Dynamic Equilibrium with Heterogeneous Agents and Risk Constraints

Rodolfo Prieto

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Abstract

We examine the impact of risk-based portfolio constraints on asset prices in an exchange economy. We show that constrained agents scale down their portfolio and behave locally like power utility investors with risk aversion that depends on current market conditions. In contrast to previous results in the literature, we show that the imposition of constraints dampens fundamental shocks, challenging the idea that risk management rules amplify aggregate fluctuations. We find that risk constraints may give rise to bubbles in asset prices, and connect these results to portfolio imbalances generated by the constraints, asset shortages and the heterogeneity across agents.

JEL Classification: D51, D52, D53, G11, G12.

Key words: Rational bubbles; Endogenous regimes; Exchange economy; Risk constraints; Stochastic volatility.

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†Department of Finance, School of Management, Boston University, 595 Commonwealth Avenue, Boston, MA 02215, USA. E-mail: rprieto@bu.edu.
1 Introduction

Despite the widespread use of (balance sheet) constraints based on risk measures in the financial industry there are, surprisingly, few academic studies analyzing their impact on prices.\footnote{Duffie and Pan (1997) and Jorion (2006) provide an overview of the use of risk measures and constraints in the financial industry.} Recent contributions\footnote{See e.g., Danielsson, Shin, and Zigrand (2011) and references therein.} consist mostly of models which point to an amplifying effect on market volatility, while assuming exogenous credit markets and risk neutrality of market participants. Our goal in this paper is to understand how the presence of risk constraints impact equilibrium quantities by analyzing a dynamic exchange economy populated by heterogeneous risk averse investors.

It is well known that taking into account portfolio constraints in an equilibrium setting is a challenging task. Earlier contributions\footnote{See e.g., Detemple and Murthy (1997), Basak and Cuoco (1998) and Basak and Croitoru (2000), among others.} focus on settings where all agents have logarithmic utility so that the stock price level and volatility are not affected by the imposition of portfolio constraints. In our baseline economy, we depart from the logarithmic utility paradigm and assume there are two classes of agents: power utility agents who are free to choose the composition of their portfolio and logarithmic utility agents who are subject to constraints.

We study two types of risk constraints which capture, parsimoniously, different evaluation and updating frequencies in risk exposures and generate equilibria with possibly multiple regimes. Both types induce shifts in portfolio choice that are locally akin to policies of agents whose higher risk aversion depends on (endogenous) market quantities. In the first example, the constraint imposes a lower bound on the portfolio’s mean-variance ratio, that is, it sets a bound on the risk-return profile which induces a constant reduction from the unconstrained benchmark portfolio. In the second example, the constraint imposes an upper bound on the portfolio’s volatility. It is a pure risk constraint (equivalent to the relative value-at-risk (VaR) or limited-expected-loss (LEL) family of constraints) and induces a time varying reduction from the benchmark unconstrained portfolio.

We construct an equilibrium by identifying a suitable consumption sharing rule which, due to nature of the constraint class, follows an autonomous process whose coefficients are obtained in closed form. The latter makes possible to determine the regions of the state space where the constraints are active independently of stock prices. Utilizing this result, we are able to fully analyze the properties of equilibrium and provide explicit existence results using pathwise comparison arguments. This is one of our methodological contributions. The interest rate and the market price of risk are determined in closed form as functions of the consumption sharing rule only, and thus, solving the model amounts to compute a single linear ordinary differential
equation which describes the price dividend ratio, avoiding the need for an approximate solution whose accuracy is difficult to assess. More importantly, the model provides the following novel insights about the impact of risk constraints on equilibrium prices.

First, negative shocks to fundamentals make risk constraints more likely to bind, lowering the interest rate and raising the market price of risk. These effects are more pronounced in bad times, in line with the empirical literature (see Ferson and Harvey (1991)).

Second, when constraints are imposed on the more risk tolerant agent, who holds a levered position in the stock, stock volatility decreases because constraints narrow the ‘effective’ distribution of risk aversion across agents when they bind, restraining an efficient risk sharing whose dynamic evolution is partly responsible for the volatility of the stock price. This insight is in contrast to recent studies that suggest that risk management rules used by active market participants, who are presumably more risk tolerant, serve to amplify aggregate fluctuations. Since constraints bind in bad times, this result also challenges the idea that capital regulations based on this type of constraints make financial crises larger and more costly.

Third, asset pricing bubbles may arise in our economy to incite unconstrained agents to hold positions that are compatible with market clearing, as local shifts in the interest rate and the market price of risk induced by the constraint may not be sufficient to reach an equilibrium. We characterize the emergence of bubbles in terms of the type of constraints and their ‘cost’. In particular, we show that mean-variance constraints do not generate bubbles whereas relative VaR constraints may give rise to bubbles, depending on their severity and the risk aversion distribution across agents. Our results connect the literature of bubbles in models with continuous trading to the macro-finance literature summarized in Caballero (2006) (see also Caballero and Krishnamurthy (2009)), where ‘bubbly prices’ may arise due to financial asset shortages and various forms of ‘imbalances’, two features that in our model correspond to fixed asset supply and portfolio constraints, respectively.

Our work is related to various strands of literature. The partial equilibrium implications of risk constraints in dynamic settings have been studied by Basak and Shapiro (2001), Cuoco, He, and Isaenko (2008) and Leippold, Trojani, and Vanini (2006), among others. Basak and Danielsson, Shin, and Zigrand (2011) and references therein. In Brunnermeier and Pedersen (2009), margins based on risk-constraints, under certain conditions, may be ‘destabilizing’. This result is novel to this paper and generalizes Hugonnier (2012), who shows that portfolio constraints can generate bubbles in securities in positive net supply even if the economy includes unconstrained investors. All agents in that economy have logarithmic preferences and are endowed with assets. Most papers in this literature have studied the impact (rather than the source) of rational asset pricing bubbles in derivative pricing (see e.g., Cox and Hobson (2005), Heston, Loewenstein, and Willard (2007), and Jarrow, Protter, and Shimbo (2010)). Bubbles are compatible with the existence of an equilibrium in our model because unconstrained agents cannot fully exploit arbitrage opportunities due to standard wealth (solvency) constraints. See Hugonnier and Prieto (2012) for a study of the impact of different credit facilities in an equilibrium model with bubbles.

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Shapiro (2001) argue that constrained investors who face VaR limits are induced to take on a larger risk exposure and experience losses in states which are more costly. A drawback of their model is given by the fact that the portfolio’s VaR is never reevaluated after the initial date. If a trader satisfies the specified risk limit at all times, as in our model, no unappealing incentives arise, and the policy generated is a dynamically consistent risk reduction process that scales down the unconstrained benchmark (see also Cuoco, He, and Isaenko (2008)). Leippold, Trojani, and Vanini (2006) study the policy of a power utility agent under VaR constraints and exogenous stochastic market coefficients. They also study the equilibrium problem using perturbation theory, an approach that is not suitable for existence results.

Within the category of dynamic equilibrium models with portfolio constraints, we highlight the models which are closest to our work. Danielsson, Shin, and Zigrand (2011) feature a risk neutral agent that is subject to volatility constraints that are always binding, value traders and an exogenous and constant interest rate. Constraints increase the volatility of stock prices due to the constrained agent’s demand response to price shocks, which shifts according to changes on his ‘risk appetite’. This behavior also arises in our model through the risk reduction process that scales down the benchmark unconstrained portfolio. In contrast to Danielsson, Shin, and Zigrand (2011), we show that the constraint dampens rather than amplifies fundamental shocks in an environment where the constrained agent is a net borrower and the interest rate is endogenous. Garleanu and Pedersen (2011) incorporate margin constraints in a setting similar to ours. They show how the presence of margins may lead to deviations of the law of one price, a feature that also arises in our model and that we associate to the severity of the risk constraint, fixed asset supply and standard solvency constraints. The regions where the constraint binds are determined jointly with the stock price (e.g., their analysis is based on a two-region conjecture), which makes existence results not readily available. In contrast to our results, this paper does not provide a prediction on stock volatility. Chabakauri (2011) focuses on a dynamic exchange economy where investors share similar CRRA preferences and constrained agents face a borrowing constraint that is always binding. Under no frictions, a no trade equilibrium obtains with constant quantities, which facilitates the comparison across constrained and unconstrained economies. In contrast to our model, all equilibrium quantities are computed numerically, and existence results are not available.

We extend our baseline model in two directions: heterogeneity of beliefs and multiple risky assets. Most of our insights hold with heterogeneous beliefs. One difference is that there may be multiple regimes with active constraints, which in turn generate a rich pattern for stock volatility. Heterogenous beliefs modify the binding regions and the direction of portfolio

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8See Kogan and Uppal (2001) for another application using this approach.
imbalances, and thus the conditions under which bubbles emerge. We show it is straightforward
to introduce multiple risky assets in our framework. In particular, the equilibrium is structurally
identical to the equilibrium in the one-risky-asset economy.

The remainder of the paper is structured as follows. Section 2 presents the model. Section
3 constructs and characterizes the equilibrium. Section 4 investigates the equilibrium implications using the two proposed examples. Section 5 extends the baseline economy to consider heterogeneous beliefs and multiple risky assets. Section 6 concludes. All proofs and technical results are gathered in A. Further details and a fully solved example with multiple risky assets are in B.

2 The economy

2.1 Information structure

We consider a continuous time economy with infinite horizon. The uncertainty in the economy is
represented by a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\) supporting a standard Brownian motion
denoted by \(B\). The filtration \(\mathcal{F} = (\mathcal{F}_t)\) is the augmentation of the filtration generated by the
Brownian motion. We let \(\mathcal{F} = \bigcup_{t \geq 0} \mathcal{F}_t\) determine the true state of nature.

2.2 Securities

There is a single perishable consumption good which serves as the numéraire. The financial
market consists of two assets: a locally riskless bond in zero net supply and one risky asset or
stock in positive supply of one unit. The price of the riskless asset evolves according to

\[
S^0_t = 1 + \int_0^t S^0_\tau r_\tau d\tau,
\]

for some instantaneous interest rate process \(r \in \mathbb{R}\) which is to be determined in equilibrium.
The risky asset is a claim to a strictly positive dividend process of the form

\[
\delta_t = \delta_0 + \int_0^t \delta_\tau (\mu_\delta d\tau + \sigma_\delta dB_\tau),
\]

for some exogenously given constants \((\delta_0, \mu_\delta, \sigma_\delta) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+\). The price process of the risky
asset is denoted by \(S\) and evolves according to

\[
S_t = S_0 + \int_0^t S_\tau (\mu_\delta d\tau + \sigma_\delta dB_\tau) - \int_0^t \delta_\tau d\tau,
\]
for some initial value $S_0 \in \mathbb{R}_+$ and some drift and volatility processes $(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}$ which are to be determined in equilibrium.

2.3 Agents

The economy is populated by two price-taking agents indexed by $k = 1, 2$ with homogeneous beliefs about the state of the economy. Agent $k$ maximizes expected utility over strictly positive consumption plans,

$$U_k(c) = E \left[ \int_0^\infty e^{-\rho t} u_k(c_{kt}) \, dt \right],$$

where $\rho > 0$ is the rate of subjective time preference and period utility is given by

$$u_1(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}, \quad u_2(c) = \log c,$$

with $\gamma > 0$. Agent 2 represents a financial institution/money manager who faces portfolio constraints which limits the amount of risk which can be held while trading. The assumption of logarithmic preferences is necessary to obtain a simple characterization of optimality under portfolio constraints.\(^9\) Importantly, in order to ensure that investors’ expected utilities are uniformly bounded given the dividend process, we impose the following growth condition

$$\rho > \max \left( 0, (1-\gamma)(\mu\delta - \frac{1}{2} \gamma \sigma^2 \delta^2) \right).$$

Agent 2 is initially endowed with $\beta \in \mathbb{R}$ units of the riskless asset and $0 \leq \alpha \leq 1$ units of the risky asset, so that his initial wealth, computed at equilibrium prices, is given by

$$w_2 = \beta + \alpha S_0.$$ (2)

Agent 1’s initial wealth, in turn, is given by $w_1 = S_0 - w_2$. Initial short positions in the riskless asset are allowed as long as initial wealth $w_k$ is strictly positive.

A trading strategy is a pair of processes $(\phi^0, \phi)$ where $\phi^0$ represents the amount invested in the riskless asset and $\phi$ represents the amount invested in the stock. An admissible trading strategy is said to be self-financing given initial wealth $w$ and consumption rate $c$ if the corresponding wealth process

$$W_t = \phi^0_t + \phi_t \geq 0$$ (3)

\(^9\)A similar assumption is used in Detemple and Murthy (1997), Basak and Cuoco (1998), Basak and Croitoru (2000), Pavlova and Rigobon (2008), Schornik (2009), Gallmeyer and Hollifield (2008), and Gärleanu and Pedersen (2011) among others.
satisfies the dynamic budget constraint

\[ W_t = w + \int_0^t (\phi^0_s r_s + \phi_s \mu_s - c_s) \, ds + \int_0^t \phi_s \sigma_s dB_s. \]

Implicit in the definition is the requirement that the trading strategy be such that the above stochastic integrals are well-defined. The nonnegative wealth requirement in (3) is standard in the literature and rules out the possibility of doubling strategies (see Dybvig and Huang (1988)). Agent 2’s trading strategy must also satisfy a portfolio constraint, which is described in the next subsection.

We complete this section by introducing the concept of equilibrium in the economy.

