Lecture 6: Dynamic models of consumption and investment
Modern perspectives

• Consumption is forward-looking because people prefer smooth consumption profiles and can manage their saving to this end.

• Investment is forward-looking because firm’s demand for new capital goods depends on a present-value of profits from such investments.
Two models

• The modern variant of the “life-cycle/permanent-income” model of consumption as developed by Hall

• The modern variant of investment with “capital adjustment costs” as developed by Lucas, Prescott and Hayashi.
General empirical problem

- Theory says that behavior depends on present-discounted value (pdv)
- Evaluating the theory requires modeling this pdv, which can be complicated.
- Two modern approaches circumvent this pdv modeling problem by clever, but different eliminations of it.
Approaches

• Consumption (Hall): work off efficiency condition ("euler equation") and use general property of expectation forecasting errors, which is that errors should be unrelated to available information

• Investment (LHP): work off efficiency condition and relate unobserved pdv to observable variable, the ratio of firm value to replacement cost of capital (Tobin’s Q)
Earlier approaches (as applied to consumption)

- “Permanent income” model relates consumption to “annuity value of future income”

\[ c_t = k \times y_{pt} \]

\[ \left[ \sum_{j=0}^{\infty} b^j \right] y_{pt} = \left[ \sum_{j=0}^{\infty} b^j E_t y_{t+j} \right] \]
Variants in prior work

• Friedman: treat permanent income as average of past income (only rational in some settings)

\[ y_{pt} = \theta y_{p,t-1} + (1 - \theta)y_t = (1 - \theta)\sum_{j=0}^{\infty} \theta^j y_{t-j} \]

• Sargent: treat permanent income as outcome of forecasting with specific model

\[ y_t = \pi s_t \quad s_t = Ms_{t-1} + Ge_t \]

\[ y_t = \sum_{j=0}^{\infty} b^j E_t y_{t+j} = \pi \sum_{j=0}^{\infty} b^j E_t s_{t+j} = \pi [I - bM]^{-1} s_t \]
A. Hall’s work on consumption

• Consider dynamic model of optimal consumption over time

\[
\max E_t \left[ \sum_{j=0}^{\infty} \beta^j u(c_{t+j}) \right]
\]

s.t. \( p(\zeta_t) a_{t+1} = a_t + y(\zeta_t) - c_t \)

where \( p_t = 1 / R_t \)

• Note that income and interest rate depend on a set of state variables, but this dependence is not made explicit.
Euler equation

• Utility consequences of a little more wealth and consumption tomorrow, at expense of consumption today, must be zero at an optimum

\[-u_c(c_t)p(c_t) + \beta E_t[u_c(c_{t+1})] = 0\]

or

\[-u_c(c_t) + \beta R(c_t)E_t[u_c(c_{t+1})] = 0\]
Hall’s observation

- Realized marginal utility is expected marginal utility plus an error term.
- Under RE, this error term should be uncorrelated with available information (prior work on “efficient markets” in finance had exploited this observation as well).
- With utility that was quadratic (or approximately so) then marginal utility would be linear (or approximately so).
- With interest rate just off-setting time preference, current and expected future consumption should then be equal under RE PIH model.
Steps

\[ u_c(c_t) = \beta R(\zeta_t) E_t[u_c(c_{t+1})] \]

\[ R \beta = 1 \Rightarrow \quad u_c(c_t) = E_t[u_c(c_{t+1})] \]

\[ \Rightarrow \quad u_c(c_{t+1}) = u_c(c_t) + \xi_{t+1} \]

\[ u_c = \phi - \gamma c_t \Rightarrow c_{t+1} = c_t - \frac{1}{\gamma} \xi_{t+1} \]
Test

• Linear regression, testing various x’s
  – Additional lags of consumption
  – Additional variables
    • Past income
    • Past wealth

\[ c_t = c_{t-1} + \theta x_{t-1} + e_t \]

\[ PIH \Rightarrow \theta = 0 \]
Initial findings

• Changes in consumption were surprisingly, largely unpredictable: Hall found consumption was “random walk”

• Sometimes written in log form: growth rate of consumption unpredictable

• Tests on micro data by Hall and Mishkin
  – Surprisingly hard to predict individual consumption changes also
Extensions

• Time varying interest rate (under loglinearity)

\[ \log(c_{t+1}) - \log(c_t) = \frac{1}{\sigma} [\log(R_t / \beta)] + e_{t+1} \]

  – Result is small intertemporal substitution elasticity $1/\sigma$ is small.
  – Estimated using an instrumental variables approach (more details in semester 2)

• Nonlinear estimation (Hansen-Singleton) applied to Euler equations for multiple assets
Challenges

• Campbell-Mankiw studied effect of a measure of expected income growth

\[
\log(c_{t+1}) - \log(c_t) = \frac{1}{\sigma} \left[ \log(R_t / \beta) \right] \\
+ \kappa \left[ E_t \log(y_{t+1} / y_t) \right] + e_{t+1}
\]

• Estimate “big” (\(\kappa\) about .4) and “significant” coefficient. Much dispute about interpretation, but rejection of basic model
B. Investment

• Consider a firm seeking to maximize its present value, subject to an accumulation equation that penalizes large movements in capital (h is positive, increasing, and strictly concave).

