination of the above well-known properties in continued fraction theory. The main outline of the proof is as follows. Equation 30 points to the fact that the type of a number controls the value of \( \phi(m \alpha) \). This motivates partitioning of the set \( S_n = \{1, 2, \ldots, j, \ldots, n\} \) based on its type and computing the contributions for each type. This leads us to defining the following sets:

\[
A_{l,c} = \{m \in S_n \mid \gamma(m) = l, \ c_{\gamma(m)+1} = c, \ \text{in the ostrowski expansion for } m\} \quad (31)
\]

\[
A_{l,c,d} = \{m \in S_n \mid \gamma(m) = l, \ c_{\gamma(m)+1} = c, \ c_{\gamma(m)+2} = d, \ \text{in the ostrowski expansion for } m\} \quad (32)
\]

Next, we denote \( \gamma^*_n \) to be the maximum possible type in \( S_n \), i.e.,

\[
\gamma^*_n = \max\{\gamma(m) \mid m \in S_n\}
\]

The following proposition provides bounds for the maximum possible type in \( S_n \)

**Proposition 4.** The maximum type, \( \gamma^*_n \), in the set \( S_n \) is smaller than \( 2 \log_2(2n) \).

**Proof.** If \( m \in S_n \) is of class \( \gamma(m) \), then, from the ostrowski decomposition it follows that

\[
m \geq B_{\gamma(m)} \geq 2^{\gamma(m)+1} \implies 2 \log_2 m + 1 \geq \gamma(m)
\]

where, we have used Equation 24 for the second inequality. Thus, the largest possible type in \( S_n \) is \( \lambda_n = 2 \log_2 n \).

Since, type is a positive integer ranging from 0 to \( 2 \log_2 n \), the result follows by inspection.

We are now left to compute the cardinality of each type. We show in the sequel that the cardinality decreases exponentially with the type.

**Lemma 3.** The cardinality of \( A_{k,c} \) in the set \( S_n \) is given by:

\[
\#A_{l,c} \leq \begin{cases} 
2 \left\lfloor \frac{n}{b_{l+1}} \right\rfloor & l < \gamma^*_n \\
1 & l = \gamma^*_n 
\end{cases}
\]

and the cardinality of the set, \( A_{l,c_1,c_2} \) is bounded by the cardinality of the set, \( A_{l+1,c_2} \).

**Proof.** The proof requires the following proposition.

**Proposition 5.** Given, two positive integers, \( q, r \) and their corresponding ostrowski representations, \( \{c_j(q)\}, \{c_j(r)\} \), it follows that:

\[
q < r \iff \exists l \in \mathbb{Z}^+ \text{ such that } c_l(q) < c_l(r) \text{ & } c_j(q) \leq c_j(r), \forall j \geq l
\]

**Remark:** The implication of the proposition is that ostrowski representation behaves as a binary/octal expansions except for the conditions of Property (C).

We now provide the proof of Lemma 3. Any integer \( q \in A_{k,c} \) has an ostrowski decomposition:

\[
q = \sum_{j=k}^{\gamma^*_n} c_{j+1}(q)B_j \equiv (0, \ldots, 0, c_{k+1}(q), c_{k+2}(q), \ldots, c_{\lambda_n}(q)), \ c_{k+1} = c
\]

where \( c_{k+1} \) is for \( l \geq 2 \) are arbitrary integers constrained only by property (C). Consequently it follows from Proposition 5, for any other integer, \( p \in A_{k,c}, \ p > q \) that,

