Epstein and Zin (1989 JPE, 1991 Ecta) following work by Kreps and Porteus introduced a class of preferences which allow to break the link between risk aversion and intertemporal substitution.

These preferences have proved very useful in applied work in asset pricing, portfolio choice, and are becoming more prevalent in macroeconomics.
Value function:

- To understand the formulation, recall the standard expected utility time-separable preferences are defined as

\[ V_t = E_t \sum_{s=0}^{\infty} \beta^{s-t} u(c_{t+s}), \]

- We can also define them recursively as

\[ V_t = u(c_t) + \beta E_t V_{t+1}, \]

or equivalently:

\[ V_t = (1 - \beta)u(c_t) + \beta E_t (V_{t+1}). \]
• EZ preferences generalize this: they are defined recursively over current (known) consumption and a certainty equivalent \( R_t (V_{t+1}) \) of tomorrow’s utility \( V_{t+1} \):

\[
V_t = F (c_t, R_t (V_{t+1})) ,
\]

where

\[
R_t(V_{t+1}) = G^{-1} (E_t G(V_{t+1})) ,
\]

with \( F \) and \( G \) increasing and concave, and \( F \) is homogeneous of degree one.

• Note that \( R_t(V_{t+1}) = V_{t+1} \) if there is no uncertainty on \( V_{t+1} \).

• The more concave \( G \) is, and the more uncertain \( V_{t+1} \) is, the lower is \( R_t(V_{t+1}) \).
Most of the literature considers simple functional forms for $F$ and $G$:

$$\rho > 0 : F(c, z) = ((1 - \beta)c^{1-\rho} + \beta z^{1-\rho})^{\frac{1}{1-\rho}},$$

$$\alpha > 0 : G(x) = \frac{x^{1-\alpha}}{1 - \alpha}.$$

For example:

$$V_t = \left( (1 - \beta)c_t^{1-\rho} + \beta (E_t V_{t+1}^{1-\alpha})^{\frac{1-\rho}{1-\alpha}} \right)^{\frac{1}{1-\rho}}.$$
Limits:

\[ \rho = 1 : F(c, z) = c^{1-\beta} z^\beta. \]
\[ \alpha = 1 : G(x) = \log x. \]

Hence

\[ \alpha > 0 : R_t(V_{t+1}) = E_t \left( V_{t+1}^{1-\alpha} \right)^{\frac{1}{1-\alpha}}, \]
\[ \alpha = 1 : R_t(V_{t+1}) = \exp \left( E_t \log (V_{t+1}) \right). \]
Proof

- Define $f(x)$
  
  $$F(c, z) = cf(x)$$

  where $x = z/c$ and

  $$f(x) = (1 - \beta + \beta x^{1-\rho})^{1/(1-\rho)}$$

- So

  $$\frac{f'(x)}{f(x)} = \frac{\beta x^{-\rho}}{1 - \beta + \beta x^{1-\rho}}$$

  and

  $$\lim_{\rho \to 1} \frac{f'(x)}{f(x)} = \frac{\beta}{X}.$$ 

- Since $f$ continuous this implies

  $$\lim_{\rho \to 1} f(x) = X^{\beta}$$

  (note this is simply the proof that a CES function converges to a Cobb-Douglas as $\rho \to 1$).
In general $\alpha$ is the relative risk aversion coefficient for static gambles and $\rho$ is the inverse of the intertemporal elasticity of substitution for deterministic variations.

Suppose consumption today is $c$ today and consumption tomorrow is uncertain: \(\{c_L, \bar{c}, \bar{c}, \ldots\}\) or \(\{c_H, \bar{c}, \bar{c}, \ldots\}\), each has prob $\frac{1}{2}$.

Utility today:

\[
V = F\left(c, G^{-1}\left(\frac{1}{2}G(V_L) + \frac{1}{2}G(V_H)\right)\right)
\]

where $V_L = F(c_L, \bar{c})$ and $V_H = F(c_H, \bar{c})$.

Curvature of $G$ determines how adverse you are to the uncertainty.

