1. \((4+4)*2 = 16\) marks/ What can you say concerning existence and uniqueness of Walrasian equilibrium in the following exchange economies with \((l = 1, \ldots, L\) goods and \(i = 1, \ldots, I\) households) where household \(i\) has:

(a) utility function 
\[ U_i = \Pi_{l=1}^{L} (x_{il} - \overline{x}_l)^{\alpha_{il}}, \text{ endowment } \omega_{il} > \overline{x}_l \text{ for all } i, l, \text{ where } \overline{x}_l > 0 \text{ is a minimum subsistence requirement for consumption of good } l, \text{ and } \alpha_{il} > 0 \text{ for all } i, l. \]

(b) utility function 
\[ U_i = \sum_{l=1}^{L} \delta^{l-1} \frac{\rho_1^{l-1}}{1-\rho} \text{ where } \delta \in (0, 1), \rho > 0, \neq 1, \text{ and endowment of good } l \text{ is equal to } i.l. \]

In each case explain your reasoning in detail. You are free to invoke theorems discussed in class.

(a) The demand function of \(i\) is
\[ x_{il} (p_1, \ldots, p_L) = \overline{x}_l + \frac{\alpha_{il}}{p_l} \left[ \sum_k p_k (\omega_{ik} - \overline{x}_k) \right], \]
which is well-defined, continuous at every \(p >> 0\), homogenous of degree 0, satisfying Walras’ Law, bounded below by \(\overline{x}_l\), and unbounded above as \(p_l \to 0\). So all five conditions for existence of Walrasian equilibrium are satisfied. Moreover the demand functions satisfy the gross substitute property, so the equilibrium is unique.

(b) For any strictly positive price vector, the demand function of \(i\) is well-defined and continuous:
\[ x_{i1} = z_i p_1^{\frac{1}{\rho}}, x_{i2} = z_i \rho_1^{\frac{1}{1-\rho}} p_2^{\frac{1}{\rho}}, x_{i3} = z_i \rho_2^{\frac{1}{1-\rho}} p_3^{\frac{1}{\rho}} \]
where
\[ z_i \equiv \frac{p_1 + 2p_2 + 3p_3}{p_1^{\frac{1}{1-\rho}} + p_2^{\frac{1}{1-\rho}} + p_3^{\frac{1}{1-\rho}}} \]
It is evident that as the price of any good goes to 0 its demand goes to $\infty$. Therefore all five conditions are met for existence of a Walrasian equilibrium. The equilibrium is unique because the utility function is homothetic and endowments of different households are proportional to one another.

2. \[3*5=15 \text{ marks}\] Are the following statements true or false? If true provide a proof; if false describe a counterexample in as much detail as possible.

(i) Every economy with a CRS production technology has a unique Walrasian equilibrium.

(ii) In a replication of an exchange economy with continuous, monotone and (weakly) convex preferences, every core allocation satisfies equal treatment in utility terms, i.e., every agent of a given type receives the same utility.

(iii) An ex post Pareto optimal allocation is always ex ante Pareto optimal.

(i) False. Consider an economy with two goods and two consumers each of whose demands satisfy WARP but so that the aggregate demand in the economy violates WARP (as shown in class) with respect to price vectors $p$ and $q$, say. Then consider production set $Y \equiv \{y \in \mathbb{R}^2 | p.y \leq 0, q.y \leq 0\}$, and check that both $p$ and $q$ are Walrasian equilibria in the economy with this CRS technology.

(ii) True. Consider the coalition of worst-off agents, one in each type, and let them construct an allocation in which any given type gets the average consumption bundle (across all agents of that type, in the original allocation). Feasibility of this allocation for the coalition is obvious from the construction. Weak convexity of preferences means that each agent in the coalition is at least as well off as before. If there was some type that was unequally treated in utility terms relative to others of the same type, then such an agent must be strictly better off. This permits the coalition to block the original allocation (since preferences are strictly monotone, the better-off agent can redistribute some of his utility gain to all the others and still come out ahead).
(iii) False. Consider an economy with one consumption good \((L = 1)\), uncertainty in endowments for every agent, and strictly concave von-Neumann-Morgenstern utility. Then any consumption allocation is \textit{ex post} Pareto optimal, since after a state is realized there is no scope for trade among agents given there is a single consumption good. But only some allocations are \textit{ex ante} Pareto optimal, those satisfying the Arrow-Borch equations.

3. \[5+7+7=19 \text{ marks}\] Consider an exchange economy with \(L\) goods distinguished by physical characteristics, \(I\) households, \(S\) states of the world \(s = 1, \ldots S\) at \(t = 1\), and trading in \(K\) financial assets at \(t = 0\), where asset \(k\) pays off \(r_{sk} > 0\) units of good \(1\) in state \(s\). All households share the same belief that state \(s\) will arise with probability \(\pi_s > 0\). Household \(i\) has a state-independent von-Neumann-Morgenstern utility function \(u_i(x_{i1}, x_{i2}, \ldots, x_{iL})\) which is strictly increasing, strictly concave, twice continuously differentiable, with the marginal utility of any good \(l\) equal to \(\infty\) if good \(l\) consumption equals zero. The economy has no aggregate risk, so while household endowment vector \((\omega_{i1s}, \ldots, \omega_{iLs}) \gg 0\) varies with the state, the economy-wide aggregate endowments of each good do not vary with the state.

