Stock prices and yield curve with nominal and real risk

Consider a representative agent economy endowment economy. The RC has expected utility and a CRRA felicity function:

$$E \sum_{t=0}^{\infty} \beta^t \frac{C_{t+1}^{1-\gamma}}{1-\gamma}.$$  

Set $\beta = .999$ and $\gamma = 4$. The process for aggregate consumption and aggregate inflation is a joint Markov chain. Let $x_t$ be the vector (consumption growth, inflation):

$$x_t = (\Delta \log C_t, \Delta \log P_t).$$

Then $x_t$ can take any of four values (states 1 through 4): high inflation and high growth, low inflation and low growth, low inflation and high growth, or high inflation and low growth. The transition matrix is as follows:

$$Q = \begin{pmatrix}
.75 & .05 & .1 & .1 \\
.05 & .75 & .1 & .1 \\
.1 & .1 & .75 & .05 \\
.1 & .1 & .05 & .75
\end{pmatrix}.$$  

High inflation means 5%, low inflation means 2%; high growth means 4%, low growth means 2%.

All calculations should be done numerically, e.g. using Matlab.

(1) Compute the real price $W_t$ of a claim to consumption, as a function of the state $s = 1, 2, 3, 4$. Hint: use the standard recursion

$$\frac{W_t}{C_t} = E_t \left( M_{t+1} \left( \frac{W_{t+1}}{C_{t+1}} + 1 \right) \frac{C_{t+1}}{C_t} \right).$$

Compute the expected return on this asset, as a function of the state.

(2) Compute the real price of a real bond of maturity 1 through 10 periods, as a function of the state. Hint: use the recursion

$$P_t^{(0)} = 1,$$
\[ P_t^{(n)} = E_t \left( M_{t+1} P_{t+1}^{(n-1)} \right). \]

Deduce the real yields \( y_t^{(n)} \). Plot the yield curve, i.e. \( y_t^{(n)} \) as a function of \( n \), for each of the four states. Compute and plot the expected returns on each of these 10 bonds, as a function of the state.

(3) Compute the nominal price of a nominal bond of maturity 1 through 10 periods, as a function of the state. Hint: use the recursion
\[
Q_t^{(0)} = 1,
Q_t^{(n)} = E_t \left( M_{t+1} Q_{t+1}^{(n-1)} \right).
\]
Deduce the nominal yields \( z_t^{(n)} \). Plot the yield curve, i.e. \( z_t^{(n)} \) as a function of \( n \), for each of the four states. Compute and plot the expected returns on each of these 10 bonds, as a function of the state.

(4) Compute average expected inflation over the next \( n \) years, for \( n = 1, 2, \ldots, 10 \), i.e.
\[
\frac{1}{n} E_t \left( \sum_{k=1}^{n} \Delta \log P_{t+k} \right),
\]
as a function of the state.

(5) Define the inflation risk premium as the difference between the expected real return on a nominal bond and a real bond, for each maturity and state. Compute the inflation risk premium. Plot on the same figure, as a function of \( n \), the inflation risk premium, the expected inflation, and the difference between nominal and real yields \( z_t^{(n)} - y_t^{(n)} \). Discuss briefly how these three lines move with the state, and with \( \gamma \).

(6) Compute the equity premium (i.e. the difference in expected returns between equity and bonds.) in two ways: equity over short-term (1-period) real bonds, and over long-term (10-period) real bonds. Show how each of these depend on the state. Explain briefly.

(7) Redo (6), by computing the equity premium over nominal bonds.

(8) Redo (1)-(7) with recursive utility, i.e.
\[
V_t = \left( (1 - \beta) C_t^{1-\rho} + \beta E_t \left( V_{t+1}^{1-\alpha} \right) \right)^{\frac{1}{1-\rho}},
\]
and

\[ M_{t+1} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\rho} \frac{V_{t+1}^{\rho-\alpha}}{E_t \left( V_{t+1}^{1-\alpha} \right)^{\frac{\rho-\alpha}{\alpha}}}, \]

and set \( \rho = 0.25 \) and \( \alpha = 20 \) to start. How are your results affected when you change \( \rho \) vs. \( \alpha \)?

Hint: first, you need to calculate \( \frac{V}{C} \) using the recursion for the value. Then, you can use this to find a simpler expression for the real SDF \( M_{t+1} \). Finally, you can use the SDF in exactly the same way as you solved questions 1-7.