A Central Limit Theorem for Belief Functions

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1. CLT for Belief Functions

The purpose of this Note is to prove a form of CLT (Theorem 1.4) that is used in Epstein and Seo (2011). More general central limit results and other applications will follow in later drafts.

Let \( S = \{B, N\} \) and \( \mathcal{K}(S) = \{\{B\}, \{N\}, \{B, N\}\} \) the set of nonempty subsets of \( S \). Denote by \( s^\infty = (s_1, s_2, \ldots) \) the generic element of \( S^\infty \) and by \( \Psi_n(s^\infty) \) the empirical frequency of the outcome \( B \) in the first \( n \) experiments in sample \( s^\infty \).

Let \( \theta \) be a belief function on \( S \), that is, there exists \( m \in \Delta(\mathcal{K}(S)) \) such that, for every \( A \subset S \),

\[
\theta(A) = m(\{K \in \mathcal{K}(S) : K \subset A\}).
\]

Its conjugate \( \theta^* \) is given by

\[
\theta^*(A) = 1 - \theta(S \setminus A),
\]

and the product \( \theta^\infty \) is the belief function on \( S^\infty \) satisfying, for every \( A \subset S^\infty \),

\[
\theta^\infty(A) = m^\infty\left(\{\tilde{K} = K_1 \times K_2 \times \ldots \in \mathcal{K}(S^\infty) : \tilde{K} \subset A\}\right). \tag{1.1}
\]

Here \( m^\infty \) is the ordinary i.i.d. product of the measure \( m \).

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The LLN asserts certainty that asymptotic empirical frequencies will lie in the interval \([\theta (B), 1 - \theta (N)]\), that is,
\[
\theta^\infty \{ s^\infty : \liminf \Psi_n (s^\infty), \limsup \Psi_n (s^\infty) \in [\theta (B), 1 - \theta (N)] \} = 1.
\]
The CLT describes (up to approximation) beliefs about finite sample frequencies. A simple CLT is provided first because it provides perspective on later results. Let \(N (\cdot)\) be the cdf of a standard normal distribution.

**Theorem 1.1.**

\[
\lim_{n \to \infty} \theta^\infty \left( \left\{ s^\infty : \sqrt{n} \frac{\Psi_n (s^\infty) - \theta^* (B)}{\sqrt{\theta (\{N\}) (1 - \theta (\{N\}))}} \leq \alpha \right\} \right) = N (\alpha)
\]
\[
\lim_{n \to \infty} \theta^\infty \left( \left\{ s^\infty : \sqrt{n} \frac{\Psi_n (s^\infty) - \theta (\{B\})}{\sqrt{\theta (\{B\}) (1 - \theta (\{B\}))}} > \alpha \right\} \right) = 1 - N (\alpha).
\]

The proof follows readily from the next lemma showing that for the events indicated, the minimizing measures are i.i.d. As a result classical limit theorems applied to these measures deliver corresponding limit theorems for the i.i.d. product \(\theta^\infty\).

**Lemma 1.2.** Let \(P^*\) and \(P_*\) be the measures on \(S\) with \(P_* (B) = \theta (B)\) and \(P^* (B) = \theta^* (B)\) respectively. Then their i.i.d. products, denoted \(P_*^\infty\) and \(P^{*\infty}\), both lie in \(\text{core} (\theta^\infty)\); and, for any \(0 \leq t \leq 1\),
\[
\theta^\infty (\{ s^\infty : t \leq \Psi_n (s^\infty) \}) = P_*^\infty (\{ s^\infty : t \leq \Psi_n (s^\infty) \}) \tag{1.2}
\]
and
\[
\theta^\infty (\{ s^\infty : \Psi_n (s^\infty) \leq t \}) = P^{*\infty} (\{ s^\infty : \Psi_n (s^\infty) \leq t \}). \tag{1.3}
\]

See the appendix for a proof.

