Lecture 4: Campbell-Shiller Decomposition

Simon Gilchrist
Boston University and NBER

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Introduce a log-linear approximation to the present-value identity (prices = Present Discounted Value [PDV] of dividends).

Use this approximation to discuss the **sources of stock price volatility**.

This formula has proven extremely useful in applied work, because it is easy to use it in conjunction with linear time series models (e.g. VARs). Also known as “Discount Rate - Cash Flow Decompositions”
Present Discounted Values

- Start from the definition of return:

\[ R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} . \]

- Rewrite this as “price equal PDV of dividends” by rewriting and iterating forward:

\[ P_t = \frac{D_{t+1} + P_{t+1}}{R_{t+1}} \]

\[ = \frac{D_{t+1}}{R_{t+1}} + \frac{D_{t+2}}{R_{t+1}R_{t+2}} + \ldots \]

\[ = \sum_{k=1}^{\infty} \frac{D_{t+k}}{R_{t,t+k}}, \]

where \( R_{t,t+k} \) is the **realized** return on the asset (not the risk-free rate!!!) from \( t \) to \( t + k \) :

\[ R_{t,t+k} = R_{t+1} \times \ldots \times R_{t+k}. \]
Log Returns.

- Use lowercase letters for logs: $r_t = \log(1 + R_t)$, $p_t = \log(P_t)$, $d_t = \log(D_t)$.

- Compute log-return as:

  $$r_{t+1} = \log(P_{t+1} + D_{t+1}) - \log(P_t)$$

  $$= p_{t+1} + \log \left( 1 + \frac{D_{t+1}}{P_{t+1}} \right) - p_t$$

  $$= p_{t+1} - p_t + \log \left( 1 + \exp (d_{t+1} - p_{t+1}) \right)$$
Do a first-order Taylor approximation to the function 
\[ f(x) = \log(1 + \exp(x)) \] around \( x = \bar{d} - \bar{p} \) where \( d \) denotes the sample average value:

\[
\log(1 + \exp(d_{t+1} - p_{t+1})) \simeq k + (1 - \rho)(d_{t+1} - p_{t+1}),
\]

with:

\[
\rho = \frac{1}{1 + \exp(d - p)}
\]

\[
k = - \log \rho + (1 - \rho) \log(1/\rho - 1).
\]

This yields:

\[
r_{t+1} = \rho p_{t+1} - p_t + k + (1 - \rho)d_{t+1},
\]

\[
p_t = \rho p_{t+1} + k + (1 - \rho)d_{t+1} - r_{t+1}.
\]
Iterating forward yields:

\[ p_t = \frac{k}{1 - \rho} + \sum_{j \geq 1} \rho^{j-1} ((1 - \rho) d_{t+j} - r_{t+j}) , \]

\[ = \frac{k}{1 - \rho} + \sum_{j \geq 0} \rho^j ((1 - \rho) d_{t+1+j} - r_{t+1+j}) . \]

This identity holds for any returns, prices and dividends ex-post. (And thus ex-ante, if you take conditional expectations of this equation.) It is an accounting identity without any behavioral assumption.
From the log-linearized return equation we have:

\[ p_t - d_t = \rho(p_{t+1} - d_{t+1}) + k + \Delta d_{t+1} - r_{t+1}. \]

Iterating forward:

\[ p_t - d_t = \frac{k}{1 - \rho} + \sum_{j \geq 0} \rho^j (\Delta d_{t+1+j} - r_{t+1+j}) \]

Variation in the price-dividend ratio occurs because of variation in dividend growth or discount factors.
Start with the return equation

\[ r_{t+1} = \rho p_{t+1} - p_t + k + (1 - \rho)d_{t+1}, \]

Now apply \( E_{t+1} - E_t \) to both sides:

\[ r_{t+1} - E_t r_{t+1} = \rho (p_{t+1} - E_t p_{t+1}) + (1 - \rho) (d_{t+1} - E_t d_{t+1}) \]

From the price-dividend expression

\[ p_{t+1} - E_t p_{t+1} = (E_{t+1} - E_t) \sum_{j \geq 0} \rho^j (\Delta d_{t+2+j} - r_{t+2+j}) \]

\[ + d_{t+1} - E_t d_{t+1} \]

which implies:

\[ r_{t+1} - E_t r_{t+1} = \rho (E_{t+1} - E_t) \sum_{j \geq 0} \rho^j (\Delta d_{t+2+j} - r_{t+2+j}) \]

\[ + (d_{t+1} - E_t d_{t+1}) \]
We then have the expression

\[ r_{t+1} - E_t r_{t+1} = (E_{t+1} - E_t) \sum_{s \geq 0} \rho^s \Delta d_{t+1+s} \]

\[ - (E_{t+1} - E_t) \sum_{s \geq 1} \rho^s r_{t+1+s}. \]

An unexpectedly good stock return must occur because either the current dividend went up, or expectations of future dividends go up, or because expectations of future returns go down.

The first two terms are a standard “cash flow effect” and the second is an expected return or risk premium effect: the price goes up if the risk premium or risk-free interest rate go down.
Hence, stock price volatility can come from either volatility of future dividends or volatility of expected future returns. Which of these terms contribute more to volatility empirically?

