Does the delay option facilitate coordination between a large number of agents?

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Draft

August 7, 2014

Abstract

The paper presents a framework with multiple periods in which the delay option makes coordination between a large number of agents with payoff strategic complementarities more difficult because it increases the incentive to free-ride on the information provided by the observation of the aggregate level of activity. It analyzes conditions under which the option to delay facilitates or hampers the coordination of agents for investment in a state where investment would be optimal under perfect information.

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1 Introduction

Does time facilitate the coordination of agents whose actions are related by payoff externalities (e.g., bank runs, speculative attacks against a currency)? Time has two effects. First, it provides the opportunity of learning, which can be critical in coordination games; second it may enable agents to choose the timing of their action. When agents have the opportunity of learning and can delay their action, they will do so in order to take advantage of new information. In that case the transmission of information through individual actions is inefficient and aggregate activity is delay, as shown by Chamley and Gale (1994) and Chamley (2004) in models that focused only on the information externality, with no payoff externality.

The case of payoff externalities where time provides the opportunity for learning but agents cannot delay has been analyzed by Chamley (1999). Under some assumptions in an analytical model, there is a unique equilibrium that is strongly rationalizable (with a unique strategy that survives the iterated elimination of dominated strategies). A relaxation of some assumptions show that a numerical model approximates well the properties of the analytical model. The analysis highlights a property that will be critical here.

In each period, there is a continuum of agents who live one period (no delay) and who learn from past aggregate activity. These agents are heterogenous and defined by an individual parameter $c$, from a hump-shaped distribution, say with a cumulative distribution function $F(c, \theta_t)$, where the state of nature $\theta_t$ evolves randomly and slowly from period to period. The equilibrium strategy is monotone: in each period $t$, any agent who takes the action 1 (instead of 0) has an individual parameter, that is that is lower than a threshold $c^*_t$. Because of the payoff externality, the equilibrium values $c^*_t$ must be in of the tails of the distribution of individual parameters. The aggregate activity conveys a signal on the state of nature $\theta$ through the mass of agents in one of the tails (either in the lower tail or in the difference between the total mass, which is known, and the mass in the upper tail who is not taking action).

Because agents have little information from the observation of the state of nature, while the state of nature evolves slowly over time, there are long intervals of time during which most of the agents invest or do not invest. These intervals are separated by sudden switches of regime when the “center” of the distribution of individual parameters approaches the threshold $c^*_t$. A switch comes as a surprise, ex ante but when it takes place, it reveals a large amount of information and it generates the effect of
“wisdom after the fact” (Caplin and Leahy, 1994).

Consider now the possibility of delay and assume that the public belief is such that the level of aggregate activity is low. The incentive to delay for more information will reduce the level of aggregate activity, as when there are only informational externalities, but that effect may be enhanced when there are payoff externalities. That is the subject of this paper.

The issue has already been addressed in Dasgupta (2007) and Esposito (2011): agents play a game with strategic complementarity in two periods. In Dasgupta (2006) the provision of an option to delay, which is costly, reduces the incidence of coordination failure relative to the static benchmark where choices are made only in one period. Also, an intermediate cost of delay minimizes the incidence of coordination failure. In Dasgupta’s framework agents are identical except for the signal that they receive and in particular they are all provided with an option to delay. The general thrust of his results is that, compared with the static case of one period, under some parameter conditions the inter-temporal context reduces the set of the values of the fundamental for which there is a coordination failure. An important step in the mechanism is that some agents, with extreme personal signals, take an action in the first period and provide information that induces other agents to follow in the second period.

Esposito (2011) relaxes the assumption that everybody has the option to delay. The population is divided in two groups, leaders and followers (in fixed proportions): the leaders have the option to delay their choices, the followers the obligation to delay. This model nests between Morris-Shin (1998) and Dasgupta (2006) benchmarks: assuming that the proportion of followers is one, i.e. agents move simultaneously in the second period, the model reduces to Morris-Shin (1998); assuming instead that the proportion of followers is zero, which means agents can choose to invest either in the first or in the second period the model reduces to Dasgupta (2006). The presence of followers in a population of agents playing a global game increases the incidence of coordination failure relative to a benchmark in which all agents can potentially be leaders. The incidence of coordination failure is still smaller relative to a static benchmark where choice are made only in one period.