A CLT for two-sided intervals is less trivial because minimizing measures are not easily identified. Thus the following theorem uses a different proof strategy and applies a version of the multidimensional Berry-Esseen Theorem (Dasgupta (2008, pp. 145-6)): If the \(d\)-dimensional random variables \(X_1, X_2, \ldots\) are i.i.d., \(E (X_1) = 0, \text{Var} (X_1)\) is the identity matrix, and if \(E (\|X_1\|)\) is finite, then there exists a constant \(K\) such that, for all \(n\),
\[
\sup_{C \subseteq \mathcal{C}} \left| \Pr \left( \frac{X_1 + \ldots + X_n}{\sqrt{n}} \in C \right) - \Pr (Z \in C) \right| \leq \frac{K}{\sqrt{n}}.
\]
Here $C$ is the collection of all convex subsets of $\mathbb{R}^d$, and $Z$ is standard normal and $\mathbb{R}^d$-valued.

Let $N_2(\cdot, \cdot; \rho)$ be the cdf for the bivariate normal with zero means, unit variances and correlation coefficient $\rho$, that is,

$$ N_2(\alpha_1, \alpha_2; \rho) = \Pr (Z_1 \leq \alpha_1, Z_2 \leq \alpha_2) $$

where $(Z_1, Z_2)$ is bivariate normal with the indicated moments.

**Theorem 1.3.** There is a constant $K$ that does not depend on $\alpha_1$, $\alpha_2$ or $n$, such that

$$ \left| \theta^\infty \left( \frac{\alpha_1}{\sqrt{n}} \right) \left( \frac{\alpha_2}{\sqrt{n}} \right) - \Psi_n(s^\infty) \right| \leq K \sqrt{n}. $$

Moreover, the same holds if $\alpha_1$ and $\alpha_2$ depend on $n$.

This theorem is a special case of the next one, but it also serves as a lemma in the proof of the more general result.

**Remark 1.** When $\theta$ is additive, the indicated correlation coefficient $\rho$ equals 1 and the inequality becomes

$$ \left| \theta^\infty \left( \frac{\alpha_1}{\sqrt{n}} - \frac{\alpha_2}{\sqrt{n}} \right) \right| \leq K \sqrt{n}, $$

where $Z$ is standard normal.

**Proof.** Define random variables$^1$

$$ X_i = I (K_i \in \mathcal{K} (S) : K_i \subset \{ B \}) \quad \text{and} \quad Y_i = 1 - I (K_i \in \mathcal{K} (S) : K_i \subset \{ N \}). \quad (1.4) $$

$^1$Note that $I (K_i \subset \{ B \}) + I (K_i \subset \{ N \}) = 1 - I (K_i \subset \{ B, N \}) \neq 1$. Thus the first two indicators are not perfectly negatively correlated.
Observe first that $X_i \leq Y_i$. Compute, using $m$, that $E(X_i) = \theta(B)$, $E(Y_i) = 1 - \theta(N)$, $Var(X_i) = \theta(B)(1 - \theta(B))$, $Var(Y_i) = (1 - \theta(N))\theta(N)$ and

$$
cov(X_i, Y_i) = E(X_i Y_i) - E(X_i)E(Y_i)
= E(X_i) - E(X_i)E(Y_i)
= \theta(B) - \theta(B)(1 - \theta(N))
= \theta(B)\theta(N).
$$

Here, the second equality follows because $X_i = 1$ implies $Y_i = 1$. Then

$$
E \left[ \left( \begin{array}{c}
X_i - \theta(B) \\
Y_i - (1 - \theta(N))
\end{array} \right) \right] = 0 \text{ and }
Var \left[ \left( \begin{array}{c}
X_i - \theta(B) \\
Y_i - (1 - \theta(N))
\end{array} \right) \right] = \left( \begin{array}{cc}
1 & \rho \\
\rho & 1
\end{array} \right),
$$

where

$$
\rho = corr(X_i, Y_i) = \frac{\theta(B)\theta(N)}{\sqrt{\theta(B)(1 - \theta(B))(1 - \theta(N))}\theta(N)}.
$$

Note that

$$
K_1 \times K_2 \times \ldots \subset \left\{ s^\infty : \beta_1 < \sum_{i=1}^n I(s_i = B) \leq \beta_2 \right\} \iff
\beta_1 < \min_{s^\infty \in K_1 \times K_2 \times \ldots} \sum_{i=1}^n I(s_i = B) \leq \max_{s^\infty \in K_1 \times K_2 \times \ldots} \sum_{i=1}^n I(s_i = B) \leq \beta_2 \iff
\beta_1 < \sum_{i=1}^n \min_{s^\infty \in K_1 \times K_2 \times \ldots} I(s_i = B) \leq \sum_{i=1}^n \max_{s^\infty \in K_1 \times K_2 \times \ldots} I(s_i = B) \leq \beta_2 \iff
\beta_1 < \sum_{i=1}^n I(K_i \subset \{B\}) \leq \sum_{i=1}^n [1 - I(K_i \subset \{N\})] \leq \beta_2 \iff
\beta_1 < \sum_{i=1}^n X_i \leq \sum_{i=1}^n Y_i \leq \beta_2
$$