- If $G$ is linear you only care about the expected value.
- If not, this is the same as the definition of a certainty equivalent:
  \[
  G(\hat{V}) = \frac{1}{2}G(V_L) + \frac{1}{2}G(V_H).
  \]
If consumption is deterministic: we have the usual standard
time-separable expected discounted utility with discount factor \( \beta \)
and IES = \( \frac{1}{\rho} \), risk aversion \( \alpha = \rho \).

Proof: If no uncertainty, then \( R_t (V_{t+1}) = V_{t+1} \) and
\( V_t = F(c_t, V_{t+1}) \). With a CES functional form for \( F \), we
recover CRRA preferences:

\[
V_t = \left((1 - \beta)c_t^{1-\rho} + \beta V_{t+1}^{1-\rho}\right)^{\frac{1}{1-\rho}}
\]

\[
W_t = (1 - \beta)c_t^{1-\rho} + \beta W_{t+1} = (1 - \beta) \sum_{j=0}^{\infty} \beta^j c_{t+j}^{1-\rho},
\]

where \( W_t = V_t^{1-\rho} \).
Similarly, if $\alpha = \rho$, then the formula

$$V_t = \left( (1 - \beta)c_t^{1-\rho} + \beta (E_t V_{t+1}^{1-\alpha})^{\frac{1-\rho}{1-\alpha}} \right)^{\frac{1}{1-\rho}}$$

simplifies to

$$V_t^{1-\rho} = (1 - \beta)c_t^{1-\rho} + \beta (E_t V_{t+1}^{1-\alpha})$$

Define $W_t = V_t^{1-\rho}$, we have

$$W_t = (1 - \beta)c_t^{1-\rho} + \beta E_t (W_{t+1}),$$

i.e. expected utility.
Simple example with two lotteries:

Lotteries:
- lottery $A$ pays in each period $t = 1, 2, \ldots, ch$ or $cl$, the probability is $\frac{1}{2}$ and the outcome is iid across period;
- lottery $B$ pays starting at $t = 1$ either $ch$ at all future dates for sure, or $cl$ at all future date for sure; there is a single draw at time $t = 1$.

With expected utility, you are indifferent between these lotteries, but with EZ lottery B is preferred iff $\alpha > \rho$.

In general, early resolution of uncertainty is preferred if and only if $\alpha > \rho$ i.e. risk aversion $> \frac{1}{I(ES)}$. This is another way to motivate these preferences, since early resolution seems intuitively preferable.
Resolution of uncertainty

- For lottery A, the utility once you know your consumption is either $c_h$, or $c_l$, since

$$V_h = F(c_h, V_h) = \left((1 - \beta)c_h^{1-\rho} + \beta V_h^{1-\rho}\right)^{\frac{1}{1-\rho}}.$$  

The certainty equivalent before playing the lottery is

$$G^{-1}\left(\frac{1}{2}G(c_h) + \frac{1}{2}G(c_l)\right) = \left(\frac{1}{2}c_h^{1-\alpha} + \frac{1}{2}c_l^{1-\alpha}\right)^{\frac{1}{1-\alpha}}.$$  

- For lottery B, the values satisfy

$$W_h^{1-\rho} = (1 - \beta)c_h^{1-\rho} + \beta \left(\frac{1}{2}W_h^{1-\alpha} + \frac{1}{2}W_l^{1-\alpha}\right)^{\frac{1-\rho}{1-\alpha}},$$

$$W_l^{1-\rho} = (1 - \beta)c_l^{1-\rho} + \beta \left(\frac{1}{2}W_h^{1-\alpha} + \frac{1}{2}W_l^{1-\alpha}\right)^{\frac{1-\rho}{1-\alpha}},$$
Resolution of uncertainty

- We want to compare \( G^{-1} \left( \frac{1}{2} G (W_h) + \frac{1}{2} G (W_l) \right) \) to \( G^{-1} \left( \frac{1}{2} G (c_h) + \frac{1}{2} G (c_l) \right) \).

