A SEMI-DEFINITE PROGRAMMING APPROACH TO SOURCE LOCALIZATION

Saligrama Venkatesh William Clem Karl
Information Systems and Sciences Laboratory
Electrical and Computer Engineering Department
Boston University
Email: {srv,wckarl}@bu.edu

ABSTRACT

A new algorithm based on semi-definite programming is presented for estimation of distributed source fields measured through a sensor array. The problem can be viewed as a subclass of inverse problems, which have been extensively investigated in the literature. Our approach is based on the so called information based complexity (IBC) paradigm, which formalizes the notion of seeking the set of all solutions that are consistent with the observed data. We formulate our problem as a question of estimating the source up to a pre-specified resolution (average source field in a neighborhood) from the observed data with optimal accuracy. We show that this problem is convex and can be re-formulated as a semi-definite program.

1. INTRODUCTION

This problem is motivated by a need for estimating the spatio-temporal density of noise fields. The problem arises in many contexts as in the need to understand the underlying noise sources that generate jet noise [1–3]. Microphone phased array processing techniques, capable of revealing the spatial distribution of noise sources, are becoming more widely used in these contexts [1, 4, 9]. The principle idea for such techniques, known as beamforming, is to coherently sum the measurements at different microphones to enhance the signal emanating from a focal position while minimizing the contribution from out-of-focus locations [5].

In the simplest delay-and-sum beamformer, the outputs of time-delayed sensor measurements are summed, with the delays as a function of focus position and sensor location, to estimate the source distribution [5]. When a source is at the focus position, the signals add coherently to produce an enhanced signal, whereas for out-of-focus positions, the signals add incoherently. The performance of this simple beamformer can be improved by weighting or shading the microphone signals before summation. While such conventional beamforming techniques work well in locating a multiple set of isolated point sources, they can perform poorly when reconstructing continuously distributed noise sources, where contributions from out-of-focus sources can lead to unacceptable estimation errors. This difficulty has been largely overlooked in the literature, and this paper presents a new beamforming approach for estimating distributed noise sources.

Our main idea is to view the reconstruction problem in the framework of inverse problems. However, unlike the regularization techniques [6] that are often used in this context, we develop a new scheme based on information based complexity paradigm [8]. This paradigm formalizes the notion of seeking the set of all solutions that are consistent with the observed data. We formulate our problem as a question of estimating the source up to a pre-specified coarse resolution from the observed data with optimal accuracy. We show that this problem is convex and can be re-formulated as a semi-definite program. In the context of source localization a meaningful notion of coarse resolution is to seek average source strength over a small region around each focus position as opposed to seeking the exact source strength at each spatial location as in conventional approaches. This overcomes the drawbacks of conventional methods of estimating continuous distributions and yields a beamformer with uniform spatial resolution and accuracy over a large frequency range. The coarse reconstruction of the source field is then recursively refined to yield finer resolutions with guaranteed degree of accuracy.

The organization of the paper is as follows. First, we discuss the problem setup and outline several issues that need to be addressed while beamforming a distribution of sources. Next, a beamforming algorithm that overcomes these problems is presented. The tradeoff between spatial resolution and accuracy of the estimate is described. Experiments to validate the processing algorithm are then discussed, where the performance of the technique developed here is compared to that from the minimum variance beamformer.

This work was partially supported by an ONR Young Investigator Grant No. N00014-02-1-0362
2. PROBLEM SETUP

In this section, we outline a new scheme for phased array processing of distributed sources. We will present the setup, outline the shortcomings of current approaches, and finally illustrate the new approach. The main ideas in the new formulation are: 1) definition of spatial resolution as a localized spatial average of the source distribution (as opposed to the traditional definition of spatial resolvability of two discrete sources), and 2) ability to tune spatial resolution for achieving desired accuracy.