Conclude from (1.1) that
\[
\theta^\infty \left( \left\{ s^\infty : \beta_1 < \sum_{i=1}^n I(s_i = B) \leq \beta_2 \right\} \right) \\
= m^\infty \left( \left\{ K_1 \times K_2 \times \ldots \in (\mathcal{K}(S))^\infty : \beta_1 < \sum_{i=1}^n X_i \leq \sum_{i=1}^n Y_i \leq \beta_2 \right\} \right).
\]

Consequently,
\[
\theta^\infty \left( \alpha_1 < \frac{n\Psi_n(s^\infty) - n\theta(B)}{\sqrt{n(1-\theta(B))\theta(B)}} \text{, } \frac{n\Psi_n(s^\infty) - n(1-\theta(N))}{\sqrt{n(1-\theta(N))\theta(N)}} \leq \alpha_2 \right)
= \theta^\infty \left( \frac{\alpha_1 \sqrt{n(1-\theta(B)) \theta(B)} + n\theta(B)}{\sqrt{n(1-\theta(N)) \theta(N)}} \leq \alpha_2 \text{ and } \frac{\sqrt{n(1-\theta(N)) \theta(N)}}{\alpha_2} + n(1-\theta(N)) \right)
= m^\infty \left( \frac{\alpha_1 \sqrt{n(1-\theta(B)) \theta(B)} + n\theta(B)}{\sqrt{n(1-\theta(N)) \theta(N)}} \leq \alpha_2 \text{ and } \frac{\sqrt{n(1-\theta(N)) \theta(N)}}{\alpha_2} + n(1-\theta(N)) \right)
= m^\infty \left( \alpha_1 < \frac{\sum_{i=1}^n X_i - n\theta(B)}{\sqrt{n(1-\theta(B)) \theta(B)}} \text{ and } \frac{\sum_{i=1}^n Y_i - n(1-\theta(N))}{\sqrt{n(1-\theta(N)) \theta(N)}} \leq \alpha_2 \right).
\]

This permits translation of the assertion to be proven into one about i.i.d. probability measures and thus classical results can be applied.

Use the Cholesky decomposition of the variance-covariance matrix to obtain \( V^* \) such that \((V^*)^{-1} \left( \begin{pmatrix} X_i - \theta(B) \\ Y_i - \theta(N) \end{pmatrix} \right) \) is standard normal (with correlation 0), and

\[
[a_1 < \frac{\sum_{i=1}^n X_i - n\theta(B)}{\sqrt{n(1-\theta(B)) \theta(B)}} \text{ and } \frac{\sum_{i=1}^n Y_i - n(1-\theta(N))}{\sqrt{n(1-\theta(N)) \theta(N)}} \leq \alpha_2] \iff
(V^*)^{-1} \left( \begin{pmatrix} \sum_{i=1}^n X_i - n\theta(B) \\ \sum_{i=1}^n Y_i - n(1-\theta(N)) \end{pmatrix} \right) \in C,
\]

for some convex \( C \subset \mathbb{R}^2 \). Therefore, by the multidimensional Berry-Esseen Theorem,

\[
\left| m^\infty \left( (V^*)^{-1} \left( \begin{pmatrix} \sum_{i=1}^n X_i - n\theta(B) \\ \sum_{i=1}^n Y_i - n(1-\theta(N)) \end{pmatrix} \right) \in C \right) - \Pr \left( \left( \begin{pmatrix} Z \\ Z' \end{pmatrix} \right) \in C \right) \right| \leq \frac{K}{\sqrt{n}}.
\]
for some constant $K$ that does not depend on $C$ or $n$, where $\left(\frac{Z}{Z'}\right)$ is standard normal.