\[
\exists l \geq k + 2 \text{ such that } c_l(p) > c_l(q), \ c_j(p) > c_j(q), \forall j \geq l
\]
This implies that,
\[
p - q = \sum_{j=k+1}^{\gamma_n^*} (c_j(p) - c_j(q))B_{j-1} = \sum_{j=k+1}^{\gamma_n^*} (c_j(p) - c_j(q))B_{j-1} \geq B_{l-1} - \sum_{j=k+1}^{l-1} c_j(q)B_{j-1} \quad (33)
\]
\[
\geq a_{l-1}B_{l-2} - \sum_{j=k+1}^{l-1} c_j(q)B_{j-1} \geq B_{l-2} - \sum_{j=k+1}^{l-2} c_j(q)B_{j-1}
\]
\[
\geq \ldots \geq B_{k+2} - c_{k+2}B_{k+1} \geq (a_{k+2} - c_{k+2})B_{k+1} \geq B_{k+1}
\]
This means that there can only be one term belonging to \( A_{k,c} \) for any sequential set of \( B_{k+1} \) integers. Now \( n \) can be written as:
\[
n = \left\lfloor \frac{n}{B_{k+1}} \right\rfloor B_{k+1} + r, \quad 0 \leq r < B_{k+1}
\]
Therefore, the remainder, \( r \), terms can contain at most one term. This implies,
\[
\#A_{k,c} \leq \left\lfloor \frac{n}{B_{k+1}} \right\rfloor + 1
\]
Now, since \( \gamma_n^* \) is largest possible type in \( S_n \) we see that \( B_{\gamma_n^*+1} \) must be larger than \( n \). This implies that,
\[
\#A_{\gamma_n^*,c} \leq 1
\]
For all other types, \( 0 \leq \gamma(m) \leq \gamma_n^* - 1 \) we have \( B_{\gamma(m)} \leq n \), which implies,
\[
\#A_{k,c} \leq 2 \left\lfloor \frac{n}{B_{k+1}} \right\rfloor + \left\lfloor \frac{n}{B_{k+1}} \right\rfloor + 1 \leq 2 \left\lfloor \frac{n}{B_{k+1}} \right\rfloor \quad 0 \leq \gamma(m) \leq \gamma_n^* - 1
\]
Finally, in order to compute the bounds for \( \#A_{k,c,d} \) we observe that the first two Ostrowski expansion coefficients for any two integers \( p, q \in A_{k,c,d} \) are identical. This implies,
\[
p - q = \sum_{j=k+1}^{\gamma_n^*} (c_j(p) - c_j(q))B_{j-1} = \sum_{j=k+3}^{\gamma_n^*} (c_j(p) - c_j(q))B_{j-1}
\]
and the rest of the proof follows as in Equation 33.

Finally, we now provide lower bounds for \( \phi(m) \) corresponding to each type in the following lemma.

**Lemma 4.**

<table>
<thead>
<tr>
<th>( \phi(m) )</th>
<th>( \gamma(m) )</th>
</tr>
</thead>
</table>
| \( \phi(m) = 0 \) | \( \gamma(m) = 0 \)
| \( c_{\gamma(m)+1}D_{\gamma(m)} + c_{\gamma(m)+2}D_{\gamma(m)+1} + c_{\gamma(m)+3}D_{\gamma(m)+2} \quad \gamma(m) = 0 \) | \( \gamma(m) = \gamma_n^* \)
| \( \frac{c_{\gamma(m)+1}}{2B_{\gamma(m)+1}} - \frac{c_{\gamma(m)+2}}{2B_{\gamma(m)+2}} \quad \gamma(m) = \gamma_n^* \) | \( \gamma(m) = \gamma_n^* \)
| \( \frac{c_{\gamma(m)+1}}{2B_{\gamma(m)+2}} + \frac{c_{\gamma(m)+2}}{2B_{\gamma(m)+3}} \quad \gamma(m) = \gamma_n^* \) | \( \gamma(m) = \gamma_n^* \)
| \( c_{\gamma(m)+1}\phi(B_{\gamma_n^*+1}) \quad \gamma(m) = \gamma_n^* \) | \( \gamma(m) = \gamma_n^* \)
| \( c_{\gamma(m)+1}D_{\gamma(m)} + c_{\gamma(m)+2}D_{\gamma(m)+1} + c_{\gamma(m)+3}D_{\gamma(m)+2} \quad \gamma(m) = \gamma_n^* \) |
| \( \gamma(m) = \gamma_n^* \) |

