Lecture 2: Stochastic Discount Factor

Simon Gilchrist
Boston University and NBER

EC 745
Fall, 2013
A stochastic discount factor is a stochastic process \( \{ M_{t,t+s} \} \) such that for any security with payoff \( x_{t+1} \) at time \( t + 1 \) the price of that security at time \( t \) is

\[
P_t = E_t M_{t,t+1} x_{t+1}
\]

Equivalently

\[
E_t M_{t,t+1} R_{t+1}
\]

for

\[
R_{t+1} = \frac{x_{t+1}}{P_t}
\]

Ex: representative agent

\[
M_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}
\]
Issues to think about:

- Heterogeneous agents with complete vs incomplete markets
- SDF must have (roughly) permanent innovations (Alvarez and Jermann, Econometrica 2004)
Basic properties of SDF:

- Risk free rate:
  \[ R^f_{t+1} = E_t M_{t+1} \]

- CAPM – SDF is linear function of market return:
  \[ \log M_{t+1} = a - b R^m_{t+1} \]

- CCAPM – SDF is a function of consumption growth:
  \[ \log M_{t+1} = \log \beta - \gamma \Delta \log C_{t+1} \]

- Compensation for risk:
  \[ E_t R^e_{t+1} - R^f_{t+1} = \frac{\sigma_t(M_{t+1})}{E_t M_{t+1}} \text{corr}_t \left( M_{t+1}, R^e_{t+1} \right) \sigma_t \left( R^e_{t+1} \right) \]

where \( \frac{\sigma_t(M_{t+1})}{E_t M_{t+1}} \) may be interpreted as the market price of risk.
Constructing SDF under complete markets:

- One period world, $s = 1$: $S$ possible states with probability $\pi_s$.
- Let $q_s$ be the state-contingent price – price of an Arrow-Debreu security that payoffs one unit in state $s$. Then the price of a security with payoffs $\{d_s\}_{s=1}^S$ is

$$p(\{d\}) = \sum_{s=1}^S q_s d_s$$

$$= \sum_{s=1}^S \pi_s \frac{q_s}{\pi_s} d_s$$

then we have defined the SDF as

$$m_s = \frac{q_s}{\pi_s}$$

so that

$$p(\{d\}) = \sum_{s=1}^S \pi_s m_s d_s$$

$$= \mathbb{E}m_s d_s$$
Incomplete markets: an example.

- One period economy with $N$ agents
- There is uncertainty about the state $s$
- Trade occurs before uncertainty is resolved.
- Securities: $L$ securities with prices $q_l$ and payoffs $d_{ls}$ in state $s$. 
Agents problem:

- Max
  \[ \max U (c_{i1}, \ldots, c_{iS}) \]

subject to
\[ c_{is} = y_{is} + \sum_{i=1}^{L} q_l \theta_{il} \leq w_i \]

where \( y_{is} \) is the endowment in state \( s \), \( \theta_{il} \) is number of shares in security \( l \) purchased by agent \( i \) and \( w_i \) is initial wealth of agent \( i \).

- FOC:
  \[ \sum_{s=1}^{S} \frac{\partial u_i}{\partial c_s} d_{ls} = q_l \lambda_i \]

where \( \lambda_i \) is agent \( i \)'s Lagrange multiplier on the wealth constraint.
Equilibrium:

- A price vector $q = \{q_l\}$ and shares $\{\theta_{il}\}_{i,l}$ such that each agent maximizes utility and markets clear:

$$\sum_{i=1}^{L} \theta_{il} = 1, \quad l = 1..L$$

- SDF: Consider any agent with $\lambda_i > 0$ then the Euler equation is

$$q_l = \sum_{s=1}^{S} m_{is} d_{ls}$$

where

$$m_{is} = \frac{\partial u_i}{\partial c_s} \frac{\lambda_i}{\lambda_i}$$

- We can use the marginal rate of substitution of any consumer who is unconstrained as an SDF. Agents may not be unconstrained in each state but as long as there are no arbitrage opportunities there exists an SDF.
The system of securities characterized by \( \{d_{ls}\} \) and \( \{q_l\} \) is arbitrage free if there is no vector of portfolio choices \( \{\theta_l\} \) such that both

\[
\sum_{l=1}^{L} q_l \theta_l \leq 0
\]

and for all \( s = 1..S \)

\[
\sum_{l=1}^{L} d_{ls} \theta_l \geq 0
\]

\( > 0 \) for some state \( s \).

In word: there does not exist a portfolio with a zero (or negative) price that gives positive payoffs in all states of the world.
Implications

- If prices $\{q_l\}$ result from a competitive equilibrium then there will be no arbitrage – otherwise demand an infinite amount of the security.

- More generally we can use no arbitrage theory to place restrictions on prices. Example if asset $x$ pays more dividends than asset $y$ in all states then the price of asset $x$ must be (weakly) greater than the price of asset $y$.