Define
\[
\left(\frac{\tilde{Z}}{\tilde{Z}'}\right) = V^* \left(\frac{Z}{Z'}\right).
\]

Then
\[
\Pr \left( \left(\frac{Z}{Z'}\right) \in C \right) = \Pr \left( \alpha_1 < \tilde{Z}, \tilde{Z}' \leq \alpha_2 \right) = \Pr \left( -\tilde{Z} < -\alpha_1, \tilde{Z}' \leq \alpha_2 \right) = N_2 \left( -\alpha_1, \alpha_2; -\rho \right).
\]

Therefore,
\[
\left| \theta^\infty \left( \alpha_1 < \frac{n\Psi_n(s^\infty) - n\theta(B)}{\sqrt{n(1-\theta(B))\theta(B)}}, \frac{n\Psi_n(s^\infty) - n(1-\theta(N))}{\sqrt{n(1-\theta(N))\theta(N)}} \leq \alpha_2 \right) \right| \leq \frac{K}{\sqrt{n}}.
\]

Finally, the same proof works when $\alpha_1$ and $\alpha_2$ are replaced by $\alpha_{1,n}$ and $\alpha_{2,n}$.

The next theorem generalizes the preceding and is the main objective of this Note.

**Theorem 1.4.** Suppose that $G : \mathbb{R} \to \mathbb{R}$ is bounded, quasi-concave and upper-semicontinuous. Then
\[
\int G \left( \frac{n}{\sqrt{n(1-\theta(B))\theta(B)}}, \frac{n(1-\theta(N))}{\sqrt{n(1-\theta(N))\theta(N)}} \right) d\theta^\infty (s^\infty) = E \left[ \min \{G(X_{1n}'), G(X_{2n}')\} \right] + O \left( \frac{1}{\sqrt{n}} \right),
\]
where $(X_{1n}', X_{2n}')$ is normally distributed with mean $(\theta(B), \theta^*(B))$ and variance
\[
\frac{1}{n} \begin{pmatrix} \theta(B) (1 - \theta(B)) & \theta(B) \theta(N) \\ \theta(B) \theta(N) & (1 - \theta(N)) \theta(N) \end{pmatrix}.
\]

That is,
\[
\limsup \sqrt{n} \int G \left( \frac{n}{\sqrt{n(1-\theta(B))\theta(B)}}, \frac{n(1-\theta(N))}{\sqrt{n(1-\theta(N))\theta(N)}} \right) d\theta^\infty (s^\infty) - E \left[ \min \{G(X_{1n}'), G(X_{2n}')\} \right] \leq K
\]
for some constant $K$. Moreover, the same holds when $G$ depends on $n$ and $\sup_{n,a} |G_n(a)| < \infty$. 

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The preceding theorem is the special case where $G(\cdot) = 1_{[a_n, b_n]}(\cdot)$ and $a_n = \theta(B) + n^{-1/2} \alpha_1 (1 - \theta(B))^{1/2} \theta(B)$, $b_n = \theta^*(B) + n^{-1/2} \alpha_2 (1 - \theta^*(B))^{1/2} \theta^*(B)$.

**Proof.** Without loss of generality, suppose that $\inf_n G(a) = 0$. Define $M = \sup_n G(a)$. Since $G$ is quasiconcave, there are two (inverse) functions $G^{-1}_L$ and $G^{-1}_R$ such that

$$\{a : G(a) \geq t \} = [G^{-1}_L(t), G^{-1}_R(t)] \text{ for all } t \in [0, M].$$

(The inverses may not be defined at $t = 0$, but this is of no consequence below.)

By the definition of Choquet integration,

$$\int G(\Psi_n(s^\infty)) \, d\theta^\infty(s^\infty) = \int_0^M \theta^\infty(G(\Psi_n(s^\infty)) \geq t) \, dt = \int_0^M \theta^\infty(G^{-1}_L(t) \leq \Psi_n(s^\infty) \leq G^{-1}_R(t)) \, dt.$$

Note that

$$\theta^\infty(G^{-1}_L(t) \leq \Psi_n(s^\infty) \leq G^{-1}_R(t)) = \theta^\infty\left(\frac{nG^{-1}_L(t) - n\theta(B)}{\sqrt{n(1 - \theta(B))} \theta(B)} \leq \frac{n\Psi_n(s^\infty) - n\theta(B)}{\sqrt{n(1 - \theta(B))} \theta(B)} \leq \frac{nG^{-1}_R(t) - n(1 - \theta(N))}{\sqrt{n(1 - \theta(N))} \theta(N)} \leq \frac{n\Psi_n(s^\infty) - n(1 - \theta(N))}{\sqrt{n(1 - \theta(N))} \theta(N)}\right).$$