The setup consists of sound incident on an n-microphone array from a broadband, radially-compact, axially non-compact, axisymmetric distribution of monopoles along the jet centerline of length $L$. The instantaneous source strength, $s(t,x)$, radiated from a source at location, $x$, at time $t$ is assumed to be a stationary random field. Based on a spherical wave propagation model at ambient sonic speed $c$, we can relate the $j$th instantaneous microphone measurement, $y_j(t)$, to the source strength, $s(t,x)$:

$$y_j(t) = \int_{\Omega} h_j(t-\tau, x)s(\tau, x)d\tau$$

where $h_j(x)$ is the propagation kernel that relates the contribution of source at the point $x$ to the $j$th microphone. We collect all the measurements at each time instant into a matrix $Y(t)$ (with $y_j(t)$ as column vectors) and compute the spectral density matrix (PSD), i.e.,

$$\Phi_y(f) = \int \int H(f,x) \Phi_s(f,\mu-x)H^*(f,x+\mu)dxd\mu$$

where $\Phi_s(\cdot,\cdot)$ is a cross spectral density of the underlying wide-sense-stationary source, $s(\cdot)$ and $H(\cdot,\cdot)$ is the column vector of fourier transforms of $h_j(\cdot,\cdot)$ for each source location. When the source is assumed to be axially uncorrelated, the above equation simplifies to:

$$\Phi_y(f) = \int H(f,x) \Phi_s(f,x)H^*(f,x)d\mu$$

where $s$ is the source strength per unit length along the jet centerline. The objective is to reconstruct the source distribution, $\Phi_s$, from the measured PSD $\Phi_y$. The observed $\Phi_y$ has $n$ rows and $n$ columns, whereas the source distribution is a function of a continuous variable $x$. Therefore, the problem involves solving a linear system of a finite number of equations with infinitely many unknowns, leading to ill-posedness. Such ill-posed problems have been studied in the general area of inverse problems [6]. An exact reconstruction of the power spectral density (PSD) of the source distribution is impossible and in general, one seeks approximate solutions. Conventional phased array processing in aeroacoustics has used well-known radar/sonar signal processing techniques. These techniques have been developed for isolation and detection of a finite number of point sources and it is our contention that such techniques are inadequate for the distributed source estimation problem.

2.1. Conventional Beamforming

Beamforming is a solution whereby, a beam is formed to focus the array along a specific direction. For instance, in the delay and sum technique, microphone measurements, $y_j(t)$, are sequentially delayed and summed to form a beam that coherently combines sources radiating from a desired focal point, while being incoherent to radiation from off-axis radiation. Additional weighting or shading can further improve the array performance. The weights for all the microphones are collected in a vector, $w(f, x)$, and the estimate for the source strength at location, $x$, and frequency, $f$, is given by

$$\hat{\Phi}(x) = w^*(x) \Phi_y w(x)$$

where, we have dropped explicit dependence on frequency for notational simplicity. In minimum variance beamforming, the objective is to minimize the total off-axis energy while letting in signals from the desired focal point. The problem can be formulated as a quadratic optimization problem. Often, $H(f,\cdot)$, $\Phi_y(f)$ is singular, so that some form of regularizaiton technique is required. Minimum variance beamforming and other forms of weighting has been used by several researchers [1, 4]. The minimum variance approach effectively estimates point sources over a broad range of frequencies as we illustrate later with numerical simulations. Such point source estimation is not sufficient for distributed sources as will be illustrated through numerical simulations in Section 4. The minimum variance technique has difficulty estimating distributed sources because of two fundamental limitations. First, it fails to account for the fact that the array is unable to discriminate among sources in the neighborhood of the focus position. This is aggravated by the fact that sources within a small region are likely to be correlated. Thus, at the very least the contribution from sources in the neighborhood of the focus position cannot be nulled. Therefore, the most that one can hope for is to find the average source strength in the neighborhood of the focus position. The second limitation of the minimum variance approach is that it is unable to control off-axis contributions leading to increased sidelobe levels (see Section 4).

2.2. New Approach

The new approach overcomes the difficulty with distributed sources by relaxing the objective of estimating the PSD, $\Phi_s(f,x)$. The relaxed objective is to estimate a smooth sound power distribution obtained by using a smoothening
kernel $Q(\cdot)$.

$$\Phi_{\delta} = \int Q\left(\frac{x_0 - \eta}{\delta}\right) \Phi_s(\eta) d\eta$$

(2)

where, $\delta$, is a scaling parameter, which controls the resolution. The resulting PSD is a local $\delta$-average over a finite region of space, which is more appropriate for distributed sources. Furthermore, unlike minimum variance beamforming, the explicit inclusion of the smoothening operator, $Q(\cdot)$ in the beamforming equations enables independent control over the spatial resolution and side-lobe levels.