- Note that the function \( x \to x^{\frac{1-\rho}{1-\alpha}} \) is concave if \( 1 - \rho < 1 - \alpha \), i.e. \( \rho > \alpha \), and convex otherwise. As a result, if \( \rho > \alpha \),

\[
\left( \frac{1}{2} W_h^{1-\alpha} + \frac{1}{2} W_l^{1-\alpha} \right)^{\frac{1-\rho}{1-\alpha}} \geq \frac{1}{2} (W_h^{1-\alpha})^{\frac{1-\rho}{1-\alpha}} + \frac{1}{2} (W_l^{1-\alpha})^{\frac{1-\rho}{1-\alpha}}
\]

\[
= \frac{1}{2} W_h^{1-\rho} + \frac{1}{2} W_l^{1-\rho}
\]

- Also

\[
W_h^{1-\rho} \geq (1 - \beta)c_h^{1-\rho} + \beta \left( \frac{1}{2} W_h^{1-\rho} + \frac{1}{2} W_l^{1-\rho} \right)
\]

\[
W_l^{1-\rho} \geq (1 - \beta)c_l^{1-\rho} + \beta \left( \frac{1}{2} W_h^{1-\rho} + \frac{1}{2} W_l^{1-\rho} \right)
\]
These results imply that if \( \rho > \alpha \) then

\[
\frac{W_h^{1-\rho} + W_l^{1-\rho}}{2} \geq \frac{c_h^{1-\rho} + c_l^{1-\rho}}{2}.
\]

in which case the certainty equivalent of lottery A is higher than the certainty equivalent of lottery B and agents prefer late to early resolution of uncertainty.

Technically, EZ is an extension of EU which relaxes the independence axiom. Recall the independence axiom is: if \( x \succeq y \), then for any \( z, \alpha : \alpha x + (1 - \alpha) z \succeq \alpha y + (1 - \alpha) z \). With EZ, “Intertemporal composition of risk matters”: we cannot reduce compound lotteries.
Euler’s Theorem:

We have

\[ V_t = \left((1 - \beta) C_t^{1-\rho} + \beta R_t (V_{t+1})^{1-\rho}\right)^{\frac{1}{1-\rho}} \]

where

\[ R_t (V_{t+1}) = \left(E_t \left(V_{t+1}^{1-\alpha}\right)\right)^{\frac{1}{1-\alpha}} \]

Since \( V_t \) is homogenous of degree one, Euler’s theorem implies

\[ V_t = MC_t C_t + E_t M V_{t+1} V_{t+1} \]
Euler equation:

Taking derivatives:

\[ MC_t = \frac{\partial V_t}{\partial C_t} = (1 - \beta) V_t^\rho C_t^{-\rho} \]

and

\[ MV_{t+1} = \frac{\partial V_t}{\partial R_t (V_{t+1})} \frac{\partial R_t (V_{t+1})}{\partial V_{t+1}} \]

where

\[ \frac{\partial V_t}{\partial R_t (V_{t+1})} = V_t^\rho \beta R_t (V_{t+1})^{-\rho} \]

and

\[ \frac{\partial R_t (V_{t+1})}{\partial V_{t+1}} = R_t (V_{t+1})^\alpha V_{t+1}^{-\alpha} \]

This implies

\[ MV_{t+1} = \beta V_t^\rho R_t (V_{t+1})^{\alpha-\rho} V_{t+1}^{-\alpha} \]
Define the intertemporal marginal rate of substitution as

\[
S_{t,t+1} = \frac{MV_{t+1}MC_{t+1}}{MC_t} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\rho} \left( \frac{V_{t+1}}{R_t(V_{t+1})} \right)^{\rho-\alpha}
\]

- The first term is familiar. The second term is next period’s value relative to its certainty equivalent.
- If \( \rho = \alpha \) or there is no uncertainty so that \( V_{t+1} = R_t(V_{t+1}) \) this term equals unity.
Household wealth:

- Start with the value function:

\[ V_t = M C_t C_t + E_t M V_{t+1} V_{t+1} \]

- Divide by \( M C_t \):

\[ \frac{V_t}{M C_t} = C_t + E_t \left( \frac{M V_{t+1} M C_{t+1}}{M C_t} \right) \frac{V_{t+1}}{M C_{t+1}} \]