In both these papers, the information that is revealed at the end of the first period is a public signal that is a transformation of the aggregate activity in the first period. Since the equilibrium strategy in the first period is monotone with a threshold $c_1$, the
aggregate activity is the value of cumulative distribution function of the private signals $Y_1 = F(c_1, \theta))$. The papers assume that agents observe the function $G(Y)$ plus a noise where the function $G$ is chosen as the inverse of the cumulative distribution function $F$: $G = F^{-1}$. That function, which hard to interpret except for convenience, nullifies, exactly, the “tail effect” that was described above: in that case, when the level of endogenous activity decreases (in the tail of the distribution) and becomes vanishing small, the endogenous part of the noisy signal becomes larger and the multiplier tends to infinity. From the previous discussion, the information content of the observation of the aggregate activity should be decreasing when the marginal agent moves toward one of the two tails of the distribution, an effect that is a the root of the results of Vives (1993, 1995). Such a property is the central assumption in this paper.

Our work is also related to that of Gale (1995) who argues that short periods improve coordination. But the result depends on two assumptions: first, there is perfect observation of other actions. More important, each agent is not small with respect to the population of players. With $N$ players, an agent who invests precipitates a sub-game with at most $N - 1$ players. With two players, the remaining agent invests immediately. The argument is generalized to $N$ players by induction.

In Chamley (2002)), agents with identical informations attack a currency in a multi-periods framework. Under the assumption that agents coordinate on the devaluation equilibrium if there are multiple equilibria, a reduction of the interest rate (equivalent to a reduction of the period length) increases the probability of a devaluation. The payoff in that game is not of pure strategic complementarity however because a large attack reduces the expected gain of each attacking agent.

Angeletos et al (2007) agents have heterogenous information in a multi-period game with pure strategic complementarity. Past the first period, the game has multiple equilibria, at least in the transition to a new regime.

The paper is organized as follows. We begin in the next section by a simple framework with two periods and agents with identical informations. The property of the increasing “multiplier” from aggregate activity to information takes a strong form. Below some threshold level, there is no information—in addition to the fact that the activity is below the threshold, which is some information—and above that threshold, there is perfect information. It is shown that if the cost of delay is not too large, there is no investment (and coordination) in any period while there can be coordination with
investment when the cost of delay is sufficiently large. The argument is then extended to an arbitrary number of periods.

It may be of interest that in a dynamic setting with delays, forward induction can provide a device for equilibrium selection when multiple equilibria occur in sub-games because of the strategic complementarities in the payoffs.

In Section 3, the strong form of the relation between activity and information is replaced by a standard form, as in Vives (1993). Agents observe the sum of the endogenous aggregate activity and a noise with a fixed variance. The model does not have an analytical solution and is analyzed by numerical computation.

In Section 4, we remove the assumption of identical information for all agents and we use a Global Game technique. With heterogeneous information, in the first period the indifference condition between investing and delay depends on agents' private information. Hence instead of having everybody that prefers to delay or everybody that prefers to invest (as in Section 2), we can pin down a threshold for the private signal such that only a fraction of agents, those whose signal is bigger than the critical threshold, prefers to invest. We are able to pin down a unique equilibrium and the regime outcome depends on the state of the economy.

2 A simple model

The structure is a canonical version of a bank run or a speculative attack against a fixed exchange rate. The standard assumption in the literature is that the reserves of the bank are not common knowledge while the mass of speculator is common knowledge. Since reserves of central banks are in general published while the mass of speculators (or the depth of their pockets) is obviously not known, we make here the assumption that the level of reserves (by a commercial bank or by a central bank) is fixed and normalized at 1. The state of nature defines the mass of the speculators, \( \theta \). A speculator has no other information besides his own existence. To simplify, there are two states of nature \( \theta_0 \) and \( \theta_1 \) with \( 0 < \theta_0 < 1 < \theta_1 \). The prior probability of the high state \( (\theta = \theta_1) \) is \( \mu \).

