Introduction to dynamic macroeconomics: Analysis of Saving and Investment Once Again
Outline of lecture

A. Investment and capital accumulation
   with given output and factor prices: partial equilibrium

B. The Solow model: exogenous saving

C. The Ramsey-Cass-Koopmans model
   in discrete time: endogenous saving and investment in general equilibrium
A. Two visions of the demand for capital

• Both visions will assume that there is
  – a constant returns to scale (CRTS) production function at the firm and aggregate level.
  – market in which capital rents at rate “q”.

• The production function will be
  – $y=F(k,n)$ in general
  – $y=ak^{1-\alpha}n^\alpha$ (Cobb-Douglas form) for examples
A1. Demand for capital with labor given

- A firm maximizes profit, $F(k,n)-qk$, taking as given the quantity of labor ($n$).
- With labor fixed, there is a diminishing marginal product of capital.
- Profit maximization requires $F_k(k,n) - q = 0$.
- The demand for capital is $k^d = \kappa(q,n)$, depending negatively on the rental price.
- Increases in the quantity of labor raise the demand for capital.
Vision 1 in a diagram
A2. Demand for capital with output given

- A firm can also buy labor at wage rate \( w \)
- A firm minimizes cost \( wn + qk \) taking factor prices and an output level \( y \) as given.
- Changes in \( w \) and \( q \) induce factor substitution (along an isoquant)
- With CRTS, the factor demands are
  - \( k^d = \kappa (q/w)y \)
  - \( n^d = \eta (q/w)y \)
- Increases in output (scale) and the wage rate (substitution) raise the demand for capital, increases in the rental rate lower it (substitution).
Vision 2 in a diagram
Investment and the demand for capital

• Suppose that a period is the length of time that it takes for investment \( (i) \) to become productive as capital \( (k) \).

• The rental market can reallocate the existing capital stock, but cannot change the predetermined quantity.

• Production: \( y_t = F(k_t, n_t) \) with \( k \) predetermined

• Capital accumulation: \( k_{t+1} - k_t = i_t - \delta k_t \), where \( \delta \) is the depreciation rate.
Investment and the demand for capital
(note that the $\kappa$ functions are different across visions)

Vision 1:

$$k_{t+1} = \kappa(q_{t+1}, n_{t+1})$$

Vision 2:

$$k_{t+1} = \kappa(q_{t+1} / w_{t+1})y_{t+1}$$

Either:  \[ i_t = (k_{t+1} - k_t) + \delta k_t \]

Total  = Net investment + replacement investment
Interest and rental rates

• An individual who invests in capital at time $t$ earns a return $q_{t+1} - \delta$. (To maintain his capital, he must deduct depreciation).

• The cost of borrowing to invest in capital is $r_t$ (sometimes this is dated as $r_{t+1}$).

• Absence of profits implies $r_t = q_{t+1} - \delta$
2. Solow’s dynamic model

- Stress on production function of neoclassical form – smooth substitution between factor inputs.
- Cambridge controversy: when is an aggregate production function useful?
- Short-cut of fixed saving rate “s” brought a great deal of tractability.
Four key equations

- Production function
  \[ y_t = F(k_t, n_t) \]

- Marginal products
  \[ w_t = F_n(k, n) \]
  \[ q_t = F_k(k, n) \]

- Capital accumulation
  \[ k_{t+1} - k_t = sf(k_t) + (1 - \delta)k_t \]

  \[ f(k) = F(k, 1) \] when assume 1 unit of labor
Stationary point

- Value of capital such that if $k_t = k$ then $k_{t+1} = k$, which is the condition that $sf(k) = \delta k$
- With Cobb-Douglas

$$0 = k_{t+1} - k_t = s[ak_t^{1-\alpha}n^\alpha] - \delta k_t$$

$$\Rightarrow \frac{k}{n} = \left(\frac{s\alpha}{\delta}\right)^\alpha$$
Stationary point: $sf(k) = \delta k$
Golden rule

- Max $c = f(k) - \delta k$ with respect to $k$
- Optimal saving rate?
Transitional Dynamics I: A Function
Transitional dynamics II: A Path

\[ k_t \text{ as ratio to } k_{ss} \]

\[ y_t \text{ as ratio to } y_{ss} \]

years

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Implications for factor prices

• Under CRTS, marginal products depend only on ratio (k/n). Hence

  – Since k grows along transition path and since marginal product of k falls, q (and r) must fall