**Proof.** Suppose, \( \gamma(m) \geq 1 \) and assume with out loss of generality that \( \phi(m) > 0 \), then applying Equation 29 we proceed as follows,

\[
\phi(m) \geq c_{\gamma(m)+1}D_{\gamma(m)} + c_{\gamma(m)+2}D_{\gamma(m)+1} + c_{\gamma(m)+3}D_{\gamma(m)+2}
\]
\[
+ \ldots + c_{\gamma(m)+2}D_{\gamma(m)+2} + c_{\gamma(m)+2}D_{\gamma(m)+3} + c_{\gamma(m)+2}D_{\gamma(m)+4} + \ldots
\]
\[
\geq c_{\gamma(m)+1}D_{\gamma(m)} + c_{\gamma(m)+2}D_{\gamma(m)+1} + a_{\gamma(m)+1}D_{\gamma(m)+2} + a_{\gamma(m)+2}D_{\gamma(m)+3} + \ldots
\]
\[
\geq c_{\gamma(m)+1}D_{\gamma(m)} + a_{\gamma(m)+2}D_{\gamma(m)+1} + (c_{\gamma(m)+2} - a_{\gamma(m)+2})D_{\gamma(m)+2} + a_{\gamma(m)+3}D_{\gamma(m)+3}
\]
\[
+ a_{\gamma(m)+4}D_{\gamma(m)+4} + \ldots
\]

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The second inequality follows from the fact that we have removed all the terms with the same sign as $D_{\gamma(m)}$ (i.e. terms of the form $D_{\gamma(m)+2j}$) while taking the maximum possible values for terms of opposite sign. The third inequality is just a restatement of the third term. Next we utilize Equation 26 to make the substitution that $D_k - D_{k-2} = a_k D_{k-1}$ and get,

$$\phi(ma) \geq c_{\gamma(m)+1}D_{\gamma(m)} + (c_{\gamma(m)+2} - a_{\gamma(m)+2})D_{\gamma(m)+1} + a_{\gamma(m)+4}D_{\gamma(m)+3} + a_{\gamma(m)+6}D_{\gamma(m)+5} + \ldots$$

$$\geq (c_{\gamma(m)+1} - 1)D_{\gamma(m)} + (c_{\gamma(m)+2} - a_{\gamma(m)+2})D_{\gamma(m)+1}$$

Finally, we know from Equation 27 that

$$D_{\gamma(m)} \geq \frac{1}{B_{\gamma(m)} + B_{\gamma(m)+1}} \geq \frac{1}{2B_{\gamma(m)+1}}$$

Next observing that $D_{\gamma(m)}$ and $D_{\gamma(m)+1}$ are of opposite signs the result follows. Now for $\gamma(m) = \gamma_n^*$ we can further refine the bound. This follows from Lemma 3 from which we know that the set $\mathcal{A}_{\gamma_n^*,c}$ is either empty or a singleton for each $c$. This implies that if $\mathcal{A}_{\gamma_n^*,c}$ is non-empty then the element, $m = cB_{\gamma_n^*}$ for some admissible $c$. Now, applying Equation 29 we see that

$$\phi(m\beta) = \phi(cB_{\gamma_n^*}\beta) = cD_{\gamma_n^*}$$

For, $\gamma(m) = 0$, the main problem with Equation 29 is that RHS expression can be larger than 1/2 in this case. This is related to the fact that $D_{-1} = -1$ (we refer to [22] for further details. Therefore, for $\gamma(m) = 0$ Equation 29 will no longer hold. To get around this problem we proceed as follows:

$$\phi(m\beta) = \phi(\sum_{j=1}^{\gamma_n^*} c_j + 1 B_j \beta) = \phi(\sum_{j=1}^{\gamma_n^*} c_j + 1 D_j)$$