**Definition 1.** An equilibrium is a price system \((S^0, S)\) and a set of consumption plans and trading strategies \(\{c_k, (\phi^0_k, \phi_k)\}\) such that: (i) The consumption plan \(c_k\) maximizes the agent’s utility in (1) and is financed by the trading strategy \((\phi^0_k, \phi_k)\) subject to admissibility and portfolio constraints and (ii) the securities and goods markets clear at all times,

\[ \phi^0_1 + \phi^0_2 = 0, \quad c_1 + c_2 = \delta. \]

### 2.4 Risk constraints

Risk constraints are represented by

\[ C = \{ \pi = \phi/W \in \mathbb{R} : a_1 \pi (\mu_t - r_t) + a_2 (\pi \sigma_t)^2 \leq a_3, \quad \forall t \in [0, \infty) \}, \tag{4} \]

where \((a_2, a_3)\) are nonnegative constants. The set in (4) is a deterministic convex function of two statistics, the portfolio’s net return, \(\pi(\mu_t - r_t)\), and the portfolio’s volatility, \(\pi \sigma_t\). The process \(\pi\) is the proportion of wealth invested in the risky asset. While constraints are not endogenized in the model, we note that their use in the financial world is due to agency problems, default risk, the need to allocate scarce capital and budgeting practices that are encouraged by regulators through capital requirement rules.

We use two examples in this paper that represent, parsimoniously, a larger constraint class. The first example consists of a lower bound on the portfolio’s mean-variance ratio, specified by

\[ (a_1, a_2, a_3) = (-1, L_1, 0), \quad C_1 = \{ \pi \in \mathbb{R} : -\pi (\mu_t - r_t) + L_1 (\pi \sigma_t)^2 \leq 0 \}, \]

with \(L_1 \in (0, \infty)\). This constraint imposes a specific risk/return profile and the quantity \(1/L_1\) is a measure of the agent’s risk bearing capacity. The second example considers a pure risk
constraint which consists of an upper bound on volatility, specified by

\[(a_1, a_2, a_3) = (0, 1, L^2_2), \quad C_2 = \{ \pi \in \mathbb{R} : (\pi \sigma_t)^2 \leq L^2_2 \}.\]

The parameter \(L_2 \in [0, \infty)\) is a measure of the agent’s risk bearing capacity. Volatility constraints are equivalent to the family of limited-expected-loss constraints (LEL) (see Pirvu and Zitković (2009)), and are also studied in Gărleanu and Pedersen (2007) and Danielsson, Shin, and Zigrand (2011) as relative value-at-risk constraints.

3 Construction of equilibrium

In this section, we gather results about individual optimality and we construct an equilibrium based on the consumption sharing rule and the optimality conditions of both agents. We also describe conditions under which prices include a bubble component.

3.1 Individual optimality

Agent 1 faces a complete financial market and thus uses the process

\[\xi_1t = e^{-\int_0^t (r_s + \frac{1}{2} \theta_s^2) ds - \int_0^t \theta_s dB_s}\]

as a state price density. The process \(\theta\) is referred to as the market price of risk and is defined by

\[\mu_t = r_t + \sigma_t \theta_t \quad (5)\]

such that \(\int_0^t \theta_s^2 ds < \infty\) for all \(t \geq 0\). It is well-known that the solution of the unconstrained agent’s problem is given by

\[c_{1t} = (e^{at} y_1 \xi_{1t})^{-\frac{1}{\gamma}}, \quad \sigma_t \pi_{1t} = \theta_t + h_{1t} (\xi_{1t} W_{1t})^{-1}, \quad (6)\]

where the process

\[W_{1t} = \frac{1}{\xi_{1t}} E \left[ \int_t^\infty \xi_{1s} c_{1s} ds \bigg| F_t \right] = \frac{1}{\xi_{1t}} \left[ H_{1t} - \int_0^t \xi_{1s} c_{1s} ds \right] \quad (7)\]
represents the agent’s wealth along the optimal path, $h_1$ is the integrand in the stochastic integral representation of the martingale $H_1$ and the strictly positive constant $y_1$ is chosen in such a way that $W_{10} = w_1$.

On the other hand, since agent 2 faces portfolio constraints, the process $\xi_1$ no longer identifies his unique arbitrage free state price density. However, due to the fact that the agent has logarithmic preferences, his optimal plans have a very familiar structure. We show in the Appendix that his optimal consumption policy is given by

$$c_{2t} = \rho W_{2t},$$

and is financed by a portfolio that solves the mean-variance program given by

$$\sup_{\pi \in C} \left\{ \pi \sigma_t \theta_t - \frac{1}{2} (\sigma_t \pi)^2 \right\}.$$  

We provide closed form solutions of this program in Proposition 1. The optimal policy is given by $\pi_{2t} = \kappa_t \pi_{mv}^{\pi}$ where $\pi_{mv}^{\pi}$ corresponds to the policy that would be chosen by an unconstrained agent, i.e., the mean-variance efficient portfolio, defined by $\sigma_t \pi_{mv}^{\pi} = \theta_t$. The process $\kappa_t$ is the reduction in risk taking induced by the constraint, as it takes values in $[0, 1]$.

**Proposition 1.** The optimal portfolio of the constrained agent is given by

$$\pi_{2t} = \kappa_t \pi_{mv}^{\pi}$$

with

$$\pi_2 \in C_1 : \kappa = \frac{1}{1 + (L_1 - 1)^\tau},$$

$$\pi_2 \in C_2 : \kappa_t = \frac{1}{1 + (|\theta_t| / L_2 - 1)^\tau}. \quad (9)$$

The mean-variance constraint gives rise to a ‘sluggish’ policy that keeps a fixed risk reduction regardless of market conditions. On the other hand, the constraint on volatility binds when the (absolute value of the) market price of risk is high relative to the agent’s risk bearing capacity. The fact that the risk reduction process depends only on the absolute value of the market price of risk, and not on the price level or stock volatility, is the key to a tractable characterization of equilibrium, as we will see next.

**Remark 1.** Interestingly, the optimal policy can be interpreted as the portfolio of a power utility agent whose (higher) relative risk aversion depends on the current market coefficients and has no hedging demand.
3.2 Determination of equilibrium

Because one of the agents faces portfolio constraints, the usual construction of a representative agent using a linear combination of the individual utility functions with constant weights is not possible. Instead, we construct an equilibrium using the consumption sharing rule

\[ s_{2t} = c_{2t}/\delta_t, \]

and the optimality conditions of both agents. We conjecture and verify that the consumption share process follows an Itô process given by

\[ ds_{2t} = s_{2t}\mu_{s_2}(\cdot)dt + s_{2t}\sigma_{s_2}(\cdot)dB_t, \]

where the coefficients \((\mu_{s_2}, \sigma_{s_2})\) are determined jointly with the interest rate and the market price of risk.

We briefly discuss the steps to construct an equilibrium. The unconstrained agent’s state price density is obtained from the first order condition of the unconstrained agent in (6),

\[ \xi_1(t, s_{2t}, \delta_t) = e^{-\rho t}y_1^{-1}(1 - s_{2t})^{\gamma} \delta_t^{-\gamma}, \]

and thus, an application of Itô’s lemma to this function identifies the market price of risk and the interest rate as functions of the drift and diffusion terms of the consumption share and the dividend dynamics. We then use the optimal consumption policy of the constrained agent in (8) and the definition of the consumption share to obtain \( s_{2t} = \rho W_{2t}/\delta_t \). From the dynamics of \( s_{2t} \) we obtain two additional equations. Since the risk reduction process \( \kappa \) depends on \( \theta \) or is a constant, the market price of risk is obtained from a nonlinear equation given by

\[ \theta_t + \gamma \frac{s_{2t}}{1 - s_{2t}} \kappa(\theta_t)\theta_t - \gamma \sigma_\delta \left( 1 + \frac{s_{2t}}{1 - s_{2t}} \right) = 0. \]

The solution in (10), \( \theta(s_{2t}) \), is then used to compute the interest rate and the pair \((\mu_{s_2}, \sigma_{s_2})\). All quantities are thus available in closed form as functions of the consumption sharing rule only. Proposition 2 formalizes the above discussion and provides expressions for the interest rate, the market price of risk and the coefficients in the consumption share dynamics.

**Proposition 2.** In equilibrium, the state price density is given by

\[ \xi_1(t, s_{2t}, \delta_t) = e^{-\rho t}y_1^{-1}(1 - s_{2t})^{-\gamma} \delta_t^{-\gamma}, \quad y_1 = (1 - s_{2t})^{-\gamma} \delta_0^{-\gamma}. \]
The market price of risk and the interest rate are given by

$$ \theta(s_{2t}) = \frac{1}{1 - (1 - \kappa(s_{2t})) R(s_{2t}) s_{2t} R(s_{2t}) \sigma_\delta}, $$

(11)

$$ r(s_{2t}) = \rho + \mu \delta R(s_{2t}) + (P(s_{2t}) - R(s_{2t})) s_{2t} \Phi(s_{2t}) \theta(s_{2t}) + \frac{P(s_{2t}) R(s_{2t})}{2} \left( (s_{2t} \Phi(s_{2t}))^2 - \sigma_\delta^2 \right), $$

(12)

where the functions $\Phi(\cdot)$, $R(\cdot)$ and $P(\cdot)$ are defined by

$$ \Phi(s_{2t}) = - (1 - \kappa(s_{2t})) \theta(s_{2t}), $$

(13)

$$ R(s_{2t}) = \frac{\gamma}{1 + (\gamma - 1) s_{2t}}, $$

(14)

$$ P(s_{2t}) = R(s_{2t})^2 \left[ 2 s_{2t} + \frac{1 + \gamma}{\gamma^2} (1 - s_{2t}) \right]. $$

(15)

The consumption share of the constrained agent obeys

$$ ds_{2t} = s_{2t} \mu_{s_2}(s_{2t}) dt + s_{2t} \sigma_{s_2}(s_{2t}) dB_t, $$

(16)

where

$$ \mu_{s_2}(s_{2t}) = g(s_{2t}) - \rho + \sigma_{s_2}(s_{2t})(\theta(s_{2t}) - \sigma_\delta), $$

(17)

$$ g(s_{2t}) = r(s_{2t}) + \sigma_\delta \theta(s_{2t}) - \mu_\delta, $$

$$ \sigma_{s_2}(s_{2t}) = \kappa(s_{2t}) \theta(s_{2t}) - \sigma_\delta, $$

(18)

and its starting point, $s_{20} \in (0, 1)$ is a solution to the equation

$$ (1 - \alpha) \rho^{-1} \delta_0 s_{20} = \beta + \alpha E \left[ \int_0^{\infty} \xi_1(t, s_{2t}, \delta_t)(1 - s_{2t}) \delta_t dt \right]. $$

(19)

The equilibrium satisfies Definition 1 when two conditions are met: (i) there exists $s_{20} \in (0, 1)$ which solves equation (19), and (ii) the consumption share process $s_2$ never reaches either zero or one in $t \in [0, \infty)$. Condition (i) implicitly restricts the size of the initial portfolio, $(\alpha, \beta)$, such that the initial wealth $w_k$ is strictly positive. Condition (ii) indicates that boundaries cannot be reached when the process starts from $s_{20} \in (0, 1)$, otherwise, the consumption policies would not be optimal and equilibrium would fail to exist.\(^{10}\)

\(^{10}\)For example, take the case in which the consumption share of agent 2 reaches 0 with positive probability, the utility of agent 2 would be minus infinity, which implies that the conjectured policy would never constitute an equilibrium.
Remark 2. The fact that equilibrium quantities depend only on the consumption share has two important consequences. First, the consumption share is an autonomous process with explicit coefficients, a fact that will allow us to show $s_2$ lives in $(0, 1)$. Second, it facilitates the computation of the stock price, as the price dividend ratio will be obtained from a single linear ordinary differential equation.

This result is in contrast to equilibrium models with position constraints, such as Gârleanu and Pedersen (2011) and Chabakauri (2011). In these papers, the evolution of $s_2$ depends explicitly on the stock price dynamics, so that the sharing process and the stock price form a system of forward-backward equations which must be solved for simultaneously in order to characterize each equilibrium quantity (and hence, the regions of the state space where the constraints are active).

3.2.1 Stock price and bubbles

Since financial markets clear in equilibrium, the value of the stock price is determined by the sum of the individual wealth processes in (7) and (8),

$$S_t = W_{1t} + W_{2t} = E\left[\int_t^\infty \frac{\xi_{1s}}{\xi_{1t}} c_{1s} ds \bigg| \mathcal{F}_t\right] + \frac{c_{2t}}{\rho}. \quad (20)$$

On the other hand, the state price density of the unconstrained agent, $\xi_{1t}$, is the unique nonnegative process such that the deflated stock price is a nonnegative local martingale and hence a supermartingale

$$\xi_{1t} S_t + \int_0^t \xi_{1s} \delta_s ds \geq E\left[\int_0^\infty \xi_{1s} \delta_s ds \bigg| \mathcal{F}_t\right].$$

The above inequality shows that the stock price must be at least as large as the expected value of future discounted dividends. Following the terminology used in the rational asset pricing bubble literature,\(^\text{11}\) if the inequality is strict then the stock price is said to be composed of two parts: a fundamental value\(^\text{12}\) given by

$$f_t(\delta) = E\left[\int_t^\infty \frac{\xi_{1s}}{\xi_{1t}} \delta_s ds \bigg| \mathcal{F}_t\right], \quad (21)$$

\(^{11}\)See e.g., Santos and Woodford (1997), Loewenstein and Willard (2000a,b) and Heston, Loewenstein, and Willard (2007), among others.

\(^{12}\)The fundamental value in (21) corresponds to the minimal amount that the unconstrained agent needs to hold (at time $t \geq 0$) in order to replicate the cash flows of the stock with an admissible portfolio strategy.
and a bubble component, given by

\[ b_t = S_t - f_t(\delta) = W_{2t} - E \left[ \int_t^\infty \xi_{1s} c_{2s} ds \middle| F_t \right]. \]  

(22)

The expression in (22) says that a bubble arises on the stock when the cost of agent 2’s optimal consumption plan is larger than the (replicating) cost of the same plan for the unconstrained agent. The intuition that the emergence of bubbles is related to how costly the constraint is for agent 2 is confirmed using the following representation.