With these comments we can state the objective precisely: determine the smallest resolution, $\delta$, that still guarantees a desired level of accuracy. Mathematically, we want an estimate, $\hat{\Phi}$ that guarantees an accuracy level, $\gamma$ across all frequencies and spatial coordinates. In general our objective reduces to the following problem:

$$\min \delta \quad \text{subject to } \|\Phi(x) - \Phi_{\delta}(x)\|_{\infty} \leq \gamma$$

Since $\Phi(x)$ is not known the problem formulated above is ill-posed. A reasonable objective would be to find an estimate, which uniformly approximates the set of all PSDs that are consistent with the observation. To this end, we define:

$$S = \{ \Phi_s(x) \ \text{consistent with Equation 1} \}$$

In general, the set is difficult to characterize and if an approximate bound on the maximum PSD is available an alternative formulation can be obtained. To this end, let

$$S = \{ \Phi_s(\cdot) \in \mathbb{C}[0, L] \mid \Phi_s(\cdot) \in [0, \alpha] \}$$

(3)

where, $\mathbb{C}[0, L]$, denotes the space of continuous functions on the interval $[0, L]$. The main problem can now defined as:

$$\min \delta \quad \text{subject to } \sup_{S_\delta} \|\Phi(x) - \Phi_{\delta}(x)\|_{\infty} \leq \gamma$$

(4)

For a fixed spatial resolution $\delta$, the solution reduces to finding the diameter of the set of all solution elements, $\Phi_s$ that are consistent with the observations. This is given by:

$$S_\delta = \{ \Phi_s(x) \ \text{consistent with Equations 1,2} \}$$

(5)

The diameter of the above set now provides a fundamental bound for the estimation error for any algorithm, i.e.,

$$\sup_{S} \|\Phi(x) - \Phi_{\delta}(x)\|_{\infty} \geq \frac{1}{2} \text{diam}(S_\delta)$$

The optimization problem has the graphical interpretation shown in Figure 2. We have a map from $\Phi(\cdot)$ to observation space $Y$. The smoothening operator maps $\Phi(\cdot)$ to $\Phi_{\delta}(\cdot)$. We seek an estimator that maps the observations to the smoothened space of solution elements. In the next section we verify conditions under which this map is guaranteed to be linear.

3. SEMI-DEFINITE PROGRAMMING

In this section we show that the problem formulated in the previous section can be recast as a semi-definite program. The reduction to a semi-definite program follows from well-known concepts in information based complexity theory [8].

The main idea is based on the so called smolyak’s theorem and its extensions [7]. The theorem states that the optimal estimator is affine if the following conditions are satisfied: (1) observations are linear functions of the underlying parameter space, (2) the parameter set is convex, (3) the cost function is a linear function of the parameter set, (4) the estimation error is measured with respect to the $\ell_\infty$ norm.

We verify these conditions for a fixed $\delta$. Equation 1 guarantees the linearity of observations as a function of $\Phi_s$. The second condition is verified by the fact that $\Phi_s(\cdot)$ belongs to the family of positive functions, which is convex. The smoothening operator is linear hence verifying the third condition. Finally, since the error is indeed measured with respect to the $\ell_\infty$ norm it follows that all the conditions of the theorem are met. Therefore, there exists a (affine) linear operator, $Q$, for each location that maps the observations to the smoothened solution estimates.

Based on these arguments we have the following theorem.