Now, we will obtain upper and lower bounds for $\sum_{j=1}^{\gamma_n^*} c_j + 1 D_j$. We note that $0 < \beta < 1/2$ we have $D_0 > 0$. Now we proceed as for the case of $\gamma(m) \geq 1$ and get

$$\sum_{j=1}^{\gamma_n^*} c_j + 1 D_j \geq c_1 \frac{1}{2B_1} + \frac{a_2 - c_2}{2B_2} \geq 0$$

An upper bound also can be obtained with similar arguments (exclude all terms of opposite sign as $D_0$ and include the maximum possible terms of the same sign) to get:

$$\sum_{j=1}^{\gamma_n^*} c_j + 1 D_j \leq c_1 D_0 + c_2 D_1 + a_3 D_2 + a_5 D_4 + \ldots \leq c_1 D_0 + (c_2 - 1)D_1$$

Next, substituting Equation 26 for $D_1 = a_1 D_0 - 1$ we get:

$$\sum_{j=1}^{\gamma_n^*} c_j + 1 D_j \leq 1 + (c_1 - a_1) D_0 + c_2 D_1$$

Now the latter term is strictly less than 1 since,

$$1 + (c_1 - a_1) D_0 + c_2 D_1 \leq 1 + ((a_1 - 1) - a_1) D_0 \leq 1 - D_0 \leq 1$$
Thus we have a upper and lower bound for $\sum_{j=1}^{n} c_{j+1}D_j$ that are both positive and smaller than 1, i.e.,

$$0 < \frac{c_1 - 1}{2B_1} + \frac{a_2 - c_2}{2B_2} \leq \sum_{j=1}^{n} c_{j+1}D_j \leq 1 + (c_1 - a_1)D_0 + c_2D_1 < 1$$

This implies that the value $\phi(m\beta)$ must satisfy:

$$\phi(m\beta) \geq \min \left( \frac{c_1 - 1}{2B_1} + \frac{a_2 - c_2}{2B_2}, 1 - (1 + (c_1 - a_1)D_0 + c_2D_1) \right)$$

Now substituting Equation 27 we obtain the desired result.

The proof of Theorem 1 now follows by combining lemmas 3, 4.

$$\frac{1}{n} \sum_{k=0}^{n} \frac{1}{1 + n\phi(ka)} = \frac{1}{n} \sum_{k=0}^{\gamma_1} \sum_{m: \gamma(m)=k} \frac{1}{1 + n\phi(m\alpha)}$$

$$= \frac{1}{n} \sum_{m \in \bigcup_{c=1}^{n+1} A_{k,c}} \frac{1}{1 + n\phi(m\alpha)} + \frac{1}{n} \sum_{k=1}^{\gamma_1} \sum_{m \in \bigcup_{c=1}^{n+1} A_{k,c}} \frac{1}{1 + n\phi(m\alpha)}$$

$$+ \frac{1}{n} \sum_{k=1}^{\gamma_1} \sum_{m \in \bigcup_{c=1}^{n+1} A_{k,c}} \frac{1}{1 + n\phi(m\alpha)}$$

Next, we simplify each of the three terms. The contribution for the second term we have:

$$\frac{1}{n} \sum_{k=1}^{\gamma_1} \sum_{m \in \bigcup_{c=1}^{n+1} A_{k,c}} \frac{1}{1 + n\phi(m\alpha)}$$

$$\leq \frac{1}{n} \sum_{k=1}^{\gamma_1} \sum_{c=2}^{k+1} \frac{\text{Card}(A_{k,c})}{1 + n \frac{c-1}{B_{k+1}}}$$

$$\leq \frac{1}{n} \sum_{k=1}^{\gamma_1} \sum_{c=2}^{k+1} \frac{n \gamma_1}{B_{k+1}} \frac{\text{Card}(A_{k,c})}{1 + n \frac{c-1}{B_{k+1}}}$$

$$\leq \frac{1}{n} \sum_{k=1}^{\gamma_1} \sum_{c=2}^{k+1} \frac{2^{\log(a_{k+1})} + n \gamma_1}{2^{\log(a_{k+1})} + n \gamma_1}$$

$$\leq \frac{1}{n} \sum_{k=1}^{\gamma_1} \frac{2^{\log(a_{k+1})} + n \gamma_1}{n} \leq \frac{4^\gamma_1 \log(n)}{n} \leq 4^{(\log(n))^2}$$