**Proposition 3.** The bubble term in (22) is given by

\[ b_t = \delta_t^\gamma (1 - s_{2t})^\gamma \int_t^\infty e^{-\rho(s-t)} (\lambda(s_{2t}, \delta_t) - E[\lambda(s_{2s}, \delta_s) | F_t]) ds, \]  

(23)

where

\[ \lambda(s_{2t}, \delta_t) = s_{2t}(1 - s_{2t})^{-\gamma} \delta_t^{1-\gamma} \]  

(24)

is a nonnegative local martingale whose dynamics obey

\[ \lambda_t = \lambda_0 + \int_0^t \lambda_s \Phi(s_{2s}) dB_s, \]  

(25)

with \( \Phi(\cdot) \) given in (13).

The function in (24) corresponds to the ratio of the agents’ marginal utilities, a process that identifies the stochastic weight of the constrained agent in the representative agent construction used in several equilibrium models with frictions.\(^{13}\) Equation (23) says that \( b_t \) is necessarily a nonnegative process, as \( \lambda_t \) is a nonnegative local martingale and hence a supermartingale. Furthermore, it shows that the equilibrium will be free of bubbles when \( \lambda_t \) is a true martingale (see also Hugonnier (2012), Theorem 1).

This characterization focuses on the bubble on the stock price, but a bubble may also arise on the riskless asset. As it follows from Loewenstein and Willard (2000b), a bubble in the

---

\(^{13}\) The functions \( R(\cdot) \) and \( P(\cdot) \) in (14) and (15) correspond to the relative risk aversion and the relative prudence of the ‘representative agent with stochastic weights’ construction, as used by Cuoco and He (1994), Basak and Cuoco (1998), Basak and Croitoru (2000), Wu (2006), Gallmeyer and Hollifield (2008), Hugonnier (2012) and others.
riskless asset for an arbitrary fixed horizon $T \geq t$, is given by

$$b_0^0(T) = S_0^0 - E \left[ \frac{\xi_1 T}{\xi_1 T_0} S_0^0 \bigg\vert F_t \right]. \quad (26)$$

This quantity\textsuperscript{14} is simply the difference between the price of the riskless asset, $S_0^0$, and the risk-adjusted present value of the cash flow $S_0^0$. (26) is consistent with the representation in (21) and (22) since the riskless asset can be viewed as a derivative security that pays a single lump dividend equal to $S_0^0$ at time $T \geq t$, and thus the quantity $E \left[ \frac{\xi_1 T}{\xi_1 T_0} S_0^0 \bigg\vert F_t \right]$ is its fundamental value at time $t \geq 0$.

In Section 4, we will show that bubble components may arise depending on the type of constraint (e.g., mean variance constraints will not generate bubbles, whereas volatility constraints might), the tightness of the constraint and, interestingly, the risk aversion distribution across agents.

### 3.2.2 Price dividend ratio and volatility of returns

The price dividend ratio is given by a function of the consumption share, $p : (0, 1) \rightarrow \mathbb{R}_+$, which solves the boundary value problem that arises from the pricing equation in (5),

$$ (g(x) - \rho)xp'(x) + \frac{1}{2}(x\sigma_s(x))^2p''(x) - g(x)p(x) + 1 = 0, \quad (27) $$

subject to boundary conditions which depend critically on the type of constraint under consideration. Equation (27) follows from the explicit dependance of the stock price in (20) on the state variables $(\delta, s_2)$ and the fact that the consumption share is an autonomous process. The volatility of returns (or stock volatility, we use it interchangeably) follows from an application of Itô’s lemma to the price process $S_t = \delta_t p(s_{2t})$

$$ \sigma_t = \sigma_\delta + s_{2t}\sigma_{s_2}(s_{2t}) \frac{p'(s_{2t})}{p(s_{2t})}. \quad (28) $$

The first term is the volatility of dividends and is commonly referred to as the fundamental component, whereas the second term is the volatility of the price dividend ratio, and is referred to as the excess volatility component. We now turn to solving for the equilibrium quantities using our two examples.

\textsuperscript{14}Note that the process $M_t = S_0^0\xi_t$ is the unique candidate for the density of the risk neutral probability measure and it follows that the existence of a bubble on the riskless asset is equivalent to the non existence of a risk neutral probability measure.
4 Analysis of equilibrium

In this section, we study the implications of the two types of constraints in equilibrium. We show that the imposition of constraints on market participants who are more risk tolerant dampens fundamental shocks. We also characterize the economic conditions by which prices include a bubble component under volatility constraints.

4.1 Equilibrium under mean-variance constraints

Using the fact that risk reduction is constant when the agent is subject to a mean-variance constraint, the market price of risk in (11) is given by the following continuous and positive function of the consumption share

$$\theta(s_{2t}) = \frac{\gamma\sigma_\delta}{1 + (\gamma\kappa - 1)s_{2t}}.$$  (29)

The expression in (29) is higher than its unconstrained counterpart for the same distribution of consumption, and interestingly, corresponds to the market price of risk of an economy with no constraints but where the risk aversion distribution across agents is given by \{\gamma, 1/\kappa\}.

Note that a high market price of risk does not imply a relatively high interest rate, unlike an unconstrained economy with heterogeneous agents (see also Kogan, Makarov, and Uppal (2007)). On the contrary, Figure 1 shows a nice feature of an economy with risk constraints: the market price of risk rises and the equilibrium interest rate decreases. These shifts incite the unconstrained agent to scale up his position in the risky asset in a way consistent with market clearing. The impact of the constraint is asymmetric, as it is more pronounced in bad times, and increases as the constraint becomes tighter.

Insert Figure 1 here.

4.1.1 The existence result and risk sharing

When \(\kappa \in [0, 1]\), the consumption share obeys

$$\frac{d s_{2t}}{s_{2t}} = \mu_{s_2}(s_{2t})dt + (\kappa\theta(s_{2t}) - \sigma_\delta) dB_t$$  (30)
with
\[
\mu_{s_2}(x) = (\gamma - 1) \frac{1 - x}{1 + (\gamma - 1)x} \mu_\delta + \frac{\gamma}{1 + (\gamma - 1)x} \sigma^2_{s_2} \frac{[1 + \gamma + (\gamma - 1)(1 + 2\gamma)x]}{2[1 + (\gamma - 1)x]^3} \sigma^2_{\delta}
\]
\[
+ (\kappa - 1)(\gamma - 1) \gamma^2 \frac{(1 - x)x^2\gamma + \gamma\kappa - 2 + 2(\gamma - 1)(\gamma\kappa - 1)x}{2[1 + (\gamma - 1)x]^2[1 + (\gamma\kappa - 1)x]^2} \sigma^2_{\delta}
\]
\[
- (\gamma\kappa - 1) \frac{(1 - x)[1 - \gamma + (\gamma\kappa - 1)x]}{[1 + x(\gamma\kappa - 1)]^2} \sigma^2_{\delta},
\]

which shows that both the drift and volatility of the consumption share are zero at the two extreme ends, i.e., when one of the agents owns the whole economy, and thus, \(\{0, 1\}\) are absorbing states for \(s_2\). We show, however, that it is possible to establish the following existence result using pathwise comparison arguments (see the Appendix).

**Proposition 4.** Suppose that the process in (30) has a starting point in \((0, 1)\), the equilibrium exists as the boundary points \(\{0, 1\}\) cannot be reached in finite time.

Intuitively, the constraint changes the portfolio allocations as if it were shifting the risk aversion distribution across agents from \(\{\gamma, 1\}\) to \(\{\gamma, 1/\kappa\}\). To see this, note from (18) that the volatility of consumption growth of the constrained agent is given by

\[
\sigma_\delta + \sigma_{s_2}(\cdot) = \kappa \theta(\cdot),
\]

which corresponds to the volatility of consumption growth of an agent with risk aversion \(1/\kappa\). This implies that when the unconstrained agent is more risk averse (\(\gamma > 1\)), the ‘effective’ risk aversion distribution across agents is narrowed with the imposition of the constraint, to the extent that if the constraint is sufficiently tight, agent 2’s relative position may change from borrower to net lender. This is depicted in Figure 2.

When the unconstrained agent is less risk averse (\(\gamma < 1\)), the constraint acts as if it were widening the risk aversion distribution, and therefore, the constrained agent is forced to hold an even smaller position in the stock. This is the mechanism behind the dampening/amplifying effect of the constraint on the volatility of returns, as we see next.

### 4.1.2 The volatility of returns

In this example, the presence of bubbles can be easily ruled out. We summarize this result in the next proposition.
Proposition 5. The stock price is given by its fundamental value for $\kappa \in (0, 1]$. The price dividend ratio is represented by

$$p(s_2) = E \left[ \int_t^\infty e^{-\int_t^s g(s_2u)du} \frac{M^s_u}{M^s_t} ds \middle| \mathcal{F}_t \right] = E \left[ \int_t^\infty e^{-\int_t^s g(s_2u)du} ds \middle| \mathcal{F}_t \right]$$  

(31)

where the density of the probability measure $\tilde{P}$ is given by the exponential martingale

$$\tilde{M}_t = e^{-\frac{1}{2} \int_0^t (\theta(s_2s) - \sigma)^2 ds - \int_0^t (\theta(s_2s) - \sigma) dB_s}.$$

The expected value in equation (31) corresponds to the Feynman-Kac representation of the solution of the ODE in (27). In order to quantify the effects of the constraint in volatility, we compute the price dividend ratio in (27) subject to two boundary conditions, which we obtain by passing to the limit in (27) at $s_2 \to \{0, 1\}$. The boundaries are given by

$$p(0) = \frac{1}{\rho - (1 - \gamma)(\mu_\delta - \frac{1}{2} \gamma \sigma^2)},$$

which corresponds to the price dividend ratio in an economy with a single unconstrained agent, and

$$p(1) = \frac{1}{\rho},$$

which corresponds to the price dividend ratio that would prevail in an economy where there is single constrained agent.\footnote{Interestingly, the mean-variance constraint allows for a single-constrained-agent economy that holds the whole stock supply. This will not necessarily be the case in our next example.}

The left panel in Figure 3 shows a key result. When the unconstrained agent is more risk averse ($\gamma > 1$), the constraint forces agent 2 to decrease his position in the stock, narrowing the 'effective' risk aversion distribution. This leads to a reduction in volatility as the constraint restrains an efficient risk sharing whose dynamic evolution is partly responsible for the volatility of the stock price.

Insert Figure 3 here.

Remark 3. Note that if the constraint is sufficiently tight ($\kappa < 1/2$ in left panel), volatility is lower than its unconstrained economy benchmark in all states of nature.
On the other hand, as seen in the right panel in Figure 3, when the unconstrained agent is less risk averse ($\gamma < 1$), fundamental shocks are amplified. This result mirrors Bhamra and Uppal (2009), who show that in an economy with no constraints and heterogeneous agents, the volatility is increasing in the dispersion of risk aversion. This result is also consistent with the empirical literature,\(^{16}\) as the economy features a smaller price-dividend ratio and higher volatility in bad times, when the share of the constrained agent is high.

4.2 Equilibrium under volatility constraints

Using the optimal portfolio policy under volatility constraints in (9) and the equation for the market price of risk in (10), gives the following result.

**Proposition 6.** Suppose the risk bearing capacity of the constrained agent is expressed in units of the volatility of dividends\(^{17}\), $L_2 = \varepsilon \sigma_2$, with $\varepsilon \geq 0$. The market price of risk is given by

$$
\theta(s_{2t}) = \begin{cases} 
\gamma \sigma_2 \frac{1 - \varepsilon s_{2t}}{1 - s_{2t}}, & R(s_{2t}) > \varepsilon, \\
\frac{\gamma \sigma_2}{1 + (\gamma - 1) s_{2t}}, & R(s_{2t}) \leq \varepsilon.
\end{cases}
$$

The constraint generates an equilibrium with two regions which are completely determined by the primitives of the economy. The first term in (32) is the market price of risk in the region where the constraint is active, whereas the second term corresponds to the market price risk in the region where the constraint does not bind. The active region corresponds to the states in which the relative risk aversion of the representative agent is relatively high, $R(s_{2t}) > \varepsilon$. This is equivalent to an upper (lower) bound on the consumption share process, depending on whether $\gamma > (\leq) 1$,

$$s_{2t} < (>) s_2^*, \quad s_2^* = \frac{\gamma - \varepsilon}{\varepsilon (\gamma - 1)}.$$

To see how the interaction between the risk bearing capacity and the risk aversion of the unconstrained agent works, take the example in which the unconstrained agent is more risk averse ($\gamma > 1$). If $\varepsilon \in [0, 1]$, the constraint is active in all states of nature. As the risk bearing capacity of agent 2 increases, such that $\varepsilon \in (1, \gamma)$, the constraint binds in the region defined by $s_{2t} < s_2^*$, since $0 < s_2^* < 1$. Finally, when $\varepsilon \geq \gamma$, the constraint is never active. The case $\gamma < 1$ can be described in similar terms.

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\(^{16}\)Mele (2007) documents, using U.S. data, that the price dividend ratio decreases more during recessions than it increases during expansions and its volatility is countercyclical. See also Schwert (1989).

\(^{17}\)This is only done for notational convenience, as the active region will depend on the risk aversion parameter ($\gamma$) and the severity of the constraint ($\varepsilon$).
Remark 4. The two-regime structure reveals an important result: the constraint limits the investor’s risk exposure in bad times. Note that when agent 1 is more risk averse \((\gamma > 1)\), the constrained agent holds most of the supply of the stock, and low total wealth states correspond to low levels of his consumption share. When agent 1 is less risk averse, \((\gamma < 1)\), the constrained agent holds most of his wealth in the riskless asset and low total wealth states correspond to the upper region.