Fit a VAR to dividend growth and returns and use the VAR to compute the implied decomposition:

\[
\begin{pmatrix}
\Delta d_{t+1} \\
r_{t+1} \\
x_{t+1}
\end{pmatrix}
= A(L)
\begin{pmatrix}
\Delta d_t \\
r_t \\
x_t
\end{pmatrix}
+ \begin{pmatrix}
\varepsilon^d_{t+1} \\
\varepsilon^r_{t+1} \\
\varepsilon^x_{t+1}
\end{pmatrix},
\]

Iterating on this VAR one can compute the forecast \( E_t \Delta d_{t+1+j} \) and then the revision of the forecast.
Companion form:

Let

\[ y_t = (\Delta d_{t+1}, r_{t+1}, x_{t+1})' \]

and define the 3x1 vector:

\[ z_t = [y_t \ldots y_{t-p}]' \]

Then \( z_t \) follows an AR1 process:

\[ z_{t+1} = A z_t + \nu_{t+1} \]

with

\[ \nu_t = [\varepsilon_t' \ 0 \ldots 0], \quad E\varepsilon_t\varepsilon_t' = \Sigma. \]
The expected PDV can be computed as:

$$E_t \sum_{j \geq 0} \beta^j z_{t+j} = \sum_{j \geq 0} \beta^j A^j z_t = (I - \beta A)^{-1} z_t$$

Premultiplying by the correct row vector to obtain the forecast of the PV of a component of $z$. 
Using this PDV we have

\[(E_{t+1} - E_t) \sum_{j \geq 0} \beta^j z_{t+j} = z_t + \beta (I - \beta A)^{-1} z_{t+1} - (I - \beta A)^{-1} z_t\]

\[= z_t + \beta (I - \beta A)^{-1} (Az_t + \varepsilon_{t+1}) - (I - \beta A)^{-1} z_t\]

The terms involving \(z_t\) cancel so that

\[(E_{t+1} - E_t) \sum_{j \geq 0} \beta^j z_{t+j} = \beta (I - \beta A)^{-1} \varepsilon_{t+1}.\]
Dividends are difficult to predict but returns are predictable as we saw before.

This implies that changes in discount rates account for all of the volatility of revisions to returns.

Since the risk-free interest rate does not move much in the data, it means the changes in expected returns are mainly changes in risk premia.

Implication – we need to study models with time-varying risk premia
  - Campbell-Cochrane, which has time-varying risk aversion.
  - Stochastic volatility in consumption growth which leads to higher risk aversion when consumption volatility is high vs low.
A Simple Model of Predictability:

- Stochastic processes:

\[
\begin{align*}
    x_t &= bx_{t-1} + \delta_t \\
    r_{t+1} &= x_t + \varepsilon_{t+1}^r \\
    \Delta d_{t+1} &= \varepsilon_{t+1}^d
\end{align*}
\]
Model Solution:

- Price-dividend ratio:
  \[ d_t - p_t = E_t \sum_{j=1}^{\infty} \rho^{j-1} (r_{t+j} - d_{t+j}) = \frac{x_t}{1 - \rho b} \]

- Returns:
  \[ R_{t+1} = \left( 1 + \frac{P_{t+1}}{D_{t+1}} \right) \frac{D_{t+1}/D_t}{P_t/D_t} \]

  Log-linearization:
  \[ r_{t+1} = \rho (p_{t+1} - d_{t+1}) + \Delta d_{t+1} - (p_t - d_t) \]

- Prices:
  \[ \Delta p_{t+1} = -(d_{t+1} - p_{t+1}) + (d_t - p_t) + \Delta d_{t+1} \]
VAR representation:

\[
d_{t+1} - p_{t+1} = b (d_t - p_t) + \frac{1}{1 - \rho b} \delta_{t+1}
\]

\[
r_{t+1} = (1 - \rho b) (d_t - p_t) + \left( \varepsilon_{t+1}^d - \frac{\rho}{1 - \rho b} \delta_{t+1} \right)
\]

\[
\Delta p_{t+1} = (1 - b) (d_t - p_t) + \left( \varepsilon_{t+1}^d - \frac{\rho}{1 - \rho b} \delta_{t+1} \right)
\]
Implications for volatility:

- Assume $\rho = 0.96$ then estimated coefficient of returns on dividend price ratio is $1 - \rho b$ where $b$ is autocorrelation of returns. This implies $b = 0.9$.

- Assume $D/P = 0.04$. Returns as a function of levels:

\[
    r_{t+1} = \frac{1 - \rho b}{D/P} \frac{D_t}{P_t} + \varepsilon_{rt+1}
\]

- If $b = 0$, a 1% rise in the D/P ratio implies that returns must rise 25%.
- If $b = 0.9$ then $1 - b = 0.1$ and $1 - \rho b = 0.14$ and a 1% rise in $D/P$ implies prices and returns rise by 2.5 and 3.4% respectively.
- If $b = 0.96$ then a 1% rise in $D/P$ implies prices and returns rise by 1% and 2% respectively. (This seems like upper-bound on persistence).

- Result: iid dividend growth and persistent price-dividend ratio implies returns must respond more than one for one to the dividend-price change.
Also

\[ \sigma (d - p) = \frac{1}{1 - \rho_b} \sigma (x) \]

\[ = 7.4 \sigma (x) \]

Dividend-price ratio should be much more volatile than returns.

Persistence in returns implies that small variation in returns can have a large effect on the dividend-price ratio – similar to Gordon growth formula

\[ P = \frac{D}{r - g} \]