Now, the values of $\phi(m\alpha)$ for the first term when $\gamma(m) = 0$ are, according to Lemma 3, characterized as a minimum over two terms. To simplify this we note that:

$$\frac{1}{1 + \phi(m\alpha)} = \frac{1}{1 + \min \left( \frac{c_1 - 1}{2B_1} + \frac{a_2 - c_2}{2B_2}, \frac{c_1 - 1}{2B_1} + \frac{c_2}{2B_2} \right) + \frac{1}{1 + \frac{c_1 - 1}{2B_1} + \frac{a_2 - c_2}{2B_2}}$$

The proof for each of the two terms now follows in an identical fashion as the steps for the first term. For the last term we need bounds for $B_{\gamma_1}$ and $a_{\gamma_1+1}$. The terms of type $\gamma_1$ are of the form $B_{\gamma_1}, 2B_{\gamma_1}, \ldots, d_{\gamma_1}B_{\gamma_1}$ of which $B_{\gamma_1}$ is the first term in $S_n$. Now since, $\alpha$, is an algebraic number, we have, for $\beta = \tau\alpha$,

$$\phi(B_{\gamma_1}^\alpha) = \phi(B_{\gamma_1}^\tau\beta) \geq \frac{C_{\beta}}{B_{\gamma_1}^\tau}$$
Since, the last term, $cB_{\gamma_n}$ of type $\gamma_n^*$ has to be smaller than $n$, we have,

$$d_* \leq \left\lfloor \frac{n}{B_{\gamma_n^*}} \right\rfloor$$

Since, $\alpha = \tau \beta$ for some $0 < \tau \leq n$, we have,

$$\sum_{m \in \bigcup \mathcal{A}_{\tau, \alpha}} \frac{1}{1 + n \phi(m \alpha)} \leq \frac{1}{n} \sum_{j=1}^{d_*} \frac{1}{1 + n j \frac{c \beta}{B_{\gamma_n^*} \gamma_n^*}} \leq \frac{1}{n} \sum_{j=1}^{d_*} \frac{1}{1 + n \frac{c \beta}{B_{\gamma_n^*} \gamma_n^*}}$$

$$= \frac{1}{n} \sum_{j=1}^{d_*} \frac{B_{\gamma_n^*}}{B_{\gamma_n^*} + C \beta j}$$

$$\leq \frac{1}{n} \min \left( d_*, B_{\gamma_n^*} \log \left( 1 + \frac{C \beta d_*}{B_{\gamma_n^*}} \right) \right)$$

$$\leq \frac{1}{n} \min \left( \left\lfloor \frac{n}{B_{\gamma_n^*}} \right\rfloor, B_{\gamma_n^*} \log \left( 1 + \frac{C \beta d_*}{B_{\gamma_n^*}} \right) \right) \leq \frac{\log(n)}{\sqrt{n}} \quad (35)$$

5 Conclusions

We have presented a unified framework for input design for system identification of uncertain systems. This work complements existing body of work on system identification with stochastic noise or bounded noise type disturbances. Our work focuses on identification of uncertain systems which naturally arise in situations where the model class is inadequate to characterize the actual complex system. We provide fundamental limits for when consistency is achievable for a wide variety of contexts in both unstructured as well structured uncertain environments. We show the optimality of well-known instrument-variable techniques in several cases and characterize the instruments and inputs that lead to robust identification. It follows that robust consistency is achievable if and only if there is an instrument that can uniformly de-correlate both the input and noise. The final part of the paper deals with the question of design of input sequences that satisfy these necessary and sufficient conditions. We present optimal deterministic sequences that meet the requirements for robust identification. We mention possibly two directions that are potentially interesting. One direction is to understand the general classes of structures for which robustly consistent algorithms can be derived. This work fits well with the needs of robust control setup. Another potential direction is to explore concepts for iterative identification and control towards a theory of indirect adaptive control.