  – Since k grows along transition path and since marginal product of labor increases in k, w must rise
Implications for factor prices

\[ w_t \text{ as ratio to } w_{ss} \]

\[ f_t = mpk_t \delta \text{ (\% per yr)} \]

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Adding growth
Adding growth
Adding growth
C. Outline of RCK model discussion

1. The Setup
2. Stationary Points
3. Dynamic paths from low initial k
4. Local dynamics
5. Properties of local, global paths
6. Market interpretations
C1. The Setup

- Social planner maximizes

\[ U = \lim_{T \to \infty} U_T = \lim_{T \to \infty} \sum_{t=0}^{T} \beta^t u(c_t) \]
Constraints

• Initial capital given

\[ k_0 \]

• Resource constraint each period

\[ \Phi(k_t) - k_{t+1} - c_t \geq 0 \]

• Terminal capital positive

\[ k_{T+1} \geq 0 \]

\[ \Phi(k_t) \text{ is short-hand for } f(k_t) + (1 - \delta)k_t \]
Finite horizon Lagrangian

• Multipliers on each constraint

\[ L = \sum_{t=0}^{T} \beta^t u(c_t) + \sum_{t=0}^{T} \beta^t \lambda_t [\Phi(k_t) - k_{t+1} - c_t] + \theta[k_{T+1}] \]

• Efficiency condition are same as for infinite horizon problem below, except for last period
Terminal capital issues

- Think about from Kuhn-Tucker perspective: don’t leave valuable stuff

\[
\frac{\partial L}{\partial k_{T+1}} = -\beta^T \lambda_T + \theta = 0
\]

\[
\frac{\partial L}{\partial \theta} \geq 0 \quad k_{T+1} \geq 0 \quad \frac{\partial L}{\partial \theta} k_{T+1} = 0
\]

Transversality condition (TC): \( \beta^T \lambda_T k_{T+1} = 0 \)
Infinite horizon Lagrangian

\[ L = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \beta^t \lambda_t [\Phi(k_t) - k_{t+1} - c_t] \]
Necessary conditions for optimum

\[ c_t : \beta^t [u_c(c_t) - \lambda_t] = 0 \]

\[ k_{t+1} : \beta^t [-\lambda_t + \beta \Phi_k(k_{t+1}) \lambda_{t+1}] = 0 \]

\[ \beta^t \lambda_t : \Phi(k_t) - k_{t+1} - c_t = 0 \]

\[ TC : \lim_{t \to \infty} \beta^t \lambda_t k_{t+1} = 0 \]
C2. Stationary points

• One feasible stationary point is to just stay with the consumption level that is “sustainable” with initial capital

\[ c_t = f(k_0) - \delta k_0 = \Phi(k_0) - k_0 \]

\[ U = \frac{1}{1 - \beta} u(c) \]
Optimal stationary point

- If the initial capital stock is such that the last condition holds with equality, then there is no contradiction. The stationary point is optimal. Thus occurs when

\[ 0 = [-1 + \beta \Phi_k(k^*)] \Leftrightarrow f_k(k^*) - \delta - \nu = 0 \]

- It is optimal to keep consumption and capital constant when the “net return” to capital is equal to the rate of time preference: these levels are called the “modified golden rule” levels of c and k. They are less than the golden rule levels.
Key economic lesson

• When current consumption must be sacrificed to form capital that yields future consumption, it is not optimal to maximize stationary utility $u(c)$.

• Generally, the solution to dynamic optimum problems does not involve maximizing the momentary objective $u$. The exception occurs if the decisionmaker does not face real intertemporal trade-offs.
Reconsidering the $k_0$ stationary point

• Is this optimal? It *is* feasible in the sense that it satisfies the resource constraint. And the TC is satisfied too. In terms of the other two conditions, we’d have

\[
\begin{align*}
    c_t : [u_c(c) - \lambda] &= 0 \\
    k_{t+1} : \lambda[-1 + \beta \Phi_k(k_0)] &> 0
\end{align*}
\]

• Constant consumption is not optimal because return to capital formation exceeds time preference. It is desirable to give up current consumption in exchange for future consumption
C. Dynamic paths from $k_0 < k^*$

- Assertion: there is only one path from each initial $k$ that (a) satisfies the FOCs and (b) has a limiting value of $k^*$.

