Lecture 8: Two period corporate debt model

Simon Gilchrist
Boston University and NBER

EC 745
Fall, 2013
A two-period model with investment

- At time 1, the firm buys capital $k$, using equity issuance $s$ and debt: the firm issues a debt with face value $b$, at unit price $q(k, b)$, hence the budget constraint:

$$\chi q(k, b) b + s = k$$

- The parameter $\chi > 1$ reflects the tax shield effect - interest expenses are subsidized. For each dollar of debt issued, the firm receives a subsidy $\chi - 1$.

- At time 2, the firm produces, and obtains a profit $\pi = zk^\alpha$, where $z$ is an idiosyncratic shock, distributed according to a cumulative distribution function $F$, with mean 1: $E(z) = \int_0^\infty s f(s) ds = 1$. 
The firm will default if its profits are not large enough to repay its debt, i.e. if \( z < z^* \), with

\[
z^* k^\alpha = b.
\]

If the firm does default, equity holders get nothing, while bondholders share the firm profits, net of bankruptcy costs.

The parameter \( 0 < \theta < 1 \) determines the share of profits which are recovered.
Debt and Equity Value

- Debt value is given by a standard pricing equation, assuming risk neutrality and a discount factor $\beta$:

$$q(k, b) = \beta \left( \int_{z^*}^{\infty} dF(z) + \int_{0}^{z^*} \theta \frac{z^{k\alpha}}{b} dF(z) \right).$$

- The firm equity value is

$$V = \beta \int_{z^*}^{\infty} (z^{k\alpha} - b) dF(z).$$
The firm chooses $k, b, s, z^*$ to maximize its present discounted value

$$\max_{k, b, z^*} \left\{ \beta \int_{z^*}^{\infty} (zk^\alpha - b) \, dH(z) - k + \chi q(k, b) \right\}$$

subject to:

$$q(k, b) = \beta \left( \int_{z^*}^{\infty} dF(z) + \int_{0}^{z^*} \theta \frac{zk^\alpha}{b} \, dF(z) \right),$$

$$z^* k^\alpha = b.$$ 

The firm solves

$$\max_{k, z^*} \beta \int_{z^*}^{\infty} k^\alpha (z - z^*) \, dF(z) - k$$

$$+ \chi \beta \left( z^* k^\alpha \int_{z^*}^{\infty} dF(z) + \int_{0}^{z^*} \theta zk^\alpha \, dF(z) \right)$$
Optimality

- This program can be solved by writing the two first order conditions, with respect to $k$ and $z^*$, which determine the optimal investment and financing. First, we have

$$1 = \beta\alpha k^{\alpha-1} \left\{ \int_{z^*}^{\infty} (z - z^*) dF(z) + \chi \left( z^* \int_{z^*}^{\infty} dF(z) + \theta \int_{0}^{z^*} zdF(z) \right) \right\}$$

which can be rearranged, given that $E(z) = 1$, as:

$$1 = \beta\alpha k^{\alpha-1} \left\{ 1 + (\chi - 1) z^* (1 - F(z^*)) + (\theta \chi - 1) \int_{0}^{z^*} zdF(z) \right\}.$$

- Interpretation: note that if $\chi = \theta = 1$, we have the frictionless limit: $1 = \beta\alpha k^{\alpha-1} E(z)$, the usual MPK = user cost condition. When $\theta < 1, \chi > 1$, this is not true anymore, and the tax shield $\chi$ and the bankruptcy cost $\theta$ both tend to decrease the user cost.
The second first-order condition, with respect to $z^*$, yields, after rearrangement

$$((1 - F(z^*)))(\chi - 1) = \chi (1 - \theta) z^* f(z^*).$$

This equation determines $z^*$, and hence the optimal leverage (increasing $z^*$ means increasing the leverage and the probability of default).

LHS: benefit of higher leverage – the tax shield conditional on no default.

RHS: The cost of higher leverage is an increase in bankruptcy costs through a higher probability of default - the change in probability of default is $f(z^*)$ at the margin, and the default costs are proportional to $z$.

If $z$ lognormal then $z \frac{zf(z)}{1-F(z)}$ is increasing, which guarantees a unique solution to this equation.
We can follow BGG notation. Let

$$\Gamma(z^*) = z^* \int_{z^*} dF(z) + \int_0^{z^*} z^* dF(z)$$

and

$$\mu G(z^*) = \mu \int_0^{z^*} z^* dF(z)$$

denote bankruptcy costs where $\mu = 1 - \theta$.