6 Appendix

(Proof of Proposition 1) First, we note that the worst case diameter of uncertainty, which is the largest distance between two elements that are indistinguishable with respect to the data, serves as a lower bound for the estimation error in Equation 7. Specifically, let $y$ denote the column vector of the output signal and $\mathcal{D}(y)$ be the set of all impulse response coefficients that are consistent with the input-output data and the noise set, i.e.,

$$\mathcal{D}(y) = \{ H \in S(1, \gamma) : y(k) = (Hu)(k) + w(k); \ w \in W^N - \mathcal{A}, k = 0, \ldots, N \}$$

We define the local diameter of uncertainty as:

$$d(y) = \max_{H_1, H_2 \in \mathcal{D}(y)} |h_1(0) - h_2(0)|$$
It is well known that,
\[
\sup_{w \in W^N} \sup_{H \in \mathcal{S}(1, \gamma)} d(\{y = Hu + w\}) \leq 2 \sup_{w \in W^N} \sup_{H \in \mathcal{S}(1, \gamma)} |h(0) - \hat{h}^N(0)|
\]

The details of this result can be found in [24]. To prove convexity, first consider the case when the unmodeled dynamics, \(\Delta\) is equal to zero. Now, by hypothesis, the diameter is smaller than \(2 \epsilon\) for an appropriate choice of the set \(\mathcal{A}\). The main idea is that we cannot do any worse by maximizing the lebesgue measure of the noise set for each \(y\) while maintaining the diameter of admissible impulse coefficients \(h(0)\) to be smaller than \(2 \epsilon\). To see this, let,

\[
y = uh(0) + w
\]

where, \(y, u, w\) are column vectors of the output, input and noise signals respectively. Now we seek the set \(A_y\)

\[
A_y = \{w \mid y = h(0)u + w, \ diam(h(0)) \leq 2 \epsilon \}
\]

so that ultimately Prob\(\bigcup A_y\) is maximized. We express these in terms of the input, \(u\), and \(n-1\) other linearly independent basis functions, \(\{v_i\}\), with coefficients, \(\{\alpha_i\}\), \(\{\eta_i\}\) respectively. The input-output equation can now be re-written as,

\[
\alpha_1 u + \sum_{k=2}^{N} \alpha_k v_k = h(0)u + \eta_1 u + \sum_{k=2}^{N} \eta_k v_k
\]

This implies that,

\[
\alpha_1 = h(0) + \eta_1; \ a_k = \eta_k, \ k \neq 1
\]

Now, the diameter of uncertainty in \(h(0)\) will be smaller than \(2 \epsilon\) if and only if the set of all values \(h(0)\) lie in some interval, \([\bar{h} - \epsilon, \bar{h} + \epsilon]\). Therefore we require that,

\[
|\eta_1 - \alpha_1 - \bar{h}| \leq \epsilon
\]

Now, there corresponds a unique set of coefficients, \(\{\eta_k\}\), for each noise realization \(w \in \mathcal{W}^N\). Since, noise is uniformly distributed the coefficient \(\eta_1\) is uniformly distributed for a given \(y\). Furthermore, \(\eta_1\) lies in the symmetric interval for each fixed value of \(\eta_k, k > 1\). In other words, for each value of \(y\) we have some symmetric interval such that:

\[
\eta_1(y) \in [-\sigma_y, \sigma_y]
\]