- Define

\[ W_t = \frac{V_t}{M C_t} \]

then

\[ W_t = C_t + E_t S_{t,t+1} W_{t+1} \]

is the present-discounted value of wealth.
Define the cum-dividend return on wealth:

\[ R_{m,t+1} = \frac{W_{t+1}}{W_t - C_t} \]

Note that

\[ W_{t+1} = \frac{V_{t+1}}{MC_{t+1}} = \frac{V_{t+1}^{1-\rho} C_{t+1}^{\rho}}{1 - \beta} \]

Hence

\[ R_{m,t+1} = \frac{V_{t+1}^{1-\rho} C_{t+1}^{\rho}}{V_t^{1-\rho} C_t^{\rho} - C_t} = \left( \frac{C_{t+1}}{C_t} \right)^{\rho} \left( \frac{V_{t+1}^{1-\rho}}{V_t^{1-\rho} - (1 - \beta) C_t^{1-\rho}} \right) \]

Now use fact that

\[ V_t^{1-\rho} = (1 - \beta) C_t^{1-\rho} + \beta R_t (V_{t+1})^{1-\rho} \]

to obtain

\[ R_{m,t+1} = \left[ \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\rho} \left( \frac{R_t (V_{t+1})}{V_{t+1}} \right)^{1-\rho} \right]^{-1} \]
Use this equation to solve for the value function relative to the certainty equivalent:

\[
R_{m,t+1}^{-1} = \left[ \beta \left( \frac{C_{t+1}}{C_{t}} \right)^{-\rho} \left( \frac{R_{t} (V_{t+1})}{V_{t+1}} \right)^{1-\rho} \right]
\]

\[
\frac{V_{t+1}}{R_{t} (V_{t+1})} = \left( \beta R_{m,t+1} \left( \frac{C_{t+1}}{C_{t}} \right)^{-\rho} \right)^{1/(1-\rho)}
\]

Comment: we can use this to directly evaluate the cost of uncertain returns and consumption.
From above:

\[ S_{t,t+1} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\rho} \left( \frac{V_{t+1}}{R_t V_{t+1}} \right)^{\rho-\alpha} \]

\[ = \beta^\theta R_{m,t+1}^{\theta-1} \frac{C_{t+1}^{-\theta}}{C_t^{\psi}} \]

where

\[ \theta = \frac{1 - \alpha}{1 - \rho} \quad \text{and} \quad \psi = 1/\rho \]

Note if \( \rho = \alpha \) we have

\[ S_{t,t+1} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\rho} \]

Now take logs

\[ \log S_{t,t+1} = \theta \log \beta - \frac{\theta}{\psi} \Delta c_{t+1} - (1 - \theta) r_{m,t+1} \]
Expected returns

- The return on the $ith$ asset satisfies:

$$E_t S_{t,t+1} R_{t,t+1}^i = 1$$

- Taking logs:

$$\log \left( \frac{E_t R_{t,t+1}^i}{R_{t+1}^f} \right) = -\text{cov}(\log S_{t,t+1}, \log R_{t,t+1}^i)$$

$$= \frac{\theta}{\psi} (\text{cov} (\Delta c_{t+1}, r_{i,t+1}))$$

$$+ (1 - \theta) \text{cov}(r_{m,t+1}, r_{i,t+1})$$

- Epstein-Zin is a linear combination of the CAPM and the CCAPM model.
For the market return we have

\[ \log \left( \frac{E R_m}{R_f} \right) = \theta \psi \text{cov} (\Delta c, r_m) + (1 - \theta) \sigma_m^2 \]

We can write as:

\[ r_m + \frac{\sigma_m}{2} = r_f + \frac{\theta}{\psi} \text{cov} (\Delta c, r_m) + (1 - \theta) \sigma_m^2 \]
Special case: $\rho = 1$