(a) Derive a set of necessary and sufficient conditions for an allocation of consumption plans in this economy to be \textit{ex ante} Pareto optimal.

(b) Show that in any \textit{ex ante} Pareto optimal allocation, the consumption of each and every good for any given household must be constant across all states.

(c) Show that if the economy has a complete set of financial assets, show that the spot commodity price vector is state independent, and assets will be priced according to their expected returns (i.e., the ratio of prices of any pair of assets will equal the ratio of their expected returns). Provide an intuitive explanation of this result.

(a) Let \(\lambda_i\) denote a positive welfare weight for household \(i\). Then an \textit{ex ante} P.O. allocation maximizes \(\sum_i \sum_s \pi_s u_i(x_{i1}, x_{i2}, \ldots, x_{iL})\) subject to the resource constraint \(\sum_i x_{ils} = W_l\) for each \(l, s\), where \(W_l\) denotes the state-independent economy-wide endowment of good \(l\). This is a concave optimization problem, and under the assumptions made all solutions are
interior, characterized by the first-order conditions: there are positive multipliers $\delta_{l}, l = 1, \ldots, L$ such that for any $i, l, s$:

$$\lambda_{i} \pi_{s} u_{il} = \delta_{l}$$

where $u_{il}$ denotes marginal utility of $i$ for good $l$ in state $s$. These imply for instance the following generalization of the Arrow-Borch equations to the multi-good case: the ratio of marginal utilities

$$\frac{u_{il}}{u_{jm}} = \frac{\delta_{l}}{\delta_{m}} \frac{\lambda_{j}}{\lambda_{i}}$$

must be constant across states, for any pair of goods $l, m$ and any pair of households $i, j$.

(b) In the single good case the Arrow-Borch equation gives the result directly, given the absence of any economy-wide risk. The extension of this argument to the multi-good case is not straightforward (unless utility is additively separable across goods, in which case the same argument goes through, commodity by commodity. But if marginal utilities of different goods are interdependent it is not evident how it generalizes).

However a different argument works quite simply. Suppose the state-contingent consumption vector $x_{js}$ of some household $j$ varied across some pair of states. Construct for every agent $i$ the alternative consumption plan consisting of a constant consumption vector $y_{i} = \sum_{s} \pi_{s} x_{is}$ across all states. This is the expected consumption vector in the previous allocation. It is easily checked that this alternative allocation is feasible for the economy:

$$\sum_{i} y_{i} = \sum_{i} \sum_{s} \pi_{s} x_{is} = \sum_{s} \pi_{s} \sum_{i} x_{is} = \sum_{s} \pi_{s} W = W$$

where $W$ denotes the constant endowment vector. Since agents are strictly risk averse, agent $j$ is strictly better off, and no one else is worse off. So we can construct an \textit{ex ante} Pareto improvement.

(c) Since the consumption vector of every household is constant across states, and preferences are strictly convex, the spot commodity price vector must be state-independent (as this price vector is proportional to ratios of marginal utility for every household at a constant consumption vector). Let $p$ denote the constant spot-commodity price vector. Then
each household $i$ will demand an asset portfolio at $t = 0$ to maximize

$$
\sum_s \pi_s V_i(p, p.\omega_i s + \sum_k r_s k z_k) \quad \text{s.t.} \quad \sum_k q_k z_k = 0.
$$

This generates the first order condition (with $\lambda_i$ denoting the multiplier on the budget constraint, and $Y_i s$ the income of $i$ in state $s$):

$$
q_k = \frac{1}{\lambda_i} \sum_s \pi_s \frac{\partial V_i}{\partial Y_i s} r_s k = \frac{1}{\lambda_i} \frac{\partial V_i}{\partial Y_i} \sum_s \pi_s r_s k
$$

because the marginal (indirect) utility of income for any $i$ will be state-independent (given that both consumption vector and the spot price are state-independent).\footnote{A full proof of this is that the \textit{ex post} allocation is \textit{ex post} Pareto optimal in the economy in any given state $s$, so by the second Welfare Theorem is supported by prices $p$ and lumpsum transfers $T_i s$. It must be the case that $p.x_i = p.\omega_i s + T_i s$. So the \textit{ex post} income $Y_i s$ of $i$ must equal $p.x_i$, which is state-independent. The asset incomes play the role of state-contingent lump sum transfers.} So asset prices are proportional to expected returns. The intuitive reason is that with a complete set of financial markets the agents are able to insure against risk perfectly, so they value assets entirely according to their expected returns.