Note that the constraint acts in the same direction of our first example, that is, increasing the market price of risk and decreasing the interest rate, as Figure 4 shows. These shifts induce the unconstrained agent to scale up his position in the risky asset.

Insert Figure 4 here.

4.2.1 The existence result and risk sharing

The two-regime structure is reflected in the dynamics of the consumption share process, which are given by

\[
\frac{ds_{2t}}{s_{2t}} = \mu_{s_2}(s_{2t})dt + \sigma_{s_2}(s_{2t})dB_t \tag{33}
\]

with

\[
\mu_{s_2}(x) = \begin{cases} 
(\gamma - 1) \frac{1-x}{1+(\gamma-1)x} \mu_\delta + (\gamma - 1) \gamma \frac{2(1-x)|x-1|}{2(1-x)|1+(\gamma-1)x|} \sigma_\delta^2 \\
+ (1-\varepsilon) \sigma_\delta^2 \left( 1 - \frac{1-x}{1+x} \right), & R(x) > \varepsilon, \\
(\gamma - 1) \frac{1-x}{1+(\gamma-1)x} \mu_\delta + \frac{\gamma}{1+(\gamma-1)x} \sigma_\delta^2 - \frac{\gamma(1+\gamma+(\gamma-1)(1+2\gamma)x}{2[1+(\gamma-1)x]^2} \sigma_\delta^2 \\
- (\gamma - 1) \frac{(1-x)|1-\gamma+(\gamma-1)x|}{|1+x(\gamma-1)|^2} \sigma_\delta^2, & R(x) \leq \varepsilon,
\end{cases}
\]

and

\[
\sigma_{s_2}(x) = \begin{cases} 
-(1-\varepsilon) \sigma_\delta, & R(s_{2t}) > \varepsilon, \\
\sigma_\delta \left[ \frac{\gamma}{1+(\gamma-1)x} - 1 \right], & R(s_{2t}) \leq \varepsilon.
\end{cases} \tag{34}
\]

Overall, (33) shares qualitative features with the dynamics in (30), since they depend on the tightness of the constraint and the risk aversion distribution across agents. Note that the drift diverges to minus infinity at \(s_2 \to 1\) if the constraint binds. This behavior counterbalances the effect of the linear diffusion in (34), such that the process never reaches 1 from the interior.
This observation, in conjunction with comparison arguments, gives us the following existence result.

**Proposition 7.** Suppose that the process in (33) has a starting point in $(0,1)$, the equilibrium exists as the boundary points $\{0,1\}$ cannot be reached in finite time.

The constrained agent’s risk reduction process

$$
\kappa(s_{2t}) = \begin{cases} 
\frac{\varepsilon}{\gamma} \frac{1 - s_{2t}}{1 - \varepsilon s_{2t}}, & R(s_{2t}) > \varepsilon, \\
1, & R(s_{2t}) \leq \varepsilon,
\end{cases}
$$

shows how constrained market participants may appear to become more risk averse in response to deteriorating market conditions. A sequence of negative shocks induce a decrease in the portfolio position with respect to an otherwise unconstrained agent.\(^{18}\)

Figure 5 depicts the corresponding portfolio strategies. When the unconstrained agent is more risk averse ($\gamma > 1$), the constrained agent curtails his position in the risky asset in states where his consumption share is low, and as the constraint tightens ($\varepsilon \downarrow$), his relative position may be changed, forcing the constrained agent to become a net lender. On the other hand, when the unconstrained agent is less risk averse ($\gamma < 1$), the constraint is active in states where the consumption share is high, forcing the constrained agent to reduce his position in the stock even further.\(^{18}\)

4.2.2 The volatility of returns

As in our first example, we solve for the price dividend ratio in (27). We obtain the boundary at 0 by passing to the limit in equation (27)

$$
p(0) = \frac{1}{\rho - (1 - \gamma)(\mu_\delta - \frac{1}{2}\gamma\sigma^2)}.
$$

This quantity corresponds to the price dividend ratio in an economy with a single unconstrained agent. If the constraint is slack as $s_2$ approaches 1, the second boundary corresponds to

$$
p(1) = \frac{1}{\rho}.
$$

\(^{18}\)This behavior is also present in Danielsson, Shin, and Zigrand (2011), where negative shocks might appear as if ‘asset sales beget asset sales’. The risk reduction goes up over time as the ‘cycle’ improves.
If the constraint binds as \( s_2 \) approaches 1, i.e., when \( \varepsilon < 1 \), the boundary will not be \( p(1) = 1/\rho \). This limit does not correspond to the price dividend ratio in an economy with a single constrained agent, because the volatility of the constrained agent’s wealth process is given by \( \varepsilon \sigma_3 \) and, at the limit, the constrained agent owns the whole supply of the stock. A second condition follows by differentiating equation (27) and evaluating at \( s_2 = 0 \), to obtain the boundary

\[
p'(0) = \gamma \left( \frac{1}{\rho} - p(0) \right).
\]

The left panel in Figure 6 displays the volatility when the unconstrained agent is more risk averse than the constrained agent (\( \gamma > 1 \)). Note that the volatility decreases with respect to the unconstrained economy.

Remark 5. If the constraint is sufficiently tight (\( \varepsilon < 1 \) in left panel), volatility is lower than its unconstrained benchmark in all states.

This result mirrors the behavior of volatility in the example with mean-variance constraints: the presence of the constraint dampens fundamental shocks in an environment where the constrained agent is a net borrower.

The right panel in Figure 6 presents the volatility when \( \gamma < 1 \) and shows a pattern that differs from the constant risk reduction case. The volatility increases in the region where the constraint is active, but as the wealth of the constrained agent increases, there is an extreme change in the discount rate \( g(\cdot) \) (a large discounting asymmetry) that turns negative the excess volatility term in (28).

4.2.3 Bubbles

As it follows from Proposition 3, we characterize the properties of the weighting process \( \lambda \) in (24) and summarize the results in the following proposition.

Proposition 8. The stock price contains a bubble if the constraint is always binding and \( \gamma \geq 1 \). The stock price is free of bubbles if (i) the constraint is not always binding and \( \gamma > 1 \), (ii) the constraint is always binding or binds with positive probability and \( \gamma < 1 \).

Proposition 8 confirms the intuition put forward in Proposition 3: the stock contains a bubble component depending on how costly the constraint is for agent 2. The latter is determined
by the severity of the constraint ($\varepsilon$) and the risk aversion distribution across agents ($\gamma$). The importance of both dimensions is evident from the fact that the region where the constraint binds is completely determined by both parameters.

Note that when both agents share the same risk aversion ($\gamma = 1$), a bubble component emerges when the constraint binds in all states ($\varepsilon < 1$). On the other hand, to illustrate the role of the risk aversion distribution, we examine the limiting case $\varepsilon = 0$. Agent 2 cannot hold the risky asset and finances his consumption plan by trading on the riskless asset, whereas agent 1 holds the whole supply of the stock and shorts the riskless asset such that it offsets the constrained agent’s position. Equilibrium portfolio positions are thus characterized by

$$\phi_2^0 = |\phi_1^0|, \quad \phi_1 = S, \quad \phi_2 = 0.$$  \hspace{1cm} (35)

Proposition 8 says that a bubble arises if agent 1 is more risk averse than agent 2. In contrast, the stock is bubble-free if agent 1 is less risk averse than agent 2, i.e., when the stockholder is predisposed to hold a larger position in the stock absent the constraints (and local shifts in the interest rate and the market price of risk are sufficient to reach market clearing). To understand this result, notice that when $\varepsilon = 0$ Proposition 8 also determines whether there is a bubble component on the riskless asset. This observation follows from the fact that the (normalized) weighting process in this limit case is also the candidate risk neutral density,

$$M_t = S_t^0 \xi_{1t} = \lambda_t/\lambda_0.$$  

The bubble in the riskless asset ($b_0^0(T)$ defined in (26)) is thus nonnegative if and only if the bubble on the stock is nonnegative. As in Hugonnier (2012), agent 1 optimally chooses to hold a levered position in the stock, even though the stock contains a bubble component, because the bubble on the riskless asset is larger.\footnote{See also Hugonnier and Prieto (2012) for an application in infinite horizon in a model with homogeneous logarithmic preferences. The assumption of logarithmic preferences delivers closed forms for $(b_t, b_t^0(T))$.} With risk aversion heterogeneity, simulations\footnote{Details of the simulations are available on request.} show that bubbles are increasing in the level of risk aversion for a fixed level of the constraint. Intuitively, as the risk aversion of the unconstrained agent increases, the bubble component on the riskless asset grows larger to incite the unconstrained agent to hold positions such that (35) is indeed an equilibrium.

Remark 6. These observations point to the fact that the emergence of bubble components in the price system responds to a clear equilibrium mechanism: since agent 1 has to clear markets with the constrained agent, bubbles arise to mitigate this ‘implicit liquidity provision constraint’. Furthermore, this result formalizes and connects the emergence of bubbles in positive net supply
securities in models with continuous trading with macro-finance models that identify (portfolio) imbalances and limited supply of financial assets as a foundation of bubbly prices.\textsuperscript{21}

5 Extensions

In this section, we show that it is straightforward to characterize the equilibrium with risk constraints in settings with heterogeneous beliefs and multiple risky assets.

5.1 Heterogeneous beliefs and volatility constraints

We introduce heterogeneity in beliefs about the evolution of the aggregate dividend in an environment with risk constraints. We assume, without loss of generality, that the beliefs of agent 1 are represented by the objective measure $\mathbb{P}$, and we let

$$
\eta_t = e^{-\int_0^t \frac{1}{2} \bar{\mu}_s^2 ds - \int_0^t \bar{\mu}_s dB_s}
$$

denote the density process of agent 2’s probability measure $\mathbb{P}_2$ with respect to $\mathbb{P}$. In the expression, $\bar{\mu}$ represents the investors’ disagreement on the mean endowment growth rate, normalized by its risk,

$$
\bar{\mu} = \frac{\sum_1 - \mu_2}{\sigma_2},
$$

where $\mu_1$ and $\mu_2$ represent the beliefs of agent 1 and 2, respectively. Note that $\bar{\mu}$ is positive when agent 1 is more optimistic.\textsuperscript{22} The divergence in beliefs is modeled in a reduced form through the system $(\eta, \bar{\mu})$ and since all computations are done under agent 1’s probability measure, the utility function of the second agent is state dependent

$$
u_2(\eta, c) = \eta \log c.
$$

The construction of equilibrium is similar to that of an economy with homogeneous beliefs. However, agents will also trade due to differences in beliefs. For simplicity, we set $\bar{\mu}$ to be constant.\textsuperscript{23} The next proposition details the market price of risk under volatility constraints.\textsuperscript{24}

\textsuperscript{21}See e.g., Caballero (2006) and Caballero and Krishnamurthy (2009).

\textsuperscript{22}The parameter $\bar{\mu}$ follows directly from the endowment process and the agents’ priors.

\textsuperscript{23}This example corresponds to what has been termed dogmatic beliefs, because it can be rationalized by assuming that each investor $k$ is so confident in his prior that he completely ignores any information from the output process, and keeps the same belief throughout.

\textsuperscript{24}The case of mean-variance constraints is solved in a similar way. The model is reminiscent of Gallmeyer and Hollifield (2008) who study the effect of short-sale constraints on asset returns. However, short-sale constraints give rise to a two-regime equilibrium.
Proposition 9. Suppose the risk bearing capacity of the constrained agent is expressed in units of the volatility of dividends, \( L_2 = \varepsilon \sigma_\delta \), with \( \varepsilon \geq 0 \). The market price of risk is given by

\[
\theta(s_{2t}) = \begin{cases} 
R(s_{2t})(\sigma_\delta + \bar{\mu}s_{2t}), & s_{2t} \in S_u = \{ x \in (0,1) : |R(x)(\sigma_\delta + x\bar{\mu}) - \bar{\mu}| \leq \varepsilon \sigma_\delta \}, \\
\gamma \sigma_\delta \frac{1 - \varepsilon s_{2t}}{1 - s_{2t}}, & s_{2t} \in S_1 = \{ x \in (0,1) : R(x)(\sigma_\delta + x\bar{\mu}) - \bar{\mu} > \varepsilon \sigma_\delta \}, \\
\gamma \sigma_\delta \frac{1 + \varepsilon s_{2t}}{1 - s_{2t}}, & s_{2t} \in S_2 = \{ x \in (0,1) : -R(x)(\sigma_\delta + x\bar{\mu}) + \bar{\mu} > \varepsilon \sigma_\delta \}. 
\end{cases}
\]

The constraint produces an equilibrium with possibly three regions which depend on the tightness of the constraint, the divergence of beliefs, the volatility of dividends and the risk aversion distribution. When the divergence of beliefs is zero, the model collapses to the baseline case of Section 4.

The set \( S_u \) describes the unconstrained region, whereas the constraint is active when the consumption share enters \( S_1 \) or \( S_2 \). The intuition of a third region is straightforward. The constraint may curtail the risk taking behavior of agent 2 also when he holds a short position in the stock, which occurs when agent 1 is more optimistic (\( \bar{\mu} > 0 \)). In order to see this, take the case of homogeneous preferences (\( \gamma = 1 \)). The constraint is active in the upper region defined by \( \{ S_1 : s_{2t} > 1 - (1 - \varepsilon) \frac{\omega}{\bar{\mu}} \} \), and in the lower region, given by \( \{ S_2 : s_{2t} < 1 - (1 + \varepsilon) \frac{\omega}{\bar{\mu}} \} \), where the agent holds a short position in the stock.

Despite the fact that heterogeneity in beliefs may generate rich patterns in volatility in economies with no portfolio constraints (e.g., it is relatively easy to obtain nonmonotonic price dividend ratios when there are divergence of beliefs), the central insight in the paper stands, in the sense that the presence of constrained agents reduces (the absolute value of) the excess volatility term in (28) when they are net borrowers. We close this section with a brief comment on bubbles.