- In this case

$$R_{m,t+1} = \left( \beta \left( \frac{C_{t+1}}{C_t} \right)^{-1} \right)^{-1}$$

and

$$\sigma_{\Delta c} = \sigma_m$$

- This implies that:

$$\log \left( \frac{ER_m}{R^f} \right) = \sigma_m^2$$
Risk free rate:

- We have:
  \[ \log R_{t+1}^f = \log E_t \exp(-\log S_{t,t+1}) \]

- In logs:
  \[ r_{t+1}^f = -\theta \log \beta + \frac{\theta}{\psi} E_t \Delta c_{t+1} + (1 - \theta) r_m \]
  \[ - \left( \frac{\theta}{\psi} \right)^2 \frac{\sigma_{\Delta c}^2}{2} - (1 - \theta)^2 \frac{\sigma_m^2}{2} - \frac{\theta (1 - \theta)}{\psi} \text{cov}(\Delta c, r_m) \]

- Substitute in for the market return to obtain:
  \[ (1 - \theta) r_m = (1 - \theta) r_f - \frac{(1 - \theta) \sigma_m}{2} + \]
  \[ + (1 - \theta)^2 \sigma_m + \frac{(1 - \theta) \theta}{\psi} \text{cov}(\Delta c, r_m) \]
Risk free rate:

- Simplify

\[ r_f^t = -\log \beta + \frac{1}{\psi} E_t \Delta c_{t+1} - \frac{\theta}{\psi^2} \frac{\sigma^2_{\Delta c}}{2} - (1 - \theta) \frac{\sigma^2_m}{2} \]

- Again if \( \rho = \alpha \) so \( \theta = 1 \) we have the standard risk-free rate equation.

- If \( \alpha > \rho \) then \( \theta < 1 \) and the volatility from the market return reduces the real interest rate.
Let

\[ \Delta C_{t+1} = g + \sigma_c \varepsilon_{t+1} \]

Let \( v_t = \frac{V_t}{C_t} \) and write value function as

\[
v_t = \left( 1 - \beta + \beta E_t \left( v_{t+1}^{1-\alpha} \left( \frac{C_{t+1}}{C_t} \right)^{1-\alpha} \right)^{\frac{1-\rho}{1-\alpha}} \right)^{\frac{1}{1-\rho}}
\]

Since consumption is iid \( v \) is constant.
With $v_t = v$

$$S_{t,t+1} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\rho} \left( \frac{V_{t+1}}{R_t (V_{t+1})} \right)^{\rho - \alpha}$$

$$= \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\alpha} \left( \frac{1}{E_t \left( \left( \frac{C_{t+1}}{C_t} \right)^{1-\alpha} \right)} \right)^{-(1-\theta)}$$

Take logs:

$$\log S_{t,t+1} = \log \beta - \alpha \Delta c_{t+1} + (1 - \theta) \log E_t \exp((1 - \alpha)\Delta c_{t+1})$$

$$= \log \beta - \alpha \Delta c_{t+1} + (\alpha - \rho) g + (1 - \theta) (1 - \alpha)^2 \frac{\sigma_c}{2}$$
Risk free rate with iid consumption

- Risk free rate:

\[ r_f = - \log E_t S_{t,t+1} = - \left( E_t \log S_{t,t+1} + \frac{\sigma_s^2}{2} \right) \]

\[ = - \log \beta + \rho g - \left[ (1 - \theta)(1 - \alpha) + \alpha^2 \right] \frac{\sigma_c^2}{2} \]

- If \( \rho = \alpha \) this is the standard expression

\[ r_f = - \log \beta + \rho g - \rho^2 \frac{\sigma_c^2}{2} \]
Log dividend price ratio: Conjecture a constant $q$

$$q = E_t S_{t,t+1} \frac{C_{t+1}}{C_t} (1 + q)$$

where

$$\log S_{t,t+1} + \Delta c_{t+1} = \log \beta + (1 - \alpha) \Delta c_{t+1}$$
$$+ (1 - \theta) \log E_t \exp((1 - \alpha) \Delta c_{t+1})$$
$$= \log \beta + (1 - \alpha) \Delta c_{t+1}$$
$$+ (\alpha - \rho) g + (1 - \theta) (1 - \alpha)^2 \frac{\sigma_c}{2}$$
So the price-dividend ratio satisfies:

\[
\log \frac{q}{1 + q} = \log \beta + (1 - \rho)g - (1 - \alpha)^2 \theta \frac{\sigma^2_c}{2}
\]

\[
= -r_f + \left( g + \frac{\sigma^2_c}{2} \right) - \alpha \sigma^2_c
\]

where the term in brackets is expected consumption growth \( \log(\text{E}_t C_{t+1}/C_t) \).