Each agent has an option to make one irreversible investment of a fixed size that is normalized to one. We first consider the case of two periods.
2.1 Two periods

The option to invest can be exercised in one of two periods, at the cost $c_t$ in period $t$, $(t \in \{1, 2\})$. The problem of optimal delay in a setting with endogenous information is interesting only if there is a premium for acting early. We therefore assume that investment is more costly in the second period:

$$c_2 = (1 + \delta)c_1, \quad \text{with} \quad \delta > 0.$$  \hspace{1cm} (1)

We could also have assumed equal investment costs in the two periods with a discount factor on the payoff of investment in the second period.

In order to avoid a trivial game, we assume that $c_1 < \mu$: if all agents invest in the first period, their expected payoff is positive.

The payoff in period $t$ of an agent who invests in period $t$ is the expected value, conditional on his information in period $t$ of the function

$$u(X) = \begin{cases} 
1 - c_t, & \text{if } X > 1, \\
-c_t, & \text{if } X \leq 1,
\end{cases}$$  \hspace{1cm} (2)

where $X$ is the total mass of agents who have invested in at the end of period 2.

The critical element of the model is the structure of information. At the end of period 1, all agents observe a public signal $y$ on the mass $X_1$ of agents who have invested in that first period. The information content of that signal increases with the mass of investment in the first period. In this section, this property is assumed to take the strong form of a step function with a threshold $\gamma$ for information: when $X_1$ is below a critical mass $\gamma \in (0, 1)$, agents observe 0, that is that observe that $X_1 < \gamma$. When $X_1 \geq \gamma$, agents observe the exact value of $X_1$. Later, we will consider a more general form of the property that information increases with activity.

**Equilibrium in the first period**

Since all agents are identical, we consider only symmetric equilibria. Let $\lambda \in [0, 1]$ be the probability of investment by agents in the first period. By the law of large numbers, $\lambda$ is also the fraction of agents who invest in the first period. The value of $\lambda$ is known by all agents. The information about the state at the end of period 1 depends on $\lambda$. 

5
Lemma 1

If $\lambda \in [0, \gamma/\theta_1)$, there is no information revealed in period 1 and the belief in period 2 is the same as in period 1. If $\lambda \in [\gamma/\theta_1, 1]$, there is perfect information on the state in period 2.

Proof: If $\lambda < \gamma/\theta_1$, the period 1 observation is 0, whatever the state. Suppose now that $\lambda \geq \gamma/\theta_1$. In the high state ($\theta = \theta_1$), the level of investment is at least equal to the threshold of information $\gamma$ and agents have perfect information.

In the low state ($\theta = \theta$), the level of investment may be higher or lower than $\gamma$. If it is at least equal to $\gamma$, there is perfect information. If it is below $\gamma$, agents observe 0. But that observation is impossible in the high state. Hence it also reveals the low state. □

If the low state is revealed, there is no investment in the second period. If the high state is revealed and if $\lambda < 1/\theta_1$, the payoff of investment in the second period sub-game is positive only if more agents invest in that period. There are two (stable) Nash-equilibria in that sub-game: either no agent invests or all agents who have not invested in period 1 invest.

In the framework of this section with identical information for all agents, a standard resolution method is to include sun spots. We will assume that when there are multiple equilibria, agents coordinate on the investment equilibrium with probability $\pi \in \{0, 1\}$.

We therefore make the following assumption for this section.

Assumption 1 (coordination)

If there are multiple equilibria in pure strategies, to invest or not invest, in the sub-game in period 2, then each player expects that with probability $\pi \in [0, 1]$, all other (remaining) players invest and with probability $1 - \pi$ no other player invests.

Definition

An equilibrium is defined as a sub-game perfect symmetric Bayesian equilibrium under Assumption 1, where the value of the parameter $\pi$ is known to all agents.

The next result highlights a central mechanism here. When the cost of delay (through
the higher cost of investment in the future) is not too large, there cannot be too many agents who invest. A high level of investment in the first period conveys a large amount of information that generates the incentive for delay.