Theorem 1. Consider the setup of Equation 5. It follows that for a fixed spatial resolution $\delta$ an estimate that minimizes the diameter of uncertainty is a linear beamformer, $Q$ satisfying:

$$1 - \beta \leq H(z)QH^*(z) \leq 1, \ |z| \leq \delta$$

and

$$\text{Tr} \left( Q \int_{|z| \geq \delta} HH^* dz \right) \leq \frac{1}{\rho} \text{Tr} \left( Q \int_{|z| \leq \delta} HH^* dz \right)$$
Fig. 2. Illustration of recursive updates with a 25KHz point source

We point out some issues pertaining to the apparent conservatism of the beamforming procedure and steps to overcome it. The conservatism stems from the fact that Equation 4 seeks an estimator that is optimal in the worst-case. The worst-case nature of the formulation can be recursively refined leading to reduced conservatism. The idea is to utilize the estimated $\hat{\Phi}$ as an initialization for computing a new beamformer. To do this requires updating the set of solution elements given in Equation 3. However, we only have estimates from the smoothened estimate, $\bar{\Phi}_s$. From positivity conditions on $\Phi_s$ it turns out that $\hat{\Phi}$ is an upper bound on $\bar{\Phi}_s$, i.e.,

$$\bar{\Phi}_s(x) \leq \hat{\Phi}(x)$$

Therefore, based on this bound and under mild regularity conditions we can update Equation 3 to:

$$S_\delta = \{ \Phi^\delta(\cdot) \in C[0, L] \mid 0 \leq \Phi_s(x) \leq C_0\bar{\Phi}_{n-1}(x) \}$$

where, $\bar{\Phi}_{n-1}$ is the estimate obtained in the $n - 1$th iteration. Theorem ?? can now be suitably modified to derive a beamformer with higher accuracy. This procedure is demonstrated in Figure ?? for a 25KHz point source. As seen the recursive algorithm significantly improves the accuracy over each update.

4. RESULTS

The minimum variance approach effectively estimates point sources over a broad range of frequencies. This is seen from numerical simulations such as in Fig 3, where a broadband point source of unit magnitude at a location 1.63m away from and 0.5m along the array axis, is accurately located and estimated at both low and high frequencies. For a single point source, when the focal position coincides with the source location, there are no other sources to corrupt the estimated source strength. However, when the focus position is not the same as the source location, erroneous reconstructed source strength, or sidelobe level, is caused by the processing scheme’s inability to completely null the source.

Fig. 3. Comparison of simulated minimum variance and new beamforming schemes for a point source at 2 kHz and 15 kHz.

Such point source estimation is not sufficient for distributed sources. This is illustrated in Fig 4 through numerical simulations for a 2kHz and 15 kHz uniformly distributed source at a distance of 1.63m from the array. The source has a uniform strength over a length of 0.5 m along the array axis. Evidently, the minimum variance technique exhibits significant errors at 15 kHz. this problem of inaccurate estimates is specific to distributed sources. The problem of spurious sources as in Fig 4 occurs because the sidelobe level is no longer a meaningful measure of the error. Rather, the integral of the aggregate source strength weighted by the sidelobe levels determines the error. The large sidelobes shown in Fig 4 at 15 kHz are a result of these limitations. Numerical simulations for the new beamforming technique at both low (2 kHz) and high (15 kHz) frequencies for point as well as distributed sources in Figs 3, 4 exhibits significant improvements in both accuracy and localization.

5. CONCLUSION

A new beamforming algorithm has been developed which allows spatial resolution to be traded with accuracy, ensuring uniform performance at different frequencies, and having higher confidence amplitude estimates than conventional beamforming approaches. This provides a promising aeracoustics measurement technique for spatially distributed sources such as those encountered in high-speed jet exhausts. With conventional beamforming, the decrease of the beamwidth with frequency implies improved spatial res-
olution but a degradation in accuracy due to higher side-lobes. Thus, it is apparently impossible to enhance signal contributions from a focal point without also enhancing the contribution from out-of-focus locations. In this study, this apparent conflict was resolved by re-defining spatial resolution in terms of localized spatial averages of the source distribution. Using such a definition a constrained optimization problem was formulated where the best spatial resolution for a guaranteed level of accuracy can be achieved, permitting a tradeoff between spatial resolution and accuracy. The new algorithm is suited to applications requiring a large dynamic range, high resolution, and high confidence over a broad range of frequencies.

6. REFERENCES