Hence this is a risk-adjusted Gordon growth formula.
The risk premium on a consumption claim is then

$$\log E_t R_{t+1} = \log E_t \frac{q + 1}{q} \frac{C_{t+1}}{C_t}$$

so that

$$r_m + \frac{\sigma_m}{2} - r_f = \alpha \sigma_c^2$$
Consumption-Wealth Ratio:

- Start with the identity:
  \[ W_{t+1} = R_{m,t+1} (W_t - C_t) \]

  to obtain the log-linear equation:
  \[ \Delta w_{t+1} = r_{m,t+1} + k + \left( 1 - \frac{1}{\rho} \right) (c_t - w_t) \]

  where \( \rho = 1 - \exp(c - w) \).

- Rearrange to obtain
  \[
  (1 - \rho) (c_t - w_t) = \rho r_{m,t+1} - \rho \Delta w_{t+1} + \rho k \\
  = \rho r_{m,t+1} + \rho [\Delta (c_{t+1} - w_{t+1}) - \Delta c_{t+1}] + \rho k
  \]

- Present value relationship:
  \[
  c_t - w_t = \rho (r_{m,t+1} - \Delta c_{t+1}) + \rho (c_{t+1} - w_{t+1}) + \rho k \\
  = \sum_{s=1}^{\infty} \rho^s [r_{m,t+s} - \Delta c_{t+s}] + \frac{\rho}{1 - \rho} k
  \]
Now combine the risk free and market rate Euler equations to obtain:

$$r_{m,t+s} - \Delta c_{t+s} = (1 - \psi) r_{m,t+s} - \mu_m$$

where $\mu_m$ is a constant that depends on conditional covariances etc..

$$c_t - w_t = (1 - \psi) E_t \sum_{s=1}^{\infty} \rho^s r_{m,t+s} + \frac{\rho (\kappa - \mu_m)}{1 - \rho}$$

The consumption-wealth ratio is an increasing function of expected future returns if the IES $< 1$.

Note, we started with an identity and combined it with the Euler equation for safe vs risky returns for a given IES. Thus these expressions are general and do not depend specifically on EZ preferences.
Unexpected changes in consumption

- Now use

\[
c_{t+1} - E_t c_{t+1} = W_{t+1} - E_t W_{t+1} \\
+ (1 - \psi) (E_{t+1} - E_t) \sum_{s=1}^{\infty} \rho^s r_{m,t+s+1} \\
= r_{m,t+1} - E_t r_{m,t+1} \\
+ (1 - \psi) (E_{t+1} - E_t) \sum_{s=1}^{\infty} \rho^s r_{m,t+s+1}
\]

- Unexpected returns increase consumption growth.
- Unexpected future returns increase current consumption growth if the IES < 1.
Some comments

- If returns are not forecastable, the consumption-wealth ratio is a constant.
- In this case, consumption volatility equals the volatility of wealth, or equivalently the market return.
- In the data this is obviously not true – hence returns must be predictable.
Asset pricing implications:

We can now compute

\[ \text{cov}_t (r_{i,t+1}, \Delta c_{t+1}) = \sigma_{ic} = \sigma_{im} + (1 - \psi) \sigma_{ih} \]

where \( \sigma_{ih} \) is the covariance of \( r_{i,t+1} \) with the surprise in future market returns:

\[ \sigma_{ih} = \text{cov}(r_{i,t+1}, (E_{t+1} - E_t) \sum_{s=1}^{\infty} \rho^s r_{m,t+s+1}) \]
Using EZ preferences, the risk premium is:

\[ E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} = \theta \frac{\sigma_{ic}}{\psi} + (1 - \theta) \sigma_{im} \]

The risk premium for asset \( i \) depends on its covariance between current returns and its covariance with news about future market returns:

\[ E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} = \alpha \sigma_{im} + (\alpha - 1) \sigma_{ih} \]

where \( \alpha \) is the coefficient of relative risk aversion.