Then the equity share is

$$1 - \Gamma(z^*) + (\chi - 1) (\Gamma(z^*) - \mu G(z^*))$$

The lender’s share is

$$\Gamma(z^*) - \mu G(z^*)$$
Firm value is

$$Q(z^*) \beta k^\alpha - k$$

where

$$Q(z^*) = 1 - \Gamma(z^*) + \chi [\Gamma(z^*) - \mu G(z^*)]$$

$$\frac{zf(z)}{1-F(z)}$$ is increasing implies that the term in brackets is strictly concave with an interior maximum at some $z^{**}$. So neither debt or equity claimants desire $z^* > z^{**}$.
Optimal leverage

- Optimality then implies choosing leverage so that the marginal value of an additional unit of debt is equal to the marginal value of an additional unit of equity:

\[ \chi \left[ \Gamma'(z^*) - \mu G'(z^*) \right] = \Gamma'(z^*) \]

- This occurs at an interior value \(0 < z^* < z^{**}\). Since \(Q(0) = 1\) and \(Q(z^{**}) < 1\) we have \(Q(z^*) > 1\). As long as \(\chi > 1\) issuing debt raises firm value. Otherwise it would be optimal to set \(z^* = 0\) to avoid bankruptcy costs.
Optimal choice of $k$

- The FOC then imply:

$$Q(z^*)\alpha k^{\alpha-1} = 1/\beta$$

which means that capital is above the efficient level owing to the tax subsidy on debt.

- We can interpret $(\beta Q(z^*))^{-1}$ as the user cost of capital.
The value of $z^*$ is determined by the parameters $\chi, \theta$, and the distribution $F$.

The comparative statics w.r.t. $\chi$ and $\theta$ is clear.

With regard to $F$, an interesting comparative static is a change in the payoff distribution $F$. If $F$ becomes more risky, at least for most distributions, we will have less debt and less capital. Intuitively, the debt contract becomes more inefficient, making the cost of the debt higher. (Note: this is ex-ante higher risk, not ex-post: this is different from risk-shifting)
Now assume that equity is fixed at some predetermined value \( n \) (Net worth) and set \( \chi = 1 \).

The firm borrows some initial amount \( qb = k - n \)

We can write the problem as:

\[
\begin{align*}
\max_{k,b,z^*} \left\{ \beta \int_{z^*}^\infty (zk^{\alpha} - b) \, dH(z) - k + n + q(k, b)b \right\}
\end{align*}
\]

subject to:

\[
q(k, b) = \beta \left( \int_{z^*}^\infty dF(z) + \int_0^{z^*} \theta \frac{zk^{\alpha}}{b} \, dF(z) \right),
\]

\[
z^* k^{\alpha} = b.
\]

\[
q(k, b)b = k - n.
\]
Maximization

- Max problem

$$\max_{k, z^*} \beta \int_{z^*}^{\infty} k^\alpha (z - z^*) \, dF(z) - k$$

$$+ \lambda \left( \beta \left( z^* k^\alpha \int_{z^*}^{\infty} dF(z) + \int_{0}^{z^*} \theta z k^\alpha \, dF(z) \right) - (k - n) \right).$$

- We can write this as

$$\max_{k, z^*} \{ \beta Q(z^*) k^\alpha - \lambda (k - n) \}$$

where

$$Q(z^*) = 1 - \Gamma(z^*) + \lambda (\Gamma(z^*) - \mu G(z^*))$$
Optimality

- FOC:

\[ Q\beta \alpha k^{\alpha-1} = \lambda \]

\[ \lambda \left[ \Gamma'(z^*) - \mu G'(z^*) \right] = \Gamma'(z^*) \]

\[ \beta \left( \Gamma(z^*) - \mu G(z^*) \right) k^\alpha = k - n \]
Entrepreneur starts period with net worth $N$.

Entrepreneur borrows:

$$B = QK - N,$$

- $Q = \text{price of capital (exogenous)}$

Project payoff:

$$\omega R^k QK^\alpha, \quad 0 < \alpha < 1$$

- $R^k = \text{aggregate (gross) rate of return on capital (exogenous)}$
- $\omega = \text{idiosyncratic shock to project’s return}$
- Assume: $\ln \omega \sim N(-\frac{\sigma^2}{2}, \sigma^2) \Rightarrow E[\omega] = 1$
Information Structure

- No asymmetric information *ex ante*:
  - $R^k$ is known to both lender and entrepreneur before investment decision.
  - $\omega$ is realized after investment decision.

- Asymmetric information *ex post*:
  - $\omega$ is freely observed by entrepreneur.
  - To observe $\omega$, lender must pay
    \[
    \mu \omega R^k Q K^\alpha
    \]

- Parameter $0 \leq \mu < 1$ measures the cost of monitoring and hence the magnitude of credit market frictions.
Entrepreneur and lender agree to a standard debt contract (SDC) that pays lender an amount $D$ as long as bankruptcy does not occur.

If $\omega R^k Q K^\alpha \geq D$:
- Entrepreneur pays $D$ to lender and keeps residual profits.