Remark 7. Heterogenous beliefs modify the binding regions and the direction of portfolio imbalances. For example, let \( \varepsilon = 1, \gamma = 2 \) and \( \bar{\mu} = \omega \sigma_\delta \), with \( \omega \in [1,3] \), so that agent 1 is more risk averse and more optimistic than the constrained agent.\(^{25}\) The function \( \Phi(s_{2t}) = -\bar{\mu} = -\omega \sigma_\delta \) is bounded, and it follows that the process in (25) is a true martingale and the price system is bubble-free.

5.2 Multiple risky assets

Contrary to models with position constraints, we show it is straightforward to introduce multiple risky assets in an environment where some agents face risk constraints. In particular, the

\(^{25}\)Recall from Proposition 8 that when agent 1 is more risk averse (\( \gamma > 1 \)) the stock contains a bubble component if the risk bearing capacity of the constrained agent is \( \varepsilon \in [0,1] \).
equilibrium is structurally identical to the equilibrium with one risky asset.\textsuperscript{26} In this economy, the \(i\)-th risky asset is a claim to a strictly positive dividend process \(\delta^i_t\), such that the price process of the \(i\)-th risky asset, denoted by \(S^i_t\), evolves according to

\[
S^i_t = S^i_0 + \int_0^t S^i_s \left( \mu^i_s ds + \sigma^i_s dB_s \right) - \int_0^t \delta^i_s ds,
\]

for some initial value \(S^i_0 \in \mathbb{R}_+\) and some drift and volatility processes \((\mu^i, \sigma^i) \in \mathbb{R} \times \mathbb{R}^n\) which are determined in equilibrium. The process \(\theta \in \mathbb{R}^n\), defined by \(\mu_t = r_t 1_n + \sigma^\top_t \theta_t\), denotes the vector of relative risk premium associated with the sources of risk in the model. Here \(\mu \in \mathbb{R}^n\) and \(\sigma \in \mathbb{R}^{n \times n}\) denote the drift and the volatility of the price vector, respectively, obtained by stacking up the individual drifts and volatilities of the stock prices.

As in the baseline model, the aggregate endowment follows a geometric Brownian motion due to the presumed i.i.d. property of aggregate consumption growth,

\[
\delta_t = \sum_{i=1}^n \delta^i_t = \delta_0 + \int_0^t \delta_s \left( \mu^\top_s ds + \sigma^\top_s dB_s \right).
\]

The dividend of security \(i\) is given by \(\delta^i_t = \delta_t x_{it}\), where \(x_{it}\) is the share in aggregate endowment of dividend \(i\). We assume that agent 2 is initially endowed with \(\beta \in \mathbb{R}\) units of the riskless asset and a fraction \(0 \leq \alpha \leq 1\) of the market portfolio, so that his initial wealth, computed at equilibrium prices, is given by

\[
w_2 = \beta + \alpha \sum_{i=1}^n S^i_0.
\]

The following proposition reveals that the equilibrium with \(n\) risky assets can be constructed in the same way of the equilibrium with one risky asset.

**Proposition 10.** The market prices of risk and the interest rate are given by

\[
\theta(s_{2t}) = \frac{1}{1 - (1 - \kappa(s_{2t})) R(s_{2t}) s_{2t}} R(s_{2t}) \delta,
\]

\[
r(s_{2t}) = \rho + \mu \delta R(s_{2t}) + \left( P(s_{2t}) - R(s_{2t}) \right) s_{2t} \Phi(s_{2t})^\top \theta(s_{2t})
\]

\[
+ \frac{1}{2} P(s_{2t}) R(s_{2t}) \left[ \| s_{2t} \Phi(s_{2t}) \|^2 - \| \delta \|^2 \right],
\]

where \(\Phi(s_{2t}) = - (1 - \kappa(s_{2t})) \theta(s_{2t})\).

\textsuperscript{26}We make use of the following vectorial notation: \(a^\top\) denotes transposition, \(\|a\|\) denotes the Euclidean norm in \(\mathbb{R}^n\) and \(1_n\) is a \(n\)-dimensional vector of ones. We denote by \(B\) an \(n\)-dimensional standard Brownian motion.
Note that the vector $\theta$ solves a system of nonlinear equation and quantities depend only on the consumption share of the constrained agent. Expected stock returns satisfy a two-factor capital asset pricing model where the weighting process, defined in (24), plays the role of the second factor

$$\mu_t^i - r_t = R(s_{2t}) \left[ \text{cov} \left( \frac{dS_t^i}{S_t^i}, \frac{d\delta_t}{\delta_t} \right) - s_{2t} \text{cov} \left( \frac{dS_t^i}{S_t^i}, \frac{d\lambda_t}{\lambda_t} \right) \right].$$

An empirical study of the effect of risk constraints in asset prices would require the identification of empirical proxies for the state variable $\lambda$, a topic that we leave for future research.\footnote{See Adrian, Etula, and Muir (2011) for a reduced form approach that studies the impact in the cross-section of stock returns of risk-constrained intermediaries.}

As in Section 3, when the equilibrium is free of bubbles, stock prices are given by their fundamental values,

$$S_t^i(s_{2t}, \delta_t, x_t) = f_t(\delta^i) = \delta_t E \left[ \int_1^\infty e^{-\rho(s-t)} \left( \frac{1 - s_{2s}}{1 - s_{2t}} \right)^{-\gamma} \left( \frac{\delta_s}{\delta_t} \right)^{1-\gamma} x_{is} ds \right] F_t.$$  

**Remark 8.** We present in B a fully solved example with two risky assets in an economy with mean-variance constraints.

On the other hand, in an economy with a bubble on the market portfolio,

$$\sum_{i=1}^n S^i_t = W_{1t} + W_{2t} = f_t(\delta) + b_t,$$

stock prices may not be uniquely determined. The intuition follows from the fact that when there is a bubble component on the market portfolio, individual stock prices can be represented by

$$S_t^i(s_{2t}, \delta_t, x_t) = f_t(\delta^i) + b_t^i,$$

where $b_t^i$ denotes the corresponding bubble component. The only equilibrium restriction on the price system is given by $\sum_{i=1}^n b_t^i = b_t$, there are no restrictions on how the value of $b_t$ is split among the risky assets. This gives rise to multiplicity of equilibria.

### 6 Concluding remarks

In this article we study a continuous-time, pure exchange economy populated by two groups of agents. Agents in the first group have logarithmic preferences and face risk-based portfolio
constraints which force them to behave locally as power utility investors with a relative risk aversion coefficient that depends on current market conditions. Agents in the second group have arbitrary CRRA preferences and are unconstrained.

The class of risk constraints in the model gives rise to a tractable equilibrium, as the consumption sharing rule follows an autonomous process whose coefficients can be determined in closed form. This allows us to provide explicit existence results and solve the model by computing a single linear ordinary differential equation which describes the price dividend ratio.

We show that the imposition of constraints on market participants which are more risk tolerant dampens fundamental shocks. This insight is in contrast to recent studies that suggest that risk management rules serve to amplify aggregate fluctuations, and also, to the belief that current capital regulation make financial crises larger and more costly.

The presence of agents who are subject to VaR-based portfolio constraints may give rise to bubbles in equilibrium prices, even though there are unconstrained agents in the economy who can exploit limited arbitrage opportunities. The emergence of bubbles depends on the risk aversion distribution across agents and the severity of the constraint. This result connects the emergence of bubbles in models with continuous trading with macro-finance models that identify portfolio imbalances and limited supply of financial assets as the source of asset pricing bubbles.

We show that it is straightforward to introduce risk constraints in settings with heterogeneous beliefs and multiple risky assets.
A Proofs

A.1 Propositions

Proof of Proposition 1. The constrained agent solves the program

\[ \sup_{c, \pi \in \mathcal{A}(w_2)} E \left[ \int_0^\infty e^{-\rho t} \log (c_{2t}) \, dt \right], \]

subject to

\[ \log (W_{2t}) = \log (W_{20}) + \int_0^t \left( r_s + \pi_s^\top \sigma_s \theta_s - \frac{1}{2} \| \sigma_s^\top \pi_s \|^2 - c_{2s}/W_{2s} \right) \, ds + \int_0^t \pi_s^\top \sigma_s dB_s \]

where \( \mathcal{A}(w_2) = \{ (\pi, c) : \pi \in \mathcal{C}_t \text{ and } W_{w_2^t, \pi, c}^2 t \geq 0 \text{ for } t \in [0, \infty) \} \). Using the objective function and the budget constraint, the problem can be expressed as the maximization of

\[ E \left[ \int_0^\infty e^{-\rho t} \left( \log (\alpha_t) + \log (W_{20}) + \int_0^t \left( r_s + \pi_s^\top \sigma_s \theta_s - \frac{1}{2} \| \sigma_s^\top \pi_s \|^2 - \alpha_s \right) \, ds \right) \, dt \right] \]

\[ = E \left[ \int_0^\infty e^{-\rho t} \left( \log (\alpha_t) + \log (W_{20}) \right) \, dt \right] 
+ \int_0^\infty \left( r_t + \pi_t^\top \sigma_t \theta_t - \frac{1}{2} \| \sigma_t^\top \pi_t \|^2 - \alpha_t \right) \left( \int_t^\infty e^{-\rho s} \, ds \right) \, dt \]

\[ = E \left[ \int_0^\infty e^{-\rho t} \left[ \log (\alpha_t) - \rho^{-1} \alpha_t + \log (W_{20}) + \rho^{-1} r_t + \rho^{-1} \left( \pi_t^\top \sigma_t \theta_t - \frac{1}{2} \| \sigma_t^\top \pi_t \|^2 \right) \right] \, dt \right] \]

where we have used a consumption policy of the form \( c_{2t} = \alpha_t W_{2t} \). The problem is solved by a pointwise optimization of

\[ \sup_{\alpha > 0} \{ \log (\alpha) - \rho^{-1} \alpha \}, \]

which admits a unique solution given by \( \alpha = \rho \), and the mean variance program

\[ \sup_{\pi \in \mathcal{C}_t} \left\{ \pi^\top \sigma_t \theta_t - \frac{1}{2} \| \sigma_t^\top \pi \|^2 \right\}. \tag{36} \]

Since \( \mathcal{C}_t \) is a closed convex subset of \( \mathbb{R}^n \), the mean variance problem in (36) admits a unique solution given by

\[ \sigma_t^\top \pi_t = \Pi \left[ \theta_t | \mathcal{C}_t \right] \]

where \( \Pi \) denotes the projection operator, defined by \( \Pi [x | y \in \mathcal{J}] = \inf_{y \in \mathcal{J}} \frac{1}{2} \| y - x \|^2 \). We solve the mean variance program in (36) for our two examples:

(i) **Mean-variance constraint.** Let \((a_1, a_2, a_3) = (-1, L, 0)\) and \( x = \sigma^\top \pi \). The Karush-Kuhn-Tucker (KKT) conditions of the projection problem are

\[- (\eta + 1) \theta + (1 + 2\eta L) x = 0, \eta \left[ x^\top \theta - L \| x \|^2 \right] = 0, \]

where...
with complementary slackness and \( \eta \geq 0 \). The problem is solved by

\[
\eta = L^{-1} (L - 1)^{+}, \quad \sigma_t^\top \pi_t = \frac{1}{1 + (L - 1)^{+}} \theta_t, \quad \kappa = \frac{1}{1 + (L - 1)^{+}}.
\]

(ii) Volatility constraint. Let \((a_1, a_2, a_3) = (0, 1, L^2)\). The KKT conditions of the projection problem are \(-\theta + (1 + 2\eta) x = 0, \eta \left(\|x\|^2 - L^2\right) = 0\), with complementary slackness and \( \eta \geq 0 \). The problem is solved by

\[
\eta_t = \frac{1}{2} \left(\frac{\|\theta_t\|}{L} - 1\right)^{+}, \quad \sigma_t^\top \pi_t = \kappa_t \theta_t, \quad \kappa_t = \frac{1}{1 + (\|\theta_t\| / L - 1)^+}.
\]

Mapping the result into the dual approach of Cvitanić and Karatzas (1992). We use this part later in the proof of Propositions 3 and 9. The agent’s problem is transformed into an unconstrained consumption and portfolio choice problem in a fictitious economy with a modified market price of risk and interest rate. Let \( \beta_t : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\} \) be the support function of the set \(-C_t\), that is, the convex function defined by \( \beta_t(\nu) = \sup_{\pi \in C_t} \{ -\pi^\top \nu : \pi \in C_t \} \), where \( B \) is its effective domain. The implicit state price density faced by agent 2 is given by

\[
\xi_{2t} = e^{-\int_0^t (r_s + \beta_s(\nu_s) + \frac{1}{2} \|\theta_{2s}\|^2) ds - \int_0^t \theta_{2s}^\top dB_s},
\]

where \( \theta_{2t} = \theta_t + \sigma_t^{-1} \nu_t \). Optimality conditions imply that \( \nu \) is defined by the relation

\[
\nu_t = \arg \min_{\nu \in B_t} \left\{ \frac{1}{2} \left\| \theta_t + \sigma_t^{-1} \nu_t \right\|^2 + \beta_t(\nu) \right\},
\]

and the optimal consumption plan and trading strategy of the constrained agent are given by

\[
c_{2t} = \frac{e^{-pt}}{\theta_{2t}}, \quad \pi_{2t} = (\sigma_t^{-1})^\top \theta_{2t},
\]

thus, we have that the wealth process along the optimal path is given by \( W_{2t} = \rho^{-1} c_{2t} \). We recover the mean-variance program of the primal problem in (36) using the definition of the support function and Fenchel’s duality theorem (see Rockafellar (1996), Theorem 31.1) which imply that

\[
\inf_{\nu \in B_t} \left\{ \frac{1}{2} \left\| \theta_t + \sigma_t^{-1} \nu_t \right\|^2 + \beta_t(\nu) \right\} = \sup_{\pi \in C_t} \left\{ \pi^\top \sigma_t \theta_t - \frac{1}{2} \left\| \sigma_t^\top \pi \right\|^2 \right\},
\]

as the conjugate functions of \( f(\nu) = \frac{1}{2} \left\| \theta_t + \sigma_t^{-1} \nu_t \right\|^2 \) and \( g(\nu) = -\beta_t(\nu) \) are \( f^*(\pi) = -\pi^\top \sigma_t \theta_t + \frac{1}{2} \left\| \sigma_t^\top \pi \right\|^2 \) and \( g^*(\pi) = 0 \), for \( \pi \in C_t \), respectively. Hiriart-Urruty and Lemaréchal (2001), Theorem 3.1.1, show that the projection operator satisfies

\[
(\sigma_t^\top w - \sigma_t^\top \pi_t)^\top (\sigma_t^\top \pi_t - \theta_t) \leq 0.
\]
Taking the maximum on the left hand side gives
\[
\max_{\omega \in C_t} \left\{ (\omega - \pi_t)^\top y_t \right\} = 0,
\]
where \( y_t = \sigma_t (-\theta_t + \sigma_t^\top \pi_t) \). In conjunction with the definition of the support function, this implies that the vector \( y_t \in B_t \). Note that the vector \( y_t \) attains the infimum on the left hand side of equation (38) and it follows that \( \nu_t = y_t = -(1 - \kappa_t)\sigma_t \theta_t \).