Figure 8 illustrates these points. Therefore, the set of values of \(\eta_1\) (and hence the noise values for each \(y\)) is maximized by taking, \(\alpha_1 = \bar{h}\). To show convexity we see that the set of admissible noise values can now be described as the intersection of two convex sets, i.e.,

\[
\mathcal{W}_0^N = \{w = \eta_1 u + \sum_{k=2}^{N} \eta_k v_k, \ |\eta_1| \leq \epsilon, \eta_k \in \mathbb{R}\} \cap \mathcal{W}^N
\]

This set is clearly convex and by construction there is no other set with larger measure such that the diameter of uncertainty is smaller than \(2 \epsilon\). Next, we proceed in a similar manner to extend this argument to the unmodeled dynamics case as well. Now, \(\tilde{\delta}_i\) be the coefficients of the unmodeled output in the n-dimensional space considered above. It is well known from elementary linear algebra that there exists a linear transformation that maps, \(\tilde{\delta}_k\) to the impulse response \(\delta(k)\) of the unmodeled error and vice versa. Furthermore, since the \(\ell_1\) norm of unmodeled error is bounded by \(\gamma\) we have,

\[
T : \tilde{\delta} \longrightarrow \delta(\cdot); \ ||T \tilde{\delta}|| \leq \gamma
\]
where, \( \tilde{\delta} \) is the column vector of coefficients \((\tilde{\delta}_1, \tilde{\delta}_2, \ldots, \tilde{\delta}_N)\). We can now re-write Equation 36 to include unmodeled error:

\[
\alpha_1 u + \sum_{k=2}^{N} \alpha_k v_k = h(0)u + \tilde{\delta}_1 u + \sum_{k=2}^{N} \tilde{\delta}_k v_k + \eta_1 u + \sum_{k=2}^{N} \eta_k v_k
\]

Proceeding as before, we have,

\[
\alpha_1 = h(0) + \tilde{\delta}_1 + \eta_1, \quad \alpha_k = \tilde{\delta}_k + \eta_k, \quad k \neq 1
\]

We observe from Equation 37 that the set of all coefficients \(\{\tilde{\delta}_k\}\) is convex and balanced. Therefore, we conclude that the diameter of uncertainty is guaranteed to be smaller than \(2\epsilon\) for every possible value of \(\delta_k\) if and only if:

\[
|\eta_1 - \alpha_1 - \bar{h}| \leq \epsilon - \max |\tilde{\delta}_1|; \quad \eta_k = \alpha_k + \tilde{\delta}_k, \quad k \neq 1, \quad \|T\tilde{\delta}\|_1 \leq \gamma,
\]

Again upon observation that \(\eta_1\) must lie in a symmetric bounded interval around zero, we see that the set of \(\eta_1\) is maximized for \(\bar{h} = \alpha_1\). This leads to the set of coefficients,

\[
\mathcal{W}_0^N = \{w = \eta_1 u + \sum_{k=2}^{N} \eta_k v_k; \quad |\eta_1| \leq \epsilon - \max |\tilde{\delta}_1|, \quad \eta_k \in \mathbb{R}\} \bigcap \mathcal{W}_N
\]

This is again a convex set and by construction is the largest set such that the diameter of uncertainty is guaranteed to be smaller than \(2\epsilon\).

**Proof of Proposition 2**

1) The fact that \(\phi(\cdot)\) is an odd function follows by definition,

\[
\phi(x) = x - p_0, \quad p_0 = \text{argmin}_{p \in \mathbb{Z}} |x - p| \implies -p_0 = \text{argmin}_{p \in \mathbb{Z}} |(-x) - p| \implies \phi(-x) = (-x) - (-p_0) = -\phi(x)
\]

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2) This follows from the observation that translations of irrational number by an integer translates the corresponding nearest integer of the irrational number by the same amount,

\[ [x + k] = [x] + k, \ \forall \ k \in \mathbb{Z}^+ \]