Thus, by Theorem 1.3,

$$\theta^\infty(G^{-1}_L(t) \leq \Psi_n(s^\infty) \leq G^{-1}_R(t)) = N_2\left(-\frac{nG^{-1}_L(t) - n\theta(B)}{\sqrt{n(1 - \theta(B))} \theta(B)}, \frac{nG^{-1}_R(t) - n(1 - \theta(N))}{\sqrt{n(1 - \theta(N))} \theta(N)}; -\rho\right) + O\left(\frac{1}{\sqrt{n}}\right)$$

with $\rho = \frac{\theta(B) \theta(N)}{\sqrt{\theta(B)(1 - \theta(B))}(1 - \theta(N)) \theta(N)}$. Because the term $O\left(\frac{1}{\sqrt{n}}\right)$ does not depend on $t$,

$$\int_0^M \theta^\infty(G^{-1}_L(t) \leq \Psi_n(s^\infty) \leq G^{-1}_R(t)) \, dt = \int_0^M N_2\left(-\frac{nG^{-1}_L(t) - n\theta(B)}{\sqrt{n(1 - \theta(B))} \theta(B)}, \frac{nG^{-1}_R(t) - n(1 - \theta(N))}{\sqrt{n(1 - \theta(N))} \theta(N)}; -\rho\right) \, dt + O\left(\frac{1}{\sqrt{n}}\right).$$
Let \( Z_1 \) and \( Z_2 \) be jointly normally distributed with mean \((0, 0)\) and variance
\[
\begin{pmatrix}
1 & \rho \\
\rho & 1
\end{pmatrix},
\]
and let \( X_{1n} \) and \( X_{2n} \) be normally distributed as in the theorem statement. Then
\[
\begin{align*}
\int_0^M & N_2 \left( -\frac{nG_L^{-1}(t) - n\theta(B)}{\sqrt{n(1 - \theta(B))\theta(B)}}, \frac{nG_R^{-1}(t) - n(1 - \theta(N))}{\sqrt{n(1 - \theta(N))\theta(N)}}; -\rho \right) \, dt \\
= & \int_0^M \Pr \left( G_L^{-1}(t) \leq \theta(B) + \frac{\sqrt{n(1 - \theta(N))\theta(N)}}{n} Z_1, \quad (1 - \theta(N)) + \frac{\sqrt{n(1 - \theta(N))\theta(N)}}{n} Z_2 \leq G_R^{-1}(t) \right) \, dt \\
= & \int_0^M \Pr (G_L^{-1}(t) \leq X'_{1n}, X'_{2n} \leq G_R^{-1}(t)) \, dt \\
= & \int_0^M \Pr (G(X'_{1n}) \geq t, G(X'_{2n}) \geq t) \, dt + O \left( \frac{1}{\sqrt{n}} \right) \\
= & \int_0^M \Pr (\min \{G(X'_{1n}), G(X'_{2n})\} \geq t) \, dt + O \left( \frac{1}{\sqrt{n}} \right) \\
= & E \left[ \min \{G(X'_{1n}), G(X'_{2n})\} \right] + O \left( \frac{1}{\sqrt{n}} \right).
\end{align*}
\]

To complete the proof, we need only prove the equality marked with an asterisk. Define random variables \((X_i, Y_i)_{i=1}^\infty\) as in (1.4). Then they are i.i.d. under \( m^\infty \) and, because \( X_i \leq Y_i \),
\[
\begin{align*}
\Pr \left( a \leq \frac{\sum_{i=1}^n X_i}{n}, \frac{\sum_{i=1}^n Y_i}{n} \leq b \right) \\
= & \Pr \left( a \leq \frac{\sum_{i=1}^n X_i}{n} \leq \frac{\sum_{i=1}^n Y_i}{n} \leq b \right) \\
= & \Pr \left( a \leq \frac{\sum_{i=1}^n X_i}{n} \leq b, a \leq \frac{\sum_{i=1}^n Y_i}{n} \leq b \right).
\end{align*}
\]
By the multidimensional Berry-Esseen Theorem,