**Proof of Proposition 2.** We construct a consumption sharing rule, \( s_{2t} \), such that \( c_{1t} = (1 - s_{2t})\delta_t \) and \( c_{2t} = s_{2t}\delta_t \) and whose dynamics follow
\[
ds_{2t} = s_{2t}\mu_{s_{2t}}dt + s_{2t}\sigma_{s_{2t}}dB_t.
\]
An application of Itô’s lemma to the process \( s_{2t} = c_{2t}/\delta_t = \rho W_{2t}/\delta_t \), where \( W_{2t} \) is the wealth process of the constrained agent along the optimal path, with dynamics
\[
dW_{2t}/W_{2t} = (r_t + \kappa_t\theta_t^2 - \rho)dt + \kappa_t\theta_t dB_t,
\]
yields,
\[
\begin{align*}
\mu_{s_{2t}} &= r_t + \sigma_t^\top \delta_t - \rho + (\kappa_t\theta_t - \sigma_\delta) (\theta_t - \sigma_\delta), \\
\sigma_{s_{2t}} &= \kappa_t\theta_t - \sigma_\delta.
\end{align*}
\]
(39)
The first order condition of the unconstrained agent in equation (6) identifies the unconstrained agent’s state price density as
\[
\xi_1(t, s_{2t}, \delta_t) = e^{-\rho t} y_1^{-1} (1 - s_{2t})^{-\gamma} \delta_t^{-\gamma},
\]
and thus, an application of Itô’s lemma to this function identifies the market price of risk and the interest rate as functions of the drift and diffusion terms in the consumption share and the dividend dynamics,
\[
\begin{align*}
\theta_t &= \gamma \left( \sigma_\delta - \frac{s_{2t}\sigma_{s_{2t}}}{1 - s_{2t}} \right), \\
rt &= \rho + \gamma \mu_\delta - \frac{1}{2}(1 + \gamma)\gamma \sigma_\delta^2 - \gamma \frac{s_{2t}\mu_{s_{2t}}}{1 - s_{2t}} + \frac{2\gamma^2\sigma_\delta(1 - s_{2t})s_{2t}\sigma_{s_{2t}} - (1 + \gamma)\gamma \sigma_{s_{2t}}^2}{2(1 - s_{2t})^2}.
\end{align*}
\]
(40)
Using equations (39) and (40) and the fact that the process \( \kappa_t \) is either constant or depends only on \( \theta_t \), we obtain a nonlinear equation for \( \theta_t \),
\[
\theta_t = \gamma \left( \sigma_\delta - \frac{s_{2t}(s_{2t}\theta_t - \sigma_\delta)}{1 - s_{2t}} \right)
\]
The solution of this problem is then used to express \( (r(\cdot), \mu_{s_{2t}}(\cdot), \sigma_{s_{2t}}(\cdot)) \) as functions of the consumption share only and correspond to equations in (12), (17) and (18), respectively. The wealth processes are
given by
\[ W_{1t} = E \left[ \int_t^\infty \frac{\xi_1(s, s_{2s}, \delta_s)}{\xi_1(t, s_{2t}, \delta_t)} (1 - s_{2s}) \delta_s ds \middle| F_t \right], \quad W_{2t} = \rho^{-1}s_{2t}\delta_t \]

and add up to the stock price due to market clearing,
\[ S_t = W_{1t} + W_{2t} \]
\[ = E \left[ \int_t^\infty \frac{\xi_1(s, s_{2s}, \delta_s)}{\xi_1(t, s_{2t}, \delta_t)} \delta_s ds \middle| F_t \right] + \rho^{-1}s_{2t}\delta_t - E \left[ \int_t^\infty \frac{\xi_1(s, s_{2s}, \delta_s)}{\xi_1(t, s_{2t}, \delta_t)} s_{2s}\delta_s ds \middle| F_t \right]. \tag{41} \]

The equation for the starting point of the consumption share process in (19) follows from using the value of the stock price in (41) and plugging the result in the definition of agent 2’s endowment, in (2). Finally, the lagrange multiplier of the unconstrained agent is set to \( y_1 = (1 - s_{20})^{-\gamma}\delta_0^{-\gamma} > 0. \]

**Proof of Proposition 3.** From the first order conditions of the representative agent’s problem
\[ u(c_t, \lambda_t) = \max_{c_{1t} + c_{2t} = c_t} \left\{ \frac{c_{1t}^{1-\gamma}}{1-\gamma} + \lambda_t \log c_{2t} \right\} \]
we obtain \( \xi_{1t} = e^{-\rho t} \frac{u_c(\delta_t, \lambda_t)}{u_c(\delta_0, \lambda_0)} \), where \( \lambda_t \) is a strictly positive process which represents the time varying weight of the constrained agent and is identified by the ratio of marginal utilities,
\[ \lambda_t = c_{2t}/c_{1t} = s_{2t}(1 - s_{2t})^{-\gamma}\delta_t^{1-\gamma}. \tag{42} \]

The bubble component follows from subtracting from the stock price its fundamental value and Fubini’s theorem,
\[ b_t = \rho^{-1}s_{2t}\delta_t - E \left[ \int_t^\infty \frac{\xi_1(s, s_{2s}, \delta_s)}{\xi_1(t, s_{2t}, \delta_t)} s_{2s}\delta_s ds \middle| F_t \right] \]
\[ = s_{2t}\delta_t E \left[ \int_t^\infty e^{-\rho(s-t)} \left( 1 - \frac{s_{2s}(1 - s_{2s})^{-\gamma}\delta_s^{1-\gamma}}{s_{2t}(1 - s_{2t})^{-\gamma}\delta_t^{1-\gamma}} \right) ds \middle| F_t \right] \]
\[ = \delta_t^{\gamma}(1 - s_{2t})^{-\gamma} \int_t^\infty e^{-\rho(s-t)} (\lambda(s_{2s}, \delta_t) - E [\lambda(s_{2s}, \delta_s) | F_t]) ds. \]

The agents’ optimal consumption plans must solve the representative agent’s utility maximization problem and it follows that \( c_{1t} = u_c(\delta_t, \lambda_t)^{-\frac{1}{\gamma}} = (1 - s_{2t})\delta_t \), and \( c_{2t} = \frac{\lambda_1 u_c(\delta_t, \lambda_t)}{u_c(\delta_0, \lambda_0)} = s_{2t}\delta_t \), and the lagrange multipliers implied by the equilibrium are given by \( y_1 = u_c(\delta_0, \lambda_0), y_2 = \frac{u_c(\delta_0, \lambda_0)}{\lambda_0}. \) The equilibrium weighting process is thus given by \( \lambda_t = \lambda_0 \xi_{1t}/\xi_{2t} \), where \( \xi_{2t} \) is given in (37). Applying Itô’s lemma in (42) gives
\[ \frac{d\lambda_t}{\lambda_t} = \left[ \beta_t (\nu) + \theta_{2t} (\theta_{2t} - \theta_1) \right] dt + (\theta_{2t} - \theta_1) dB_t, \tag{43} \]
and from Proposition (1), $\sigma_t \pi_t = \theta_{2t}$, $\theta_{2t} = \kappa_t \theta_t$, $\nu_t = -(1 - \kappa_t) \sigma_t \theta_t$. Replacing in the drift of equation (43) gives

$$\beta_t(\nu) + \theta_{2t} (\theta_{2t} - \theta_t) = \kappa_t (1 - \kappa_t) (\sigma_t^{-1} \theta_t) \sigma_t \theta_t = \kappa_t (1 - \kappa_t) \theta_t \theta_t = 0$$

which implies that $\lambda$ is a nonnegative local martingale.

**Proof of Propositions 4 and 7.** Set $I = (0, 1)$. (i) the drift and diffusion functions have continuous derivatives in $I$ and (ii) $(s_2 \sigma_{s_2})^2 > 0$ in $I$. We also verify a (iii) local integrability condition, for all $x \in I$, there exists $\epsilon > 0$ such that $\int_{x-\epsilon}^{x+\epsilon} \frac{1 + |\mu_{s_2}(y)|}{\sigma_s(x)^2} dy < \infty$. It is known that (i) implies that the coefficient are local Lipschitz, a sufficient condition for pathwise uniqueness of the solution (see Karatzas and Shreve (1991), Theorem 5.2.5). Also, conditions (i), (ii) and (iii) guarantee the existence of a weak solution (see Karatzas and Shreve (1991), Theorem 5.5.15) up possibly to an explosion time. The existence of a weak solution combined with pathwise uniqueness imply that equation (16) admits a strong solution up possibly to an explosion time, i.e., when $s_{2t}$ hits one of the boundaries (the endpoints of $I$).

Define the stopping times

$$T_\Delta = \inf \{ t \geq 0 : s_{2t} \geq \Delta \},$$

with $\Delta \in I$, and let

$$T_1 = \lim_{\Delta \uparrow 1} T_\Delta, \quad T_0 = \lim_{\Delta \downarrow 0} T_\Delta.$$

To rule out explosions, we proceed as follows. From equation (43), the weighting process is a nonnegative supermartingale under $\mathbb{P}$,

$$E[\lambda_T] \leq \lambda_0 > 0, \quad \forall T \in [0, \infty),$$

consequently, it is a.s. finite under the objective probability measure, which implies

$$\mathbb{P}[T_1 < T] = 0, \quad \forall T \in [0, \infty).$$

Let $s_{2t} = s_{2t \wedge T_\Delta}$. Using a comparison argument (see Proposition 5.2.18 in Karatzas and Shreve (1991)), we bound the stopped process $s_{2t}$ from below by a process $s_{\ell t}$, with dynamics

$$ds_{\ell t} = \mu_{s_{\ell t}}(s_{\ell t}) s_{\ell t} dt + \sigma_{s_{\ell t}}(s_{\ell t}) s_{\ell t} dB_t.$$

This process, by construction, never reaches the left boundary. Its diffusion is given by $x \sigma_{s_{x}}(x)$. We fix $s_{t0} = s_{20}$ and set the drift of $s_t$ such that $\mu_{s_{t}}(x) \leq \mu_{s_{\ell}}(x)$, which implies that $\forall \Delta \in (0, 1)$,

$$\mathbb{P}[T_0 < T_\Delta] = E[1_{\{T_0 < T_\Delta\}}] = 0,$$

(44)
that is, the probability of the consumption share process of hitting 0 before it reaches an arbitrary $\Delta$ in $(0, 1)$ is zero.

In order to show that $\mathbb{P}[T_0 < T] = 0$, $\forall T \in [0, \infty)$, it suffices to note that

$$E[\lim_{\Delta \uparrow 1} 1_{\{T_0 < T_\Delta\}}] \leq \lim_{\Delta \uparrow 1} E[1_{\{T_0 < T_\Delta\}}] = 0,$$

which follows from Fatou’s lemma, and (44), implying $\mathbb{P}[T_0 < T] = 0$, $\forall T \in [0, \infty)$, since the probability of reaching 1 in finite time is zero a.s.. To close the proof, we need to find candidate processes $s_t$ for each type of constraint.

(i) Mean-variance constraint. The natural candidate for $s_t$ corresponds to the consumption share process of a log agent in an unconstrained economy with risk aversion distribution $\{\gamma, 1\}$, time discount rate $\rho > 0$ and a dividend process with parameters $(\mu_0, \sigma)$. If parameters are such that utilities and price processes are finite, the consumption share process takes values in the set $(0, 1)$ if started in $(0, 1)$. One simple way to show it is by noticing that the process $s_t(1 - s_t) = \lambda_0 \delta_{-1}^{\gamma} a.s.$.