- Easier to understand this feature in continuous time because we can draw “phase plane”.
Generating a path

• Start at any shadow price, initial capital

• This implies consumption (from FOC: c)

• Consumption and capital imply next period’s capital (resource constraint)

• Current shadow price and next period’s capital imply next period’s shadow price.

• Continue…
Suboptimality of alternative paths

• Assume that utility and production are both strictly concave

\[ u(c) < u(c^*) + u_c(c^*)[c - c^*] \]
\[ f(k) < f(k^*) + f_k(k^*)[k - k^*] \]

• This implies that

\[ U(\{c_t^*\}_{t=0}^{\infty}) - U(\{c_t\}_{t=0}^{\infty}) > 0 \]
Why? Diagram
Why? Algebra

\[ 0 = \sum_{t=0}^{\infty} \beta^t \lambda^*_t \{ [\Phi(k^*_t) - c^*_i - k^*_{t+1}] - [\Phi(k^*_t) - c_i - k^*_{t+1}] \} \]

\[ = \sum_{t=0}^{\infty} \beta^t \lambda^*_t [c_i - c^*_i] \]

\[ + \sum_{t=0}^{\infty} \beta^t \lambda^*_t [\Phi(k^*_t) - \Phi(k_t) + \Phi_k (k^*_t)(k_t - k^*_t)] \]

\[ + \sum_{t=0}^{\infty} \beta^t \lambda^*_t [(k^*_{t+1} - k^*_t) - \Phi_k (k^*_t)(k_t - k^*_t)] \]
Why? (algebra cont’d)

• Last line is zero because feasible paths must start from the same point $k_0$ and optimal paths have a specific growth in shadow prices (red components equal zero)

$$\sum_{t=0}^{\infty} \beta^t \lambda_t^* [(k_{t+1} - k_{t+1}^*) - \Phi_k(k_t^*)(k_t - k_t^*)]$$

$$= \Phi_k(k_t^*)(k_0 - k_0^*)$$

$$+ \sum_{t=0}^{\infty} \beta^t [\lambda_t^* - \beta \lambda_{t+1}^* \Phi_k(k_{t+1}^*)](k_{t+1} - k_{t+1}^*) = 0$$
Why? The result

- Concavity in $u$ and $f$ deliver implication using $\lambda_t^* = u_c(c_t^*)$

$$U(\{c_t^*\}_{t=0}^\infty) - U(\{c_t\}_{t=0}^\infty)$$

$$= \sum_{t=0}^\infty \beta^t [u(c_t^*) - u(c_t) + u_c(c_t^*)(c_t - c_t^*)]$$

$$+ \sum_{i=0}^\infty \beta^i u_c(c_t^*)[\Phi(k_t^*) - \Phi(k_t) + \Phi_k(k_t^*)(k_t - k_t^*)] > 0$$
D. Local dynamics

- Taylor series approximation of FOCs

\[ u_{cc}(c_t - c^*) = (\lambda_t - \lambda^*) \]
\[ (\lambda_{t+1} - \lambda^*) + \beta \Phi_{kk} \lambda^* (k_{t+1} - k^*) = (\lambda_t - \lambda^*) \]
\[ (k_{t+1} - k^*) = \Phi_k (k_t - k^*) - (c_t - c^*) \]
Substitute out for “c”

\[
(c_t - c^*) = \frac{1}{u_{cc}} (\lambda_t - \lambda^*)
\]

\[
(\lambda_{t+1} - \lambda^*) + \beta \Phi_{kk} \lambda^* (k_{t+1} - k^*) = (\lambda_t - \lambda^*)
\]

\[
(k_{t+1} - k^*) = \Phi_k (k_t - k^*) - \frac{1}{u_{cc}} (\lambda_t - \lambda^*)
\]
Local dynamics in matrix form (of capital and shadow price, after substituting out for consumption)

\[
\begin{bmatrix}
1 & \beta \Phi_{kk} \lambda^* \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
(\lambda_{t+1} - \lambda^*) \\
(k_{t+1} - k^*)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 \\
-\frac{1}{u_{cc}} & \Phi_k
\end{bmatrix}
\begin{bmatrix}
(\lambda_t - \lambda^*) \\
(k_t - k^*)
\end{bmatrix}
\]

\[AY_{t+1} = BY_t \Rightarrow Y_{t+1} = MY_t\]
Eigenvalues

\[
0 = \begin{bmatrix}
1 & \beta \Phi_{kk} \lambda^* \\
0 & 1
\end{bmatrix} z - \begin{bmatrix}
1 & 0 \\
-\frac{1}{u_{cc}} & \Phi_k
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(z - 1) \beta \Phi_{kk} z \\
\frac{1}{u_{cc}} & z - \Phi_k
\end{bmatrix}
\]

\[
= (z - 1)(z - \Phi_k) - \frac{u_c}{u_{cc}} \beta \Phi_{kk} z
\]
Eigenvalues (cont’d)

• The eigenvalues can be “located” in terms of size and the influence of various factors explored, as follows.
  