**Proposition 1**

*Under Assumption 1, if \( \delta < \delta^* = (1 - \mu)/\mu \), for any value of \( \pi \), in an equilibrium, the fraction \( \lambda \) of agents who invest in the first period is such that \( \lambda < \gamma/\theta \). No information is revealed in the first period.*

**Proof:** The proof is by contradiction. There are two cases. Assume first that \( \lambda \geq \gamma/\theta \). Under there is perfect information in period 2, with multiple equilibria in the high state. Under Assumption 1, the payoff in period 1 of an agent who delays his decision until period 2, where he gets perfect information, is

\[
W = \mu_1 \pi (1 - c_2).
\]

The payoff of investing in the first period is \( U = \mu_1 \pi - c_1 \) because if the state is high, it is revealed in period 2 and with probability \( \pi \), all agents invest in that period.

Agents strictly prefer to delay if \( W > U \), or \( \mu_1 \pi (1 - c_2) > \mu_1 \pi - c_1 \), which is equivalent to \( c_1 > \mu_1 \pi c_2 \), or \( \delta < (1 - \mu_1)/(\mu_1 \pi) \), which holds because of the assumption in the Proposition. \( \square \)

Proposition 1 leads to the next result.

**Proposition 2**

*Under Assumption 1, if \( \delta < \delta^* = (1 - \mu)/\mu \), for any value of \( \pi \), there is a unique equilibrium. In that equilibrium no agent invests in any of the two periods.*

**Proof:** Because of Proposition 1, \( \lambda < \gamma/\theta \). Nothing is learned in the first period and the belief is unchanged in the second period: \( \mu' = \mu \). Because of the inequality in the Proposition, \( c_2 < \mu \). Following our previous assumption, agents in period 1 know which equilibrium is chosen in the second period. If \( \pi > 0 \), and agents happen to coordinate on investment in period 2, then they should invest in period 1 when the cost \( c_1 \) is lower than \( c_2 \). That contradicts the inequality \( \lambda < \gamma/\theta \). The only equilibrium value is therefore \( \pi = 0 \) and \( \lambda = 0 \). \( \square \)
The proof of the proposition uses an argument of forward induction for the choice of equilibria in the sub-game in period 2. The equilibrium with investment is ruled out because it is incompatible with the choice of agents in period 1. If they had anticipated the equilibrium with investment, they would not have chosen the action that led to the possibility of this equilibrium. The other case, where \( \delta > (1 - \mu)/\mu \), is considered in the next result.

**Proposition 3**

If \( \delta > \delta^* = (1 - \mu)/\mu \), for any value of \( \pi \), there are two equilibria: either all agents invest in the first period, or they do not invest in any period.

**Proof:** There are three cases to consider depending on the value of the strategy \( \lambda \).

1. Suppose that \( \lambda < \gamma/\theta_1 \). There is no learning in period 1 and \( \mu' = \mu \). The condition of the Proposition is equivalent to \( c_2 < \mu \). The period 2 sub-game has two non random equilibrium strategies, with and with no investment, respectively. If \( \pi = 1 \), and agents coordinate on the equilibrium with investment, then investment is cheaper in the first period with \( c_1 < c_2 \). That contradicts the assumption \( \lambda < \gamma/\theta_1 \). As in the case of Proposition 2, the argument of forward induction rules out the equilibrium with coordination. If \( \pi = 0 \), then there should not be investment in the first period, which is compatible with \( \lambda < \gamma/\theta_1 \).

2. If \( \gamma/\theta_1 \leq \lambda < 1/\theta_1 \), there is perfect information on the state in period 2. In the high state, the sub-game has two equilibria with non random strategy, with and without coordination, respectively. By the same argument as in the previous case, in an equilibrium all agents should invest in period 1 which contradicts \( \lambda < 1/\theta_1 < 1 \). We don’t even need to consider the case where the low state is revealed.