The processes $(s_t, s_2)$ share the same functional form in the diffusion term, and have different drifts. The dynamics of $s_t$ are given by

$$ds_t = \mu_s(s_t) s_t dt + \left[\frac{\gamma}{1 + (\gamma - 1)x} \right] s_t dB_t,$$

with

$$\mu_s(x) = \frac{\gamma(1 - x)(1 - \kappa)}{[1 + (\gamma - 1)x][1 + (\gamma - 1)x]} \times \left[ \mu_{\delta \delta} - \frac{\sigma^2_{\delta}}{2[1 + (\gamma - 1)x]} \left( 1 + \gamma - \frac{\gamma}{1 + (\gamma - 1)x} \right) \right].$$

The proof of Proposition 5 follows from Lemma 1 shows that the ratio of marginal utilities,

$$\lambda_t = c_{2t}/c_{1t} = s_{2t}(1 - s_{2t})^{-\gamma} \delta_{-1}^{\gamma}$$
is a true martingale when the agent is subject to a mean variance constraint, the bubble term in (23) is
zero. This implies that the price dividend ratio can be written as
\[ p(s_t) = E \left[ \int_t^\infty e^{-\int_t^u (r_u + \theta_u \sigma_u - \bar{\mu}_u)} du - \int_t^\infty (\theta_u - \bar{\mu}_u) dB_u \bigg| \mathcal{F}_t \right] \]
\[ = E \left[ \int_t^\infty e^{-\int_t^u (r_u + \theta_u \sigma_u - \mu_u)} du - \int_t^\infty (\theta_u - \mu_u) dB_u \bigg| \mathcal{F}_t \right] \]
\[ = \bar{E} \left[ \int_t^\infty e^{-\int_t^u (r_u + \theta_u \sigma_u - \mu_u)} du - \int_t^\infty (\theta_u - \mu_u) dB_u \bigg| \mathcal{F}_t \right] \]
where the density of the measure \( \bar{\mathbb{P}} \) is defined by the exponential martingale
\[ \frac{d\bar{\mathbb{P}}}{d\mathbb{P}} \bigg| \mathcal{F}_t = \bar{M}_t = e^{-\frac{1}{2} \int_0^t (\theta_s - \sigma_s)^2 ds - \int_0^t (\theta_s - \sigma_s) dB_s}. \]

**Proof of Proposition 6.** Using the portfolio choice of the constrained agent in (9) in the market price of risk in equation (11) gives,
\[ \theta = \left[ 1 - \left( 1 - \frac{1}{1 + (\theta/\varepsilon \sigma - 1)^\gamma} \right) R(s_2) \right]^{-1} R(s_2) \sigma_\delta \]
which is uniquely solved by the positive, continuous and piecewise differentiable function
\[ \theta(s_{2t}) = \gamma \sigma_\delta \frac{1 - \varepsilon s_{2t}}{1 - s_{2t}} 1_{\{R(s_{2t}) > \varepsilon\}} + \frac{\gamma \sigma_\delta}{1 + (\gamma - 1) s_{2t}} 1_{\{R(s_{2t}) \leq \varepsilon\}}. \]

**Proof of Proposition 8.** Assume that the parametric condition in (47) holds. The proof follows from
Lemma 2 and Novikov’s condition.

**Proof of Proposition 9.** The optimal consumption plan and trading strategy of the constrained agent are given by
\[ c_{2t} = \frac{e^{-\rho t \eta_t}}{y_2 \xi_2 t}, \quad \sigma_t \pi_{2t} = \theta_{2t} - \bar{\mu}_t. \]
Optimality of the constrained agent portfolio choice follows from solving
\[ \sup_{\pi \in \mathcal{C}_t} \left\{ \pi \sigma_t (\theta_t - \bar{\mu}_t) - \frac{1}{2} (\sigma_t \pi)^2 \right\}, \]
where \( \mathcal{C}_t = \{ \pi : (\sigma_t \pi)^2 \leq L^2 \} \). Applying Itô’s lemma to the process \( \lambda_t = \lambda_0 \eta_t \xi_{1t}/\xi_{2t} \), gives
\[ \frac{d\lambda_t}{\lambda_t} = \left[ \beta_t (\nu) + (\theta_{2t} - \bar{\mu}_t) (\theta_{2t} - \theta_t) \right] dt + (\theta_{2t} - \bar{\mu}_t - \theta_t) dB_t, \]
with \( \beta_t (\nu) + (\theta_{2t} - \bar{\mu}_t) (\theta_{2t} - \theta_t) = 0 \), and hence, the process \( \lambda_t \) is a local martingale. An application of Itô’s lemma to the state price density \( \xi_{1t} = e^{-\rho t u_2 (\delta_u, \lambda_t)} \) identifies the market price of risk as \( \theta_t = \)
\( R_t (\sigma_\delta - s_{2t} \Phi_t) \) and the interest rate as in (12), whereas the dynamics of the weighting process shows that \( \Phi_t = \theta_{2t} - \bar{\mu}_t - \theta_t \). On the other hand, from the portfolio policy of agent 2 one gets

\[
\sigma_t \pi_{2t} = \Pi [\theta_t - \bar{\mu}_t | \sigma_t C_t] = \frac{1}{1 + (|\theta_t - \bar{\mu}| / \varepsilon \sigma_\delta - 1)} (\theta_t - \bar{\mu}_t)
\]

\[
= \theta_{2t} - \bar{\mu}_t
\]

where we have set \( L = \varepsilon \sigma_\delta \), with \( \varepsilon \geq 0 \), and therefore, putting all together gives a nonlinear equation for the market price of risk,

\[
\theta = R(s_{2t}) \sigma_\delta - R(s_{2t}) s_2 \left[ \frac{1}{1 + (|\theta - \bar{\mu}| / \varepsilon \sigma_\delta - 1)} (\theta - \bar{\mu}) \right].
\]

The solution to this equation is uniquely given by the continuous function

\[
\theta(s_{2t}) = R(s_{2t}) (\sigma_\delta + s_{2t} \bar{\mu}) 1_{\{s_{2t} \in S_u\}} + \gamma \sigma_\delta \left[ \frac{1 - \varepsilon s_{2t}}{1 - s_{2t}} \right] 1_{\{s_{2t} \in S_1\}} + \sigma_\delta \left[ \frac{1 + \varepsilon s_{2t}}{1 - s_{2t}} \right] 1_{\{s_{2t} \in S_2\}},
\]

where \( S_u = \{ x \in (0, 1) : |R(x) (\sigma_\delta + x \bar{\mu}) - \bar{\mu}| \leq \varepsilon \sigma_\delta \} \), \( S_1 = \{ x \in (0, 1) : R(x) (\sigma_\delta + x \bar{\mu}) - \bar{\mu} > \varepsilon \sigma_\delta \} \), and \( S_2 = \{ x \in (0, 1) : -R(x) (\sigma_\delta + x \bar{\mu}) + \mu > \varepsilon \sigma_\delta \} \). The dynamics of the consumption share process obeys

\[
d s_{2t} = s_{2t} \mu_{s_2} (s_{2t}) dt + s_{2t} \sigma_{s_2} (s_{2t}) dB_t
\]

where

\[
\mu_{s_2} (s_{2t}) = \gamma (s_{2t}) - \rho + (\Phi(s_{2t}) + \theta(s_{2t}) - \sigma_\delta) (\theta(s_{2t}) - \sigma_\delta),
\]

\[
\Phi(s_{2t}) = -\bar{\mu}_1 1_{\{s_{2t} \in S_u\}} + \left[ \varepsilon \sigma_\delta - \gamma \sigma_\delta \left[ \frac{1 - \varepsilon s_{2t}}{1 - s_{2t}} \right] \right] 1_{\{s_{2t} \in S_1\}} - \left[ \varepsilon \sigma_\delta + \gamma \sigma_\delta \left[ \frac{1 + \varepsilon s_{2t}}{1 - s_{2t}} \right] \right] 1_{\{s_{2t} \in S_2\}},
\]

\[
\sigma_{s_2} (s_{2t}) = \Phi(s_{2t}) + \theta(s_{2t}) - \sigma_\delta
\]

\[
= -(1 - s_{2t}) \frac{\bar{\mu}_1 + (1 - \gamma) \sigma_\delta}{1 + (\gamma - 1) s_{2t}} 1_{\{s_{2t} \in S_u\}} - (1 - \varepsilon) \sigma_\delta 1_{\{s_{2t} \in S_1\}} - (1 + \varepsilon) \sigma_\delta 1_{\{s_{2t} \in S_2\}}.
\]

Following an argument similar to the one used in the proof of propositions 4 and 7, it can be shown that this process, if started in \((0, 1)\), does not reach either zero or one in finite time.

\[\blacksquare\]

**Proof of Proposition 10.** We explicitly solve the individual agent’s problem for \( n \) risky assets in the proof of Proposition 1. Following an argument similar to the one used in the proof of Propositions 2 and 3, we construct a consumption sharing rule and identify the state price density from agent 1’s optimality condition. We assume that the volatility matrix is invertible at all times, and hence, markets are dynamically complete for the unconstrained agent. The market prices of risk are the solution to a system of nonlinear equations of the form

\[
\theta_t = R(s_{2t}) (\sigma_\delta + s_{2t} (1 - \kappa_t) \theta_t),
\]

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which once solved, allows us to identify all equilibrium quantities as functions of the consumption share. Note that the solution to the above equation is particularly simple for the mean-variance constraint. The ratio of marginal utilities is given by
\[ \lambda_t = c_t^2 / c^\gamma_t, \]
applying Itô’s lemma to \( \lambda \) gives
\[ \frac{d\lambda_t}{\lambda_t} = -(1 - \kappa(s_{2t})) \theta(s_{2t})^\top dB_t, \]
which implies that \( \lambda \) is a local martingale, and thus, the test for the presence of bubbles in the market portfolio is performed exactly as in the one risky asset case. The starting point and dynamics of the consumption share of the constrained agent is obtained using the same procedure of Proposition 2. We omit the details.

A.2 Testing for the presence of bubbles

Lemma 1. (Martingality of \( \lambda_t \) with constant risk reduction) The equation
\[ \lambda_t = \lambda_0 - \int_0^t \lambda_s \frac{(1 - \kappa)\gamma \sigma_\delta}{1 + (\gamma \kappa - 1)s_{2s}} dB_s, \]
with \( \kappa \in (0, 1] \), admits a unique and strong solution and is a strictly positive martingale.

Proof. The volatility of the logarithm of the weighting process given by function
\[ \Phi (x) = -\frac{(1 - \kappa)\gamma \sigma_\delta}{1 + (\gamma \kappa - 1)x} \]
which is continuous and bounded in \([0, 1]\), for \( \kappa > 0 \). The existence of a unique strong solution follows from Theorem V.6 in Protter (2004), in particular, note that the diffusion term \( \lambda \Phi(\cdot) \) is (random) Lipschitz. From Proposition 2, the process \( \lambda \) is a local martingale. To establish that \( \lambda \) is a strictly positive martingale, it suffices to show the process as an exponential martingale with initial datum \( \lambda_t = s_{20}(1 - s_{20})^{-\gamma} \mu_0^{-\gamma} > 0 \) and verify the Novikov condition, which follows from the boundedness of \( \Phi(\cdot) \).

Lemma 2. (Martingality of \( \lambda_t \) with time varying risk reduction when constraints are always binding) The equation
\[ \lambda_t = \lambda_0 + \int_0^t \lambda_s \left[ \varepsilon \sigma_\delta - \gamma \sigma_\delta \frac{1 - \varepsilon s_{2s}}{1 - s_{2s}} \right] dB_s, \] (45)
with \( \varepsilon \in [0, 1] \) and \( \sigma_\delta > 0 \), admits a unique and strong solution, and is a positive local martingale but fails to be a true martingale when \( \gamma \geq 1 \). If \( \gamma < 1 \) and \( (1 - \gamma) (\mu_\delta - \frac{1}{2} \gamma \sigma_\delta^2) - \frac{1}{2} (1 - \varepsilon)^2 \sigma_\delta^2 > 0 \), the process is a true martingale.

Proof. We use the exponential local martingale
\[ M_t^\lambda = \frac{\lambda_t}{\lambda_0} = e^{-\int_0^t \Phi(s_{2s})^2 ds + \int_0^t \Phi(s_{2s}) dB_s}, \]
where
\[
\Phi(x) = \varepsilon \sigma_\delta - \gamma \sigma_\delta \frac{1 - \varepsilon x}{1 - x}
\]
as the density of a candidate equivalent change of measure \( \mathbb{P}^\lambda \). We verify the properties of the consumption share process, whose dynamics under \( \mathbb{P}^\lambda \) follows
\[
ds_{2t} = \mu^\lambda_{s_2}(s_{2t})s_{2t}dt - (1 - \varepsilon)\sigma_\delta s_{2t}dB_t^\lambda
\]
with
\[
\mu^\lambda_{s_2}(x) = (\gamma - 1) \frac{1 - x}{1 + (\gamma - 1)x} \mu_\delta + (\gamma - 1) \gamma \frac{2 - \varepsilon (2 - \varepsilon) x}{2(1 - x)} \sigma_\delta^2 + (1 - \varepsilon)^2 \sigma_\delta^2,
\]
where \( dB^\lambda_t = dB_t - \Phi(s_{2t}) dt \) is a \( \mathbb{P}^\lambda - \) Brownian motion. In equilibrium, the consumption share lives in \((0, 1)\), therefore, if under \( \mathbb{P}^\lambda \) the process hits one of the boundaries with positive probability, \( \mathbb{P}^\lambda \) could not be equivalent to \( \mathbb{P} \) because this behavior is not possible under the objective measure. As shown by Heston, Loewenstein, and Willard (2007) (Th. A.1.), this is equivalent to testing for the martingality of \( \lambda \).

When \( \gamma = 1 \), the process in (46) under \( \mathbb{P}^\lambda \) corresponds to a geometric Brownian motion, and hence, it reaches 1 in finite time with positive probability, \( \mathbb{P}^\lambda[ T_1 < T ] > 0 \). This contradicts the equivalence between \( \mathbb{P} \) and \( \mathbb{P}^\lambda \), and thus, the solution in equation (45) is a strictly positive local martingale but fails to be martingale.

When \( \gamma > 1 \), the drift diverges to plus infinity when \( s_2 = 1 \) (note that the numerator in the second term is positive as \( s_2 \to 1 \)), we get \( \mathbb{P}^\lambda[ T_1 < T ] > 0 \) by using a standard comparison argument (the drift can be bounded from below using a linear function). This contradicts the equivalence between \( \mathbb{P} \) and \( \mathbb{P}^\lambda \), and thus, the solution in equation (45) is a strictly positive local martingale but fails to be martingale.