\[
\Pr \left( a \leq \frac{\sum_{i=1}^{n} X_i}{n}, \frac{\sum_{i=1}^{n} Y_i}{n} \leq b \right) - \Pr(a \leq X'_{1n}, X'_{2n} \leq b) \leq \frac{K}{\sqrt{n}},
\]

and similarly

\[
\Pr \left( a \leq \frac{\sum_{i=1}^{n} X_i}{n}, \frac{\sum_{i=1}^{n} Y_i}{n} \leq b, a \leq X'_{1n}, a \leq X'_{2n} \leq b \right) - \Pr(a \leq X'_{1n} \leq b, a \leq X'_{2n} \leq b) \leq \frac{K}{\sqrt{n}}.
\]

It follows that, for all \(a \leq b\) and \(n\),

\[
\Pr(a \leq X'_{1n}, X'_{2n} \leq b) - \Pr(a \leq X'_{1n} \leq b, a \leq X'_{2n} \leq b) \leq \frac{2K}{\sqrt{n}}.
\]

This proves the marked equation.

Finally, the above proof works also if \(G\) is replaced by \(G_n\) such that \(\sup_{n,a} |G_n(a)| \leq \infty\). \(\Box\)

A. Appendix: Proof of Lemma 1.2

Prove (1.2).

Step 1: \(K_1 \times K_2 \times ... \subset \{ s^\infty : \sum_{i=1}^{n} I(s_i = B) \geq t \} \) iff \(\sum_{i=1}^{n} I(K_i \subset \{B\}) \geq t\).

Here is a proof:

\[
K_1 \times K_2 \times ... \subset \left\{ s^\infty : \sum_{i=1}^{n} I(s_i = B) \geq t \right\} \iff \min_{s^\infty \in K_1 \times K_2 \times ...} \sum_{i=1}^{n} I(s_i = B) \geq t \iff \sum_{i=1}^{n} \min_{s_i \in K_i} I(s_i = B) \geq t \iff \sum_{i=1}^{n} I(K_i \subset \{B\}) \geq t.
\]
Step 2: Let $m^\infty \in \Delta (\mathcal{K}(S^\infty))$ be the measure for $\theta^\infty$. Then
\[
\theta^\infty \left( \left\{ s^\infty : \sum_{i=1}^{n} I (s_i = B) \geq t \right\} \right) \\
= m^\infty \left( \left\{ K_1 \times K_2 \times \ldots \in \mathcal{K}(S^\infty) : \sum_{i=1}^{n} I (K_i \subset \{B\}) \geq t \right\} \right).
\]

Argue as follows:
\[
\theta^\infty \left( \left\{ s^\infty : \sum_{i=1}^{n} I (s_i = B) \geq t \right\} \right) \\
= m^\infty \left( \left\{ K \in \mathcal{K}(S^\infty) : K \subset \left\{ s^\infty : \sum_{i=1}^{n} I (s_i = B) \geq t \right\} \right\} \right) \\
= m^\infty \left( \left\{ K_1 \times K_2 \times \ldots \in \mathcal{K}(S^\infty) : K_1 \times K_2 \times \ldots \subset \left\{ s^\infty : \sum_{i=1}^{n} I (s_i = B) \geq t \right\} \right\} \right).
\]

Next apply Step 1.
Step 3: Complete the proof. By Step 2,
\[
\theta^\infty \left( \left\{ s^\infty : \Psi_n (s^\infty) \geq t \right\} \right) \\
= \theta^\infty \left( \left\{ s^\infty : \frac{1}{n} \sum_{i=1}^{n} I (s_i = B) \geq t \right\} \right) \\
= m^\infty \left( \left\{ K_1 \times K_2 \times \ldots \in \mathcal{K}(S^\infty) : \frac{1}{n} \sum_{i=1}^{n} I (K_i = \{B\}) \geq t \right\} \right) \\
= \Pr \left( \frac{1}{n} \sum_{i=1}^{n} Y_i \geq t \right),
\]

where $Y_i = 0$ or $1$, $\Pr (Y_i = 1) = \theta (B)$ and the $Y_i$’s are i.i.d. Therefore, the preceding equals $P_\ast^\infty \left( \left\{ s^\infty : \Psi_n (s^\infty) \geq nt \right\} \right)$.

To prove (1.3), reverse the roles of $B$ and $N$ in the preceding argument.  

References