3. If \( 1/\theta_1 \leq \lambda \leq 1 \), then there is a unique equilibrium in the sub-game of that period, with all remaining players, if any, investing. The payoff of investment in that period is \( 1 - c_2 \). In that case, it is optimal to invest in the first period. The optimal strategy \( \lambda = 1 \), is compatible with the assumption \( 1/\theta_1 \leq \lambda \leq 1 \).

The previous two results indicate in a simple way that when the period becomes sufficiently short, there cannot be a coordination on the equilibrium with investment.
2.2 Multiple periods

In this section, we extend the previous model to an arbitrary number of periods. The payoff function is the same as in (2), where $X$ is the mass of investment at the end of the game. (In an equilibrium, the game will be played in a final number of periods). We consider the more interesting case of a small cost of delay in the sense of Proposition 1. We will assume that the cost of investment in period $t$ is $c_t$ is an increasing function of time with

$$c_{t+1} = (1 + \delta)c_t.$$  \hfill (3)

Define $N$ such as

$$N = \text{Max} \{n | (1 + \delta)^n c_1 < \mu\}.$$ \hfill (4)

We assume that $N \geq 2$.

**Proposition 4**

extension of Prop 3 and 4.

3 Gaussian aggregate noise with two periods

We keep the same structure of state of nature and payoffs as in the previous section. Recall that all agents have the same information at the beginning of period 1. We consider symmetric equilibria. The strategy of each agent is to invest with probability $\lambda$. At the end of period 1, agents observe the level of aggregate in period 1 with some additional noise. By the law of large numbers, the mass of players' investment in the first period is $\lambda \theta_i$. We assume that at the end of the first period all agents observe this mass through a noise. The public signal at the end of the first period is

$$y = \lambda \theta_i + \sigma_\eta \eta,$$

where $\lambda$ is the fraction of agents who invest in the first period and $\eta$ is a standardized Gaussian random variable. We defined $\rho_\eta = 1/\sigma_\eta^2$. The information content of the public signal is equivalent to that of the variable

$$z = \frac{y}{\lambda} = \theta_i + \frac{\sigma_\eta}{\lambda} \eta.$$

One sees immediately that the signal to noise ratio increases with the fraction $\lambda$ of agents who invest in the first period. A higher level of investment generates more information. That is the fundamental property that was discussed previously.
Let $\psi$ be the density function of the random variable $\eta$, which is normal with mean 0 and precision $\rho_\eta$. Let $\mu$ and $\mu'$ at the belief (probability) of the good state ($\theta = \theta_1$) at the beginning and the end of period 1, respectively.

By Bayes’ rule,

$$\frac{\mu'}{1 - \mu'} = \frac{\mu}{1 - \mu} \frac{\phi(\lambda\sqrt{\rho_\eta}(z - \theta_1))}{\phi(\lambda\sqrt{\rho_\eta}(z - \theta_0))},$$

(5)

where $\phi$ is the standardized Gaussian density.

Let $L$ and $L'(z)$ the Log likelihood ratio (LLR) between the good and the bad state before and after the observation of $z$, respectively.

$$L'(z) = L + \lambda^2 \rho_\eta (\theta_1 - \theta_0) \left(z - \frac{\theta_1 + \theta_0}{2}\right).$$

(6)

This equation shows how in the updating of the belief, the impact of the observation $z$ increases with the mass of investment. That impact is unbounded: extreme values of that investment have an extremely strong discriminating power in differentiating between the two states.

**Investment in period 2**

Let $\hat{L} = Log(c_2/(1-c_2))$ where $c_2$ is the cost of investment in period 2. There is an equilibrium with investment by all remaining agents in period 2 if and only if $\mu' > c_2$, which is equivalent to

$$L'(z) > \hat{L},$$

or

$$z > \hat{z}(\lambda) = \frac{\theta_1 + \theta_0}{2} + \frac{\hat{L} - L}{\lambda^2 \rho_\eta (\theta_1 - \theta_0)}.\tag{7}$$

which is equivalent to

$$\eta > \hat{\eta} = \lambda\sqrt{\rho_\eta} \left(\frac{\theta_1 + \theta_0}{2} - \theta_i\right) + \frac{\hat{L} - L}{\lambda\sqrt{\rho_\eta} (\theta_1 - \theta_0)}.$$

(8)

We make the following assumption.