Note we don’t need to know the IES or consumption growth to price risk in this framework.
EZ preferences and the Equity premium puzzle:

- For EZ preferences we can now write the risk premium on the market return as:

\[ E_t r_{m,t+1} - r_{f,t+1} + \frac{\sigma^2_m}{2} = \alpha \sigma^2_m + (\alpha - 1) \sigma_{mh} \]

- If returns are unforecastable, \( \sigma_{ih} = 0 \). Given \( \sigma_{im} = 0.17 \) we need \( \alpha = 2 \) to obtain a risk premium of 6%. So we succeed in matching the risk premium with low relative risk aversion but fail on the fact that the consumption-wealth ratio will be a constant, and consumption volatility should equal wealth volatility.

- If there is mean reversion and future returns are negatively correlated with current returns then \( \sigma_{mh} < 0 \) we would need a higher \( \alpha \). Since mean-reversion is difficult to determine, the estimate could be substantially higher.
Predictable consumption growth:

- Consumption growth:
  \[
  \Delta c_{t+1} = g + x_t + u_t \\
  x_t = \phi x_{t-1} + v_t
  \]

- Again use Campbell-Shiller decomposition:
  \[
  r_{t+1} = \rho q_{t+1} - q_t + \Delta c_{t+1}
  \]
  where \( q_t = P_t / C_t \) the price of a consumption claim and
  \[
  \rho = \frac{q}{1 + q} < 1
  \]

- Solving forward
  \[
  q_t = E_t \sum_{s=1}^{\infty} \rho^s [\Delta c_{t+s+1} - r_{t+s+1}]
  \]
Euler equation

$$\Delta c_{t+s} = \psi r_{t+s}$$

where $\psi$ is the IES so that

$$[\Delta c_{t+s+1} - r_{t+s+1}] = \left[ 1 - \frac{1}{\psi} \right] \Delta c_{t+s+1}$$

The price-dividend ratio satisfies

$$q_t = \left( \frac{\rho \phi}{1 - \rho \phi} \right) \left[ 1 - \frac{1}{\psi} \right] x_t$$
Implications

- If the $IES > 1$ then an increase in current consumption growth causes an increase in the price-dividend ratio.

- Intuition: A persistent increase in consumption growth provides news about future cash flows and discount rates that go in opposite directions.

- If the IES is high, interest rates don’t need to move very much in response to the change in consumption growth. The cash flow effect dominates.
We need a high $\phi$ to get large volatility in the price-dividend ratio.

But this comes from predictable dividend growth not from time-varying returns (the risk free rate is moving but the risk premium is not).
Time-varying volatility:

- Now add time-varying volatility:

\[ x_t = \phi x_{t-1} + \sigma_t u_t \]
\[ \sigma_t = (1 - \gamma)\sigma + \gamma \sigma_{t-1} + v_t \]

We then have a solution of the form

\[ q_t = \left( \frac{\rho \phi}{1 - \rho \phi} \right) \left[ 1 - \frac{1}{\psi} \right] x_t + a \sigma_t^2 \]

where \( a > 0 \) if \( IES > 0 \) and risk-aversion > 0.

- We will also get time-varying risk premia – “discount rate news” that offsets the “cash flow” news of the consumption growth shock.

- In other words, we need time-varying volatility to match the equity premium combined with persistent movements in consumption growth to match the volatility of the price dividend ratio.
Calibration and empirical implementation

- **Calibration:**
  - $\text{IES} = 1.5, \ \alpha = 10$
  - Very high persistence and volatility for shocks to volatility and persistent consumption-growth process.

- **Issues to think about:**
  - $\text{IES} > 1$ is controversial.
  - Difficult to estimate long-run risk.