• Firm faces time-varying productivity of capital and time-varying investment good price (p). Constant discount factor for simplicity only

\[
\max \sum_{j=0}^{\infty} b^j E_t (a_t k_t - p_t i_t)
\]

s.t.

\[
k_{t+1} - k_t = h\left(\frac{i_t}{k_t}\right) k_t
\]
Key features

• Homogeneity of firm’s problem: a kind of dynamic constant-returns to scale.
  – Motivated by applied work showing growth rates of firms do not depend importantly on level.

• Implications for value of firm: \( v_t = w_t k_t \) with \( w_t \) not depending on capital stock \( k_t \).
Dynamic program

\[ v(k_t, \zeta_t) = \max_{k_{t+1}, i_t} \left[ a(\zeta_t) k_t - p(\zeta_t) i_t \right] \]

\[ + bE_t v(k_{t+1}, \zeta_{t+1}) \]

\[ s.t. \quad k_{t+1} - k_t = h\left(\frac{i_t}{k_t}\right) k_t \]
Lagrangian

\[ L = [a(\zeta_t)k_t - p(\zeta_t)i_t] + bE_t v(k_{t+1}, \zeta_{t+1}) \]

\[ + \lambda_t [h(\frac{i_t}{k_t})k_t + k_t - k_{t+1}] \]
FOCs and ET

\[ i_t : -p(\zeta_t) + \lambda_t h_z(z_t) = 0 \quad \text{with} \quad z_t = \frac{i_t}{k_t} \]

\[ k_{t+1} : -\lambda_t + bE_t v_k(k_{t+1}, \zeta_{t+1}) \]

\[ ET : v_k(k_t, \zeta_t) = a(\zeta_t) + \lambda_t [1 + h(z_t) - z_t h_z(z_t)] \]
So \( z = (i/k) \)

- Can be “explained” entirely by \( p/\lambda \)
  - Multiplier is “marginal value” of another unity of capital tomorrow, \( p \) is current cost of terms of investment

- Depends on form of \( h(z) \), particularly its extent of decreasing returns to \( z \).

- Problem: multiplier is unobservable
Market value

• Suppose firm pays out all profits as dividends. Then, its “ex dividend” market value is

\[ v_t - [a_t k_t - p_t i_t] = bE_t v_{t+1} \]

Ex dividend value

\[ = bk_{t+1} E_t v_{k,t+1} \]

Homogeneity

\[ = k_{t+1} \lambda_t \]

Efficient Investment
Why was it ok to assume that value was proportional to $k$?
If future value is proportional to $k'$
then current value will be proportional to $k$.

\[
\frac{v(k_t, \varsigma_t)}{k_t} = \max_{\frac{k_t+1}{k_t}, \frac{i_t}{k_t}} \left[ a(\varsigma_t) - p(\varsigma_t) \frac{i_t}{k_t} \right] \\
+ bE_t \frac{v(k_{t+1}, \varsigma_{t+1})}{k_{t+1}} \frac{k_{t+1}}{k_t} \\
\text{s.t.} \quad \frac{k_{t+1}}{k_t} = [h(\frac{i_t}{k_t}) + 1]
\]
Tobin’s Q

- James Tobin hypothesized that investment would be related to

\[ Q = \frac{\text{market value of firm}}{\text{replacement cost of capital}} \]

- This model (and others like it) delivers Tobin’s view since

\[ Q = \frac{v_t - \pi_t}{p_t k_{t+1}} = \frac{\lambda_t k_{t+1}}{p_t k_{t+1}} = \frac{\lambda_t}{p_t} \]
Theory and tests

• Theory predicts \( i/k \) depends on \( Q \) and only \( Q \) (perfect fit)

• Tests find some association between \( i/k \) and \( Q \), but far from perfect fit
  – Mismeasurement of capital stock?
  – Firm value depends on other factors (e.g., patents)
  – Model wrong on other dimensions (homogeneity, instant adjustment of \( i \), …)
“Blackboard” work

- Adding a stochastic discount factor
- Implications for DP
- Implications for investment rule
- Links to Lucas-Prescott