**Assumption 1**

$$c_1 < \mu < c_2.$$

The condition $c_1 < \mu$ is necessary for any action to take place. When $\mu < c_2$, investment takes place in period 2 only after some “good news” in period 1, that is if
the mass of first period investment above some threshold. The LLR for investment in period 2, \( \hat{L} \), is greater than the prior LLR, \( L \). In equation (8), the minimum level of the shock, \( \hat{\eta} \), increases when the mass of investment \( \lambda \) decreases. If \( \lambda \to 0 \), the second term in (8) tends to infinity and there is investment in period 2 only if the shock in period 1 is very high.

If the condition for investment in period 2, \( \mu' > c_2 \), is satisfied, then there is an equilibrium in which all remaining agents in period 2 invest. As in the previous section, we focus on the case where agents coordinate on the “most optimistic” equilibrium, in which all remaining players invest.

The payoff of no delay

Consider an agent who invests in period 1.

- If the strategy of all agents is such that \( \lambda \theta_1 > 1 \), which is possible only in the good state, the critical mass is attained in the first period. Note that the observation in period 1 may be such that the belief (in the good state) at the end of period 1 is below the investment cost in period 2, \( c_2 \). In that case there is no investment in period 2. However, after period 2, agents learn the true state \( \theta_1 \). The action in period 2 is irrelevant for the payoff of the agents who invest in period 1. That payoff is \( \mu - c_1 \), where \( \mu \) is the belief at the beginning of period 1.

- If \( \lambda \theta_1 < 1 \), a positive payoff of investment in period 1 requires that: (i) the true state is good; (ii) more agents invest in period 2, that means that the belief in period 2 is greater than \( c_2 \): \( \mu'(z) > c_2 \). Recall that in this case we assume that the period 2 sub-game, agents coordinate on the equilibrium where all remaining players invest. The condition (ii) is equivalent to \( z > \hat{z}(\lambda) \). The expected payoff for investing in period 1 is therefore

\[
U(\lambda) = -c_1 + \mu \int_{\hat{z}(\lambda)} \theta_1 g(z|\theta_1)dz, \tag{9}
\]

where \( g(z|\theta_1) \) is the density of \( z \) in the good state:

\[
g(z) = \lambda \sqrt{\rho_\eta} \phi \left( \lambda \sqrt{\rho_\eta} (z - \theta_1) \right). \tag{10}
\]

Using the symmetry of the cumulative distribution function \( \Phi \) of the normal distribution,

\[
U(\lambda) = -c_1 + \mu \Phi \left( \lambda \sqrt{\rho_\eta} (\theta_1 - \hat{z}(\lambda)) \right). \tag{11}
\]
Using the expression of $\hat{z}(\lambda)$ in (7), we have

$$U(\lambda) = -c_1 + \mu \Phi \left( \lambda \sqrt{\rho_\eta} \left( \frac{\theta_1 - \theta_0}{2} - \frac{\hat{L} - L}{\lambda^2 \rho_\eta (\theta_1 - \theta_0)} \right) \right).$$

(12)

Because of Assumption 1, $\hat{L} - L > 0$. The payoff of no delay, $U(\lambda)$, is an increasing function of the mass of investment in the first period. This property has an intuitive interpretation.

If $\lambda = 0$, $U(0) = -c_1$. If no agent invests in the first period, there is no learning in that period $\mu' = \mu < c_2$ and not agent invests in the second period. There is no externality and the gross payoff of investment is zero. An agent who invests in period 1 only pays the cost $c_1$.

If $\lambda$ is near (but smaller than) $1/\theta_1$, the second term is positive but smaller than $\mu$ and depends on the parameters of the model. If $\rho_\eta \to \infty$, then the observation in period 1 asymptotically provides perfect information and in the good state, all remaining agents invest. In that case the payoff of no delay tends to $\mu - c_1$.