- Assets that are redundant, i.e. can be replicated or “spanned” by other assets can be priced even if markets are incomplete.
Example: S=6, l=3

\[ D = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \]

Since the third asset is the first minus the second it can be priced.
Black and Scholes – under NAO, options can be priced as linear combinations of equity and debt payouts.
Some general results on SDF:

- The law of one price holds iff there is (at least) one stochastic discount factor.

- LOOP implies prices are linear functions of payoffs:
  \[ P(x + y) = P(x) + P(y) \]. This implies that for a security that paysoffs \( x \)
  \[ P(x) = \sum w_s x_s \]

Define

\[ m_s = \frac{w_s}{\pi_s} \]

then

\[ P(x) = \sum_s \pi_s m_s x_s \]

so that \( m_s \) is an SDF. Converse is obvious.
Some general results on SDF:

- There are no arbitrage opportunities iff there exists (at least) one strictly positive SDF.
  - NAO: if $x \geq 0$, $p(x) \geq 0$ and if $x_s > 0$ for some $s$ then $p(x) > 0$, use separating-hyperplane argument to argue that SDF exists.
Some general results on SDF:

- The stochastic discount factor is unique iff markets are complete.
  - Suppose markets are complete then there is an asset that pays off one unit in state $s$ where the price of that asset $p_s$ satisfies:
    \[ p_s = m_s \pi_s \]
    This is true for any SDF:
    \[ p_s = m^*_s \pi_s \]
    therefore
    \[ m^*_s = m_s \]
  - Suppose the SDF $m_s$ is unique, assume that the market is incomplete. Then there is a state $j$ which is non-tradeable. Define
    \[ m^*_s = \begin{cases} m_s & \text{if } s \neq j \\ m_s + 1 & \text{if } s = j \end{cases} \]
    The SDF $m^*_s$ will price all tradeable securities hence $m_s$ is not unique.
Hansen-Jagannathan Bounds

Consider any risky return \( R \) and risk-free return \( R^f \) then

\[
E(mR) = 1 \\
E(m) R^f = 1 \\
E(mR) = E(m) R^f
\]

This implies:

\[
cov(m, R) = E(mR) - E(m) E(R) \\
= E(m) R^f - E(m) E(R)
\]
Hansen-Jagannathan Bounds

- Now use bound \((-1 \leq \text{corr}(x, y) \leq 1)\):

\[
-\sigma(x)\sigma(y) \leq \text{cov}(x, y) \leq \sigma(x)\sigma(y)
\]

to obtain

\[
-\sigma(m)\sigma(R) \leq E(m)R^f - E(m)E(R) \leq \sigma(m)\sigma(R)
\]

- Reverse inequality to obtain:

\[
-\frac{\sigma(m)}{E(m)} \leq \frac{E(R) - R^f}{\sigma(R)} \leq \frac{\sigma(m)}{E(m)}
\]
\[
\frac{E(R) - R^f}{\sigma(R)} \text{ is the Sharpe ratio and } \frac{\sigma(m)}{E(m)} \text{ is the market price of risk.}
\]

The Sharpe ratio for the market is \(0.06/0.17 = 35\%\).

This applies to any asset – Sharpe ratios can easily be on the order of 100\% so the price of risk must be very high.

To obtain tight bounds, consider a portfolio with very high sharp ratio.

Under complete markets, the price of risk is the slope of the mean-variance frontier – this defines parabola in \(E(m), \sigma(m)\) space.
For CRRA and random-walk consumption:

\[ \Delta \ln c_t = \mu + \varepsilon_t \]

where

\[ \varepsilon_t \sim N(0, \sigma^2_\varepsilon) \]

Results from log-normal:

\[
E\left( \frac{C_{t+1}}{C_t} \right) = \exp \left( \mu + \frac{1}{2} \sigma^2_\varepsilon \right)
\]

\[
\text{var}\left( \frac{C_{t+1}}{C_t} \right) = \left( e^{\sigma^2_\varepsilon} - 1 \right) e^{2\mu + \sigma^2_\varepsilon}
\]
These results imply:

\[ E(m) = \beta \exp \left[ \gamma \left( -\mu + \gamma \frac{\sigma^2}{2} \right) \right] \]

and

\[ \frac{\sigma(m)}{E(m)} = \left( \exp (\sigma^2 \gamma^2) - 1 \right)^{1/2} \]

where \( \gamma \) is CRRA coefficient.

**Quarterly calibration:** \( \sigma_{\text{var}\epsilon} = 0.036/4 \)

- If \( \gamma = 1 \) price of risk = 0.01
- If \( \gamma = 10 \) price of risk = 0.09
- If \( \gamma = 20 \) price of risk = 0.18
- If \( \gamma = 50 \) price of risk = 0.47