If $\omega R^k Q K^\alpha < D$:
- Entrepreneur declares bankruptcy and gets nothing.
- Lender pays bankruptcy cost to monitor entrepreneur and keeps profits net of bankruptcy cost.
Bankruptcy occurs if $\omega \leq \bar{\omega}$:
\[
\bar{\omega} \equiv \frac{D}{R^k Q K^\alpha}
\]

Expected payoff to entrepreneur:
\[
\int_{\bar{\omega}}^{\infty} \omega R^k Q K^\alpha d\Phi(\omega) - \bar{\omega} \int_{\bar{\omega}}^{\infty} R^k Q K^\alpha d\Phi(\omega)
\]

Expected payoff to lender:
\[
(1 - \mu) \int_{0}^{\bar{\omega}} \omega R^k Q K^\alpha d\Phi(\omega) + \bar{\omega} \int_{\bar{\omega}}^{\infty} R^k Q K^\alpha d\Phi(\omega)
\]

Competitive loan market: Lender must earn an expected (gross) rate of return $R$ on the loan amount $B$. 
Payoffs as a Share of Expected Profits ($R_k Q K$)

- Define:

\[
\Gamma(\omega) \equiv \int_{0}^{\omega} \omega d\Phi(\omega) + \omega \int_{\omega}^{\infty} d\Phi(\omega)
\]

\[
\mu G(\omega) \equiv \mu \int_{0}^{\omega} \omega d\Phi(\omega)
\]

- Entrepreneur’s expected share of profits:

\[
1 - \Gamma(\omega)
\]

- Lender’s expected share of profits:

\[
\Gamma(\omega) - \mu G(\omega)
\]
Choose $K$ and $\bar{\omega}$ to solve:

$$\max_{K,\bar{\omega}} \left[ 1 - \Gamma(\bar{\omega}) \right] R^k Q K^\alpha$$

subject to the lender’s participation constraint:

$$[\Gamma(\bar{\omega}) - \mu G(\bar{\omega})] R^k Q K^\alpha = R(QK - N)$$

Lagrangean:

$$\max_{K,\bar{\omega}} \left\{ [(1 - \Gamma(\bar{\omega})) + \lambda (\Gamma(\bar{\omega}) - \mu G(\bar{\omega}))] R^k Q K^\alpha - \lambda R(QK - N) \right\}$$

- $\lambda = \text{Lagrange multiplier on the lender’s participation constraint}$
  - and hence measures the shadow value of an extra unit of net worth to the entrepreneur.
- The term in brackets reflects total firm value when valued using the shadow price of external funds.
Optimality Conditions

- FOC w.r.t. $\bar{\omega}$:
  \[ \lambda = \frac{\Gamma'(\bar{\omega})}{[\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})]} \geq 1 \]

- FOC w.r.t. $K$:
  \[ \alpha [(1 - \Gamma(\bar{\omega})) + \lambda (\Gamma(\bar{\omega}) - \mu G(\bar{\omega}))] R^k Q K^{\alpha-1} = \lambda R Q \]

- FOC w.r.t. $\lambda$:
  \[ [\Gamma(\bar{\omega}) - \mu G(\bar{\omega})] R^k Q K^\alpha = R (Q K - N) \]
FOCs imply:

\[ \alpha R^k Q K^{\alpha-1} = \rho(\bar{\omega}) R Q \]

- \( \rho(\bar{\omega}) = \left[ \frac{\lambda}{[1-\Gamma(\bar{\omega})]+\lambda[\Gamma(\bar{\omega})-\mu G(\bar{\omega})]} \right] \geq 1 \)
- \( \rho(\bar{\omega}) = \text{external finance premium (EFP)} \)
- EFP is increasing in the default barrier \( \bar{\omega} \):

\[ \rho'(\bar{\omega}) > 1 \]
The default barrier $\bar{\omega}$ is **increasing** in leverage:

$$\frac{QK}{N} = \frac{\psi(\bar{\omega})}{1 - (1 - \alpha)\psi(\bar{\omega})}$$

where

$$\psi(\bar{\omega}) = \left[1 + \frac{\lambda \left[ \Gamma(\bar{\omega}) - \mu G(\bar{\omega}) \right]}{1 - \Gamma(\bar{\omega})} \right] \geq 1$$

$$\psi'(\bar{\omega}) > 0$$

**Intuition:**

- An increase in leverage requires a higher default barrier to increase the payoff to the lender relative to the entrepreneur.
- The increase in the default barrier also implies a higher shadow value of external funds $\lambda$.
- An increase in net worth reduces the default barrier and lowers the premium on external funds.
The default barrier is determined by the rate of return on capital relative to the risk-free rate of return:

\[
\frac{R^k}{R} = \rho(\bar{\omega})
\]

Given \(\bar{\omega}\), capital expenditures are determined by available net worth:

\[
\frac{QK}{N} = \psi(\bar{\omega})
\]

Combining these, we obtain a positive relationship between the premium on external funds and leverage:

\[
\frac{R^k}{R} = s \left( \frac{QK}{N} \right), \quad s' > 0
\]