Now the proof follows in the following fashion,

\[
\phi(kx) = kx - [kx] = kx - [kx - k[x] + k[x]] = kx - [k(x - [x])] - k[x] = k(x - [x]) - [k(x - [x])]
\]

3) Suppose, \( \phi(x) < 0 \)

\[ |\phi(kx)| = |\phi(k\phi(x))| = | - \phi(k(\phi(x))| = |\phi(k|\phi(x))| \]

For \( \phi(x) > 0 \), it follows that \( |\phi(kx)| = |\phi(k|\phi(x))| \)

4) Finally, to obtain bounds for the sinusoid function, we observe that:

\[
\sin(2\pi x) = \sin(2\pi \phi(x)) \in [0.5\phi(x), \phi(x)]
\]

where the last assertion is a standard result in elementary calculus.

**Proof of Proposition 3**

Consider the function, \( H_L(\cdot) \):

\[
H_L(x) = \frac{3L}{1 + L |\phi(x)|}
\]

Since, \( |\sin(2\pi L x)/\sin(2\pi x)| \leq L \), it follows

\[
\left| \frac{(1 + L \phi(x)) \sin(2\pi L x)}{\sin(2\pi x)} \right| \leq L + L \frac{|\phi(x)|}{|\sin(2\pi x)|} |\sin(2\pi L x)| \leq 3L
\]

where for last inequality we have used the fourth property of Proposition 2 Next, we show that \( H_L(\cdot) \) is monotonic in \( L \). Suppose, we have given two integers, \( n, m \), with \( n < m \), it follows that:

\[
m + mn\phi(x) \leq n + mn\phi(x) \Longleftrightarrow \frac{m}{1 + m\phi(x)} \leq \frac{n}{1 + n\phi(x)}
\]

The result now follows by inspection.

**Proof of Proposition 5**

\( (\Leftarrow \text{ case}) \) Suppose there is an \( l \) satisfying the hypothesis. It then follows that,

\[
r - q = \sum_{j=1}^{\lfloor 2\log_2 n \rfloor} c_j(r)B_{j+1} - \sum_{j=1}^{\lfloor 2\log_2 n \rfloor} c_j(q)B_{j+1}
\]

\[
= \sum_{j=1}^{l-1} (c_j(r) - c_j(q))B_{j+1} + \sum_{j=l}^{\lfloor 2\log_2 n \rfloor} (c_j(r) - c_j(q))B_{j+1}
\]

\[
> - \sum_{j=1}^{l-1} c_j(q)B_{j+1} + B_{l+1}
\]

We are left to show that:

\[
B_{l+1} \geq \sum_{j=1}^{l-1} c_jB_{j+1}
\]
for all admissible \( c_j \)s. We show this by means of an induction argument. For \( l = 2 \) we have:

\[ B_2 \geq c_1 B_0, \quad \forall \ c_1 \leq a_1 \]

Suppose, the induction hypothesis is true for \( l = k \), then for \( l = k + 1 \), we have:

\[ B_{k+1} = a_{k+1} B_k + B_{k-1} \geq \begin{cases} (a_{k+1} - 1)B_k + \sum_{j=1}^{k-1} c_{j+1} B_j, & c_k \neq 0 \\ a_{k+1} B_k + \sum_{j=1}^{k-2} c_{j+1} B_j, & c_k = 0 \end{cases} \]

where, the inequalities follow from induction hypothesis and Equation 22. The RHS corresponds to all the admissible \( c_j \)s and the proof follows.

\((\Rightarrow \text{ case})\) Suppose, there does not exist any such \( l \). Then it follows that either \( c_j(q) = c_j(r) \) which would imply \( q = r \) violating the hypothesis, or there is an \( l \) such that \( c_l(q) > c_l(r) \) with \( c_j(q) \geq c_j(r), \forall j > l \). Then, by following along the lines of the proof for the only if case, we come to the conclusion that \( r < q \) again violating our hypothesis.

\(\blacksquare\)

References


