Lecture 3: Euler-Equation Estimation

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Euler equation tests

- This lecture considers some further implications and tests of the “standard” asset pricing model: for any gross return $R_{t+1}$, we have the equation:

$$E_t \left( \frac{\beta u'(C_{t+1})}{u'(C_t)} R_{t+1} \right) = 1,$$

- Hence for any gross return $R_{t+1}$, we have

$$E_t \left( \frac{\beta u'(C_{t+1})}{u'(C_t)} \left( R_{t+1} - R_{f_{t+1}} \right) \right) = 0.$$

- These equations hold for any asset which you can buy or sell (at the margin): stocks, bonds, options, commodities, etc.

- Key intuition: assets are priced according to their covariance with consumption growth. Assets which pay off when consumption growth is high (in good times) are risky, hence they will need to have high expected returns for investors to want to hold them.
Earlier lecture studies the implication of the Euler equation, plus an assumption that consumption growth is iid and dividends = consumption.

Euler equation methods requires very few assumptions - no need to specify the shocks, the technology (as you would in a GE model), or the endowment process.

The logic of Euler equation methods applies to models others than the CRRA representative agent model.

Example: Suppose utility depends on $c_t$ and other observable $x_t$, say $U(c_t, x_t)$, then we have

$$M_{t+1} = \frac{\beta U_1(c_{t+1}, x_{t+1})}{U_1(c_t, x_t)},$$

and $E_t (M_{t+1} R_{t+1}) = 1$ can similarly be tested.
Euler equation tests

Specify $u(C) = \frac{C^{1-\gamma}}{1-\gamma}$. For any variable $Z_t$ known at time $t$, we have:

$$Z_tE_t \left( \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1} - 1 \right) = 0,$$

Since $Z_t$ known at $t$:

$$E_t \left( Z_t \left( \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1} - 1 \right) \right) = 0$$

Take unconditional expectations:

$$E \left( Z_t \left( \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1} - 1 \right) \right) = 0.$$

This is a set of restrictions that the theory imposes on the joint distribution of consumption growth, returns and $Z_t$. 
Express the Euler equation as:

\[
\left( \frac{\beta u'(C_{t+1})}{u'(C_t)} R_{t+1} - 1 \right) = \varepsilon_{t+1},
\]

where

\[
\varepsilon_{t+1} = \left( \frac{\beta u'(C_{t+1})}{u'(C_t)} R_{t+1} - 1 \right) - E_t \left( \frac{\beta u'(C_{t+1})}{u'(C_t)} R_{t+1} - 1 \right)
\]

By construction \( \varepsilon_{t+1} \) is a forecast error – the difference between realized and expected values. Therefore it is uncorrelated with any variable known at time \( t \):

\[
E(Z_t \varepsilon_{t+1}) = 0
\]

No need for distributional assumption, except that returns and consumption growth must be stationary.
Any stationary variable $Z_t$ realized at time $t$ is a valid instrument. If $Z_t$ is uncorrelated with future returns and consumption, then it does not add any restriction since

$$0 = E\left(Z_t \left( \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1} - 1 \right) \right)$$

$$= E(Z_t)E\left( \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1} - 1 \right).$$

Instrument relevance: $Z_t$ must forecast future returns or future consumption growth.
Empirical Implementation

- This equation holds for any returns $R_{t+1}, Z_t$, so we have $NK$ equations, where $N = \#$ of returns and $K = \#$ of “instruments” $Z_t$, and we have 2 parameters ($\beta, \gamma$).

- Intuitively, we simply pick $\beta$ and $\gamma$ to set the sample version of these equations equal to zero, i.e.

$$\frac{1}{T} \sum_{t=1}^{T} Z_t \left( \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1} - 1 \right) = 0.$$
Exact identification:

- Let $Z_t = 1$ and apply to excess aggregate stock returns: choose $\gamma$ such that

$$\frac{1}{T} \sum_{t=1}^{T} \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \left( R_{t+1}^e - R_{t+1}^f \right) = 0.$$ 

Note: this may not be possible! i.e. with the data there may not be a $\gamma$ s.t. this holds.

- Let $Z_t = 1$ find $\beta$ and $\gamma$ to match both stock returns and the risk-free rate:

$$\frac{1}{T} \sum_{t=1}^{T} \left( \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1}^e - 1 \right) = 0,$$

$$\frac{1}{T} \sum_{t=1}^{T} \left( \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1}^f - 1 \right) = 0.$$

Two eqns in two unknowns. Again, this may not be possible to find both $\beta$ and $\gamma$ in sample.
If more than equations, then of course (generically) cannot set all the equations equal to zero. In practice minimize the weighted errors:

$$\min_{\beta, \gamma} g'_T W_T g_T,$$

where $g_T = \frac{1}{T} \sum_{t=1}^{T} Z_t \left( \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1} - 1 \right)$ and $W_T$ is a weighting matrix.