When \( \gamma < 1 \), unlike the previous cases, the behavior of the process under \( \mathbb{P}^\lambda \) resembles its behavior under \( \mathbb{P} \), that is, its drift diverges to negative infinity when \( s_2 \to 1 \).

In order to obtain information about the behavior of the consumption share as it approaches 1, we construct its scale function, \( S_c(x) \),
\[
S_c(x) = \int_c^x \exp \left[ -2 \int_c^y \frac{z \mu_{s_2}(z)}{z^2 \sigma_{s_2}} dz \right] dy
\]
\[
= \frac{(1 - x) x}{b_3 - 1} \left[ \frac{1 - x}{1 - c} \right]^{-(1 + \gamma)} \left[ \frac{x}{c} \right]^{b_3 - 1} \left[ \frac{1 + (\gamma - 1)x}{1 + (\gamma - 1)c} \right]^{-b_2} F_1 \left[ 1, b_1, b_2, b_3, x, \frac{\gamma x}{1 + (\gamma - 1)c} \right]
\]
\[
- \frac{(1 - c) c}{b_3 - 1} F_1 \left[ 1, b_1, b_2, b_3, c, \frac{\gamma c}{1 + (\gamma - 1)c} \right]
\]

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where \( c \) is an arbitrary constant in \( I \) and \( F_1(\cdot) \) denotes the (Appell) hypergeometric function of two variables,\(^{28}\) with

\[
\begin{align*}
  b_1 &= \frac{2\mu_s - \gamma [2 - \varepsilon (2 - \varepsilon)] \sigma_\delta^2}{(1 - \varepsilon)^2 \sigma_\delta^2}, \\
  b_2 &= -2\gamma \mu_s - \frac{1}{2} \gamma \sigma_\delta^2 - 1, \\
  b_3 &= 2(1 - \gamma) \mu_s - \frac{1}{2} \gamma \sigma_\delta^2 \\
  &\quad \quad \quad \quad \left(1 - \gamma \right) \left( \mu_s - \frac{1}{2} \gamma \sigma_\delta^2 \right) - \frac{1}{2} (1 - \varepsilon)^2 \sigma_\delta^2 > 0,
\end{align*}
\]

When \( b_3 - 1 > 0 \) or equivalently,

\[
(1 - \gamma) \left( \mu_s - \frac{1}{2} \gamma \sigma_\delta^2 \right) - \frac{1}{2} (1 - \varepsilon)^2 \sigma_\delta^2 > 0,
\]

we obtain \( \lim_{\Delta \to 1} Sc(\Delta) = \infty \), which is a sufficient condition that guarantees that starting from any point in \( I \), the right boundary cannot be reached in finite time, i.e., \( \mathbb{P}^{\lambda}[T_1 < T] = 0 \) (see Karatzas and Shreve (1991), p. 348). To ensure that the share process does not reach 0, we use a comparison argument similar to that of Proposition 4, we omit the details.

\(\blacksquare\)

**B Examples**

B.1 Solving for the stock prices

B.1.1 One risky asset: boundary value problems with homogeneous beliefs

When the constraints are always binding, the coefficients in equation (27) are:

(i) Mean variance constraint.

\[
\begin{align*}
  g(x) &= \rho + (\gamma - 1) \frac{1 - x}{1 + (\gamma - 1) x} \mu_s + \frac{\gamma}{1 + (\gamma - 1) x} \sigma_\delta^2 - \frac{\gamma [1 + \gamma + (\gamma - 1)(1 + 2\gamma) x]}{2[1 + (\gamma - 1) x]^2} \sigma_\delta^2 \\
  &\quad \quad \quad \quad \left(1 - \gamma \right) \left( \mu_s - \frac{1}{2} \gamma \sigma_\delta^2 \right) - \frac{1}{2} (1 - \varepsilon)^2 \sigma_\delta^2
\end{align*}
\]

\[
\sigma_{s_2}(x) = \left[\frac{\gamma \varepsilon}{1 + (\gamma \varepsilon - 1) x} - 1\right] \sigma_\delta.
\]

(ii) Volatility constraint.

\[
\begin{align*}
  g(x) &= \rho + (\gamma - 1) \frac{1 - x}{1 + (\gamma - 1) x} \mu_s + (\gamma - 1) \gamma \frac{[2 - \varepsilon (2 - \varepsilon)x] x - 1}{2[1 - x][1 + (\gamma - 1) x]} \sigma_\delta^2, \\
  \sigma_{s_2}(x) &= -(1 - \varepsilon) \sigma_\delta.
\end{align*}
\]

\(^{28}\)See Whittaker and Watson (1990), Ex. 22, p. 300. and [http://functions.wolfram.com/HypergeometricFunctions/AppellF1/](http://functions.wolfram.com/HypergeometricFunctions/AppellF1/)
B.1.2 n risky assets: cash flow model and approximation method

Cash flow model. The share \( x_{it} \) follows the mean reverting process,\(^{29}\)
\[
\begin{align*}
\frac{dx_{it}}{x_{it}} &= \eta(\bar{x}_i - x_{it})dt + x_{it}\sigma^i_t (x_t) dB_t, \\
\end{align*}
\]
where \( \bar{x}_i \in (0, 1) \) represents the long run mean, \( \eta > 0 \) is the speed of mean reversion, and the term \( \sigma^i_t (x_t) \) in the volatility is determined by \( \sigma^i_t (x_t) = v^i - x^i_t v = v^i - \sum_{j=1}^{n} x_{jt} v^j \), with
\[
\begin{align*}
x_t &= \begin{bmatrix} x_{1t} \\ \vdots \\ x_{nt} \end{bmatrix}, \\
v &= \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}, \\
v^i &= \begin{bmatrix} v^i_1 & \cdots & v^i_n \end{bmatrix}.
\end{align*}
\]

The term \( \sigma^i_t (x_t) \) ensures that \( \sum_{i=1}^{n} x_{it} = 1 \), while the mean reverting drift implies that no asset will dominate the economy. Note that the process \( x \) lies on the unit simplex at all times. By applying Itô’s lemma to \( \delta^i_t = \delta_i x_{it} \), the dynamics of the dividends of asset \( i \) follow \( \frac{d\delta^i_t}{\delta^i_t} = \left[ \mu_i + \eta \left( \frac{\bar{x}_i}{x_{it}} - 1 \right) + \theta^i_{CF} - x^i_t \theta_{CF} \right] dt + \left[ \sigma_\delta + \sigma^i_t (x_t) \right] dB_t \), where \( \theta^i_{CF} = v^i \sigma_\delta \).

Approximation method. One equilibrium that can be easily characterized is the no-bubble equilibrium with mean-variance constraints and \( \kappa = 1/\gamma \), with \( \gamma > 1 \). Using the fact that the consumption share process is deterministic \( (\sigma_{s_2}(\cdot) = 0) \) and applying Fubini’s theorem, stock prices are given by
\[
S_t^i = s_{2t} \delta_i \int_t^\infty e^{-\rho(s-t)} s_{2s}^{-1} E^\lambda [x_{is} | \mathcal{F}_t] ds
\]
where \( E^\lambda \) is the expected value operator under the probability measure whose density is defined by the exponential martingale \( \lambda/\lambda_0 = e^{-\frac{1}{2} \|\Phi\|^2 + \Phi^\top B_t} \) with \( \Phi = (1-\gamma)\sigma_\delta \in \mathbb{R}^n \).

The value of the consumption share at time \( t \) can be obtained by inverting the price dividend ratio function of the baseline case model. The expected value in \((48)\) solves a linear PDE that can be approached numerically. However, we describe a polynomial approximation that performs well when compared to Monte Carlo simulations. We describe the method for two assets, which can be easily generalized to \( n > 2 \). In the example, the matrix \( v \), which contains the row vectors \( v^i \), is a diagonal matrix with \( ii \)-element given by \( v^i \).

The term \( E^\lambda [x_{i,t+1} | \mathcal{F}_t] \) is computed by adapting a method suggested in Gabaix (2008) to approximate arbitrary processes with linearity-generating processes, which fits here nicely as powers of \( x \) can be ordered decreasingly up to arbitrary order. We use a third order polynomial to approximate the expected value. Let \( Y = (y_1, y_2, y_3)^\top \), where \( y_t = x^f_t \). Setting \( \gamma = 2 \), \( y_3 = y_0 = 0 \) and \( \psi = \theta^1_{CF} - \theta^2_{CF} \), and using the slaving principle (see Gabaix (2008)), an application of Itô’s lemma shows that \( Y \) follows approximately
\[
dY_t = (A_0 + A_1 Y_t) dt + \Sigma (Y_t) dW^\lambda_t,
\]
\(^{29}\)The use of share process to distribute the aggregate endowment across firms was first introduced in Menzly, Santos, and Veronesi (2004).
with
\[
A_0 = \begin{bmatrix} \eta \bar{x} & 0 & 0 \end{bmatrix}^\top,
\]

\[
A_1 = \begin{bmatrix}
- (\eta + \psi) & \psi & 2 (\psi - \|v\|^2) + \frac{\eta \|v\|^2}{(\eta + \psi - \frac{3}{2} \|v\|^2)} \\
2 \eta \bar{x} & -2 (\eta + \psi - \frac{1}{2} \|v\|^2) & 2 \left( \psi - \|v\|^2 \right) + \frac{\eta \|v\|^2}{(\eta + \psi - \frac{3}{2} \|v\|^2)} \\
0 & 3 \eta \bar{x} & -3 \left( \eta + \psi - \|v\|^2 \right) - \left( \psi - 2 \|v\|^2 \right) \left( \eta + \psi - \frac{3}{2} \|v\|^2 \right)
\end{bmatrix}
\]

The approximate dynamics of \( Y \) imply
\[
E^\lambda [Y_{t+\tau}] | F_t = \Psi (\tau) Y_t + \int_0^\tau \Psi (\tau - s) A_0 ds,
\]
where \( \Psi (\tau) \), a 3 \times 3 matrix, solves the homogeneous linear system with constant coefficients. If \( A_1 \) has real and distinct eigenvalues \( \varpi_i \), then \( \Psi (\tau) = U \left[ e^{\varpi \cdot \tau} \right] U^{-1} \), where \( \varpi \) is the vector of eigenvalues, \( e^{\varpi \cdot \tau} \) is a diagonal matrix whose \( ii \)–element is given by \( e^{\varpi_i \tau} \) and \( U \) is the matrix of associated eigenvectors (the three linearly independent eigenvectors of \( A_1 \) are set up as columns). The expected value in (48) is computed as
\[
E^\lambda [x_{t+\tau}] | F_t = E^\lambda [e_1 Y_{t+\tau}] | F_t
\]
where \( e_1 = (1, 0, 0) \).

References


Figure 1: (Market price of risk and interest rate with mean-variance constraints). The figure plots the market price of risk and the interest rate in an economy with mean-variance constraints. Parameters are set to $\mu = 0.02$ and $\sigma = 0.05$, left panel $\gamma = 2$, right panel $\gamma = 1/2$, and $\rho = 0.03$. The solid line corresponds to an unconstrained economy.
Figure 2: (Portfolio policy with mean-variance constraints). The figure plots the portfolio policy of the constrained agent, $\pi_2$, in an economy with mean-variance constraints. Parameters are set to $\mu_\delta = 0.02$, $\sigma_\delta = 0.05$, left panel $\gamma = 2$, right panel $\gamma = 1/2$, and $\rho = 0.03$. The fraction of wealth invested in the risky asset decreases with the severity of the constraint. The solid line corresponds to an unconstrained economy. When $\kappa = \frac{1}{\gamma}$, agents find optimal to trade only in the stock and the optimal consumption policy is a deterministic proportion of the aggregate endowment. The market price of risk is given by the constant $\theta = \gamma \sigma_\delta$ and the volatility of the stock price equals its fundamental term.
Figure 3: (Volatility of returns with mean-variance constraints). The left panel shows how the volatility decreases as the dispersion in ‘effective’ risk aversion is narrowed. The right panel shows that the volatility increases as the dispersion in ‘effective’ risk aversion increases. The volatility attains its maximum when the second agent has a larger share of aggregate consumption. Parameters are set to $\mu_3 = 0.02$, $\sigma_3 = 0.05$, left panel $\gamma = 2$, right panel $\gamma = 1/2$, and $\rho = 0.03$. The solid line corresponds to an unconstrained economy.
Figure 4: (Market price of risk and interest rate with volatility constraints). The market price of risk diverges to plus infinity and the interest rate diverges to minus infinity as the consumption share approaches one if the constraint is active. Parameters are set to $\mu_\delta = 0.02$, $\sigma_\delta = 0.05$, left panel $\gamma = 2$, right panel $\gamma = 1/2$, and $\rho = 0.03$. The solid line corresponds to an unconstrained economy.
Figure 5: (Portfolio policy with volatility constraints). The figure plots the portfolio policy of the constrained agent, $\pi_2$, in an economy with volatility constraints. The fraction of wealth invested in the risky asset decreases with the severity of the constraint. Parameters are set to $\mu_\delta = 0.02$, $\sigma_\delta = 0.05$, left panel $\gamma = 2$, right panel $\gamma = 1/2$, and $\rho = 0.03$. The solid line corresponds to an unconstrained economy.
Figure 6: *(Volatility of returns with volatility constraints).* The figure plots the volatility of returns in an economy with volatility constraints. The left panel shows how the volatility decreases as the dispersion in ‘effective’ risk aversion is narrowed. Parameters are set to $\mu_{d} = 0.02$, $\sigma_{d} = 0.05$, left panel $\gamma = 2$, right panel $\gamma = 1/2$, and $\rho = 0.03$. The solid line corresponds to an unconstrained economy.