  • First, note that $1 = \beta \Phi_k$ or that $\Phi_k = 1 + \nu > 1$ where $\nu > 1$
  
  • Second, note that the equation above can be written as the intersection of a quadratic and a line with positive slope

$$(z - 1)(z - (1 + \nu)) = \frac{u_c}{u_{cc}} \beta \Phi_{kk} z$$
Eigenvalues cont’d (graph)

one root is $0 < \mu_s < 1$ and one root is $\mu_u > 1 + \nu$
Solutions for $k$ and $\lambda$

\[ k_t - k^* = q_{ku} \mu_u^t + q_{ks} \mu_s^t \]

\[ \lambda_t - \lambda^* = q_{\lambda u} \mu_u^t + q_{\lambda s} \mu_s^t \]

\[ \lim_{t \to \infty} [\beta^t \lambda_t k_{t+1}] = 0 \]

\[ \Rightarrow \lim_{t \to \infty} [\beta^t (\lambda_t - \lambda^*)(k_{t+1} - k^*)] = 0 \]

\[ \Rightarrow q_{\lambda u} = 0; \quad q_{ku} = 0 \]
Stable dynamics

\[ k_t - k^* = q_{ks} \mu_s^t = (k_0 - k^*) \mu_s^t = \mu_s (k_{t-1} - k^*) \]

\[ \lambda_t - \lambda^* = q_{\lambda s} \mu_s^t = (\lambda_0 - \lambda^*) \mu_s^t \]

\[ (k_{t+1} - k^*) = (1 + \nu)(k_t - k^*) - \frac{1}{u_{cc}}(\lambda_t - \lambda^*) \]

\[ \mu_s (k_0 - k^*) = (1 + \nu)(k_0 - k^*) - \frac{1}{u_{cc}}(\lambda_0 - \lambda^*) \]

\[ (\lambda_0 - \lambda^*) = u_{cc}[(1 - \mu_s) + \nu](k_0 - k^*) \]
E. Properties of global, local paths

- Capital rises from low initial level to optimal stationary level, as in Solow
- Marginal product of capital falls through time (with capital stock), as in Solow. So too does the market rental and interest rate, \( r_t = q_{t+1} - \delta \)
- Marginal product of labor rises through time (with capital stock), as in Solow,
- Shadow price \( \lambda \) falls from high initial level to optimal stationary level
- Consumption rises from low initial level to optimal stationary level (inversely related to shadow price)
- With power utility, consumption growth rate declines over time (as a result of Fisher’s rule)
- Saving rate \( (i/y) \) may either rise or fall (or remain constant if there is log utility, Cobb-Douglas production, and \( \delta = 1 \)).
Using Fisher’s rule

\[ c_t - c^* = [(1 - \mu_s) + \nu](k_t - k^*) \]

\[ \frac{c_t - c^*}{c^*} = [(1 - \mu_s) + \nu]\left[\frac{k^*}{c^*}\right]\left(\frac{k_t - k^*}{k^*}\right) \]

\[ \log\left(\frac{c_t}{c^*}\right) = [(1 - \mu_s) + \nu]\left[\frac{k^*}{c^*}\right]\log\left(\frac{k_t}{k^*}\right) = \theta \log\left(\frac{k_t}{k^*}\right) \]

\[ r_t - \nu = \sigma[\log(c_{t+1}/c^*) - \log(c_t/c^*)] = \sigma(\mu_s - 1)\theta \log\left(\frac{k_t}{k^*}\right) \]
F. Market interpretations

• Shadow prices from planner’s problem can be used to decentralize this “optimal allocation” as a competitive equilibrium for an infinitely lived representative individual
  – Of a Hicksian form with initial date markets
    Intertemporal prices are \( P_t = \beta^t p_t = \beta^t \lambda_t \)
  – Of a Fisherian form with sequential markets
    Interest rates are \( (1+r_t) = \lambda_t / \beta \lambda_{t+1} \)