Can pick $W_T = I$, or $W_T$ so that moments have the same scale, or choose $W$ in a statistically optimal way. Formula for $W_T$ is the inverse of the asymptotic variance of the moment vector:

$$W = \left[ \text{Var} \left( \frac{1}{T} \sum_{t=1}^{T} \left( \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1} - 1 \right) \right) \right]^{-1}$$
Over-identifying tests

- Overidentification: if $NK > \# \text{ of parameters}$ the system is over-identified.

- Test that $\frac{1}{T} \sum_{t=1}^{T} Z_t \left( \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1} - 1 \right)$ is close to zero.
Express a model as
\[ E[f(x_t, b)] = 0 \]

The general GMM estimate \( \hat{b} \) is given by:
\[ a_T g_T(\hat{b}) = 0, \]

where
\[ g_T(b) \equiv \frac{1}{T} \sum_{t=1}^{T} f(x_t, b), \]

and \( a_T \) is a matrix that defines which linear combination of \( g_T(b) \) will be set to zero.
The asymptotic distribution of the GMM estimate is:

\[ \sqrt{T}(\hat{b} - b) \rightarrow N[0, (ad)^{-1}aSa'(ad)^{-1'}], \]

where

\[ d \equiv E \left[ \frac{\partial f}{\partial b'}(x_t, b) \right] = \frac{\partial g_T(b)}{\partial b'}, \]
\[ a \equiv a_T, \]
\[ S \equiv \sum_{j=-\infty}^{\infty} E[f(x_t, b)f(x_{t-j}, b')]. \]

Hansen (1982) gives the sampling distribution of the moments \( g_T(b) \):

\[ \sqrt{T}g_T(\hat{b}) \rightarrow N[0, (I - d(ad)^{-1}a)S(I - d(ad)^{-1}a')]. \]
The efficient estimate is obtained by setting:

\[ a = d' S^{-1} \]

In this case,

\[ \sqrt{T}(\hat{b} - b) \to N[0, (d' S^{-1} d)^{-1}] \]

The quadratic objective function provides a test of over-identifying restrictions:

\[ T J_T = T g_T (\hat{b})' S^{-1} g_T (\hat{b}) = \chi^2(NK - x) \]
Testing parameter restrictions

- Suppose we estimate a model that imposes $R$ restrictions on $b$.
- We can test these restrictions using:

$$T(J_{T,R} - J_T) = \chi^2(R).$$

where $J_{T,R}$ is the value of objective estimated under the restriction.

Comment: when imposing the restrictions we should use the unrestricted weighting matrix. Otherwise the test statistic has a non-central $\chi^2$ distribution.

- The test statistic is constructed as a quadratic function of the difference between the two moment vectors:

$$(g_{T,R} - g_T)W_T(g_{T,R} - g_T)$$

- If we used alternative weighting matrices the difference in the $J$ statistics is not guaranteed to be positive.
Ordinary Least Squares

- $\beta$ solves:

$$\text{Min}_\beta E_T[(y_t - \beta' x_t)^2]$$

- In this case:

$$f(x_t, \beta) = x_t(y_t - \beta' x_t) = x_t \epsilon_t,$$

$$g_T(\beta) = E_T[x_t(y_t - \beta' x_t)],$$

$$a_T = I,$$

$$d = -E_T(x_t x'_t),$$

$$\hat{\beta} = E_T[x_t x'_t^{-1}] E_T(x_t y_t)$$

$$\text{var}(\hat{\beta}) = \frac{1}{T} E_T[x_t x'_t^{-1}] \sum_{j=-\infty}^{\infty} E(\epsilon_t x_t x'_{t-j} \epsilon'_{t-j}) E_T[x_t x'_t^{-1}]$$
An implication of Euler equations

- With Euler equations we have

\[ g_T(b) = \frac{1}{T} \sum_{t=1}^{T} Z_t \varepsilon_{t+1}(b) \]

- We then choose \( b \) to minimize the quadratic

\[ g_T(b) W_T g_T \]

- This is equivalent to choosing \( b \) so that

\[ \frac{d(g_T(b))'}{db} W_T g_T = 0 \]

- In this case, the optimal weighting matrix is \( W_T = V_{gT}^{-1} \) where

\[ V_{gT} = E \left( g_T(b) g_T(b)' \right) \]
Note that in Euler equation case:

\[ E \left( f(x_t, b) f(x_{t-s}, b) \right) = E \left( Z_t \varepsilon_{t+1} \varepsilon'_{t-s+1} Z'_{t-s} \right) \]

If we confine attention to one period returns then \( \varepsilon_t \) is serially uncorrelated and this expression is zero for \( s \neq 0 \). In this case, the variance-covariance matrix can be constructed as

\[ E \left( g_T(b) g_T(b)' \right) = \frac{1}{T} \sum_{t=1}^{T} Z_t \varepsilon_{t+1} \varepsilon'_{t+1} Z'_t \]

If we use multi-period returns then we induce serial correlation and we need a Newey-West style estimator of the variance of the moment matrix.