The option value of delay

The fraction of agents who invest in period 1 is $\lambda$. Given the value of $\lambda$, the remaining players invest in period 2 only if $z > \hat{z}(\lambda)$. Under our coordination assumption, all remaining players invest in this case. If $z > \hat{z}(\lambda)$, $\mu'(z) > c_2$ and a delaying agent invests in period 2. But his ex post gross payoff of an agent who delays in period 1 is one if and only if $\theta = \theta_1$. The expected payoff in period 1 of a delaying agent is

$$W(\lambda) = \int_{\hat{z}(\lambda)} (\mu'(z) - c_2) g(z) dz,$$

(13)

Using the same method as for the payoff with no delay, we have

$$W(\lambda) = \int_{\hat{z}(\lambda)} (P(\theta = \theta_1|z) - c_2) g(z) dz = \int_{\hat{z}(\lambda)} \left( g(z|\theta_1) \mu - c_2 g(z) \right) dz.$$

Define $D(\lambda) = W(\lambda) - U(\lambda)$. Using the expression of $U$ in (11) and $W$ in (13),

$$D(\lambda) = W(\lambda) - U(\lambda) = c_1 - c_2 \int_{\hat{z}(\lambda)} g(z) dz.$$ 

(14)

The difference $W - U$ measures the option value of delay. Given our assumptions about the behavior of agents in the second period, the option value is the difference
between the cost of investment in the first period and the cost of investment in the second period only after the news are sufficiently good to raise the belief above $c_2$.

In the second term of the option value, the delay reduces the probability of investing when the state is bad but the delay also increases the cost of investment from $c_1$ to $c_2$.

$$D(\lambda) = c_1 - c_2 \left( \mu \Phi \left( \lambda \sqrt{\eta} (\theta_1 - \hat{z}(\lambda)) \right) + (1 - \mu) \Phi \left( \lambda \sqrt{\eta} (\theta_0 - \hat{z}(\lambda)) \right) \right).$$

$$D(\lambda) = c_1 - c_2 \left( \mu \Phi \left( \lambda \sqrt{\eta} \left( \frac{\theta_1 - \theta_0}{2} - \frac{\hat{L} - L}{\lambda^2 \eta \theta_0 (\theta_1 - \theta_0)} \right) \right) 
+ (1 - \mu) \Phi \left( \lambda \sqrt{\eta} \left( - \frac{\theta_1 - \theta_0}{2} - \frac{\hat{L} - L}{\lambda^2 \eta (\theta_1 - \theta_0)} \right) \right).$$

If $\rho_\eta \to \infty$ and $\lambda > 0$, $D(\lambda) \to c_1 - \mu c_2$. The limit has a simple interpretation. When $\rho_\eta$ tends to infinity, the observation in the first period asymptotically reveals the good state. A delaying agent pays the investment cost only if the good state is revealed. He saves on avoiding to invest if the state is bad, but if the state is good, he pays a cost higher than in the first period.

**Proposition 5**

For given $\lambda > 0$, if $\rho_\eta \to \infty$,

$$\lim U(\lambda) = \mu - c_1, \quad \lim W(\lambda) = \mu (1 - c_2).$$

We have $\lim W(\lambda) > \lim U(\lambda)$ if $\mu c_2 < c_1$ or $\mu < 1/(1 + \delta)$. We will consider parameters such that

$$1 < \frac{\mu}{c_1} < 1 + \delta, \quad \mu < \frac{1}{1 + \delta}.$$ 

The second condition requires that the belief $\mu$ should not be too large. In that case, the information gain effect dominates the effect of the higher cost in the second period.

Figure 1 represents the graph of the payoffs with no delay, $U(\lambda)$, and with delay, $W(\lambda)$, as function of the fraction $\lambda$ of agents who invest in the first period. The payoff of delay is greater than that of no delay for any value of $\lambda$. The only equilibrium is that with $\lambda = 0$. No agent invests in any period.
Figure 1: Values of no delay ($U$) and delay ($W$) as function of the fraction $\lambda$ of agent who do not delay

Parameters: $\theta_0 = 0, \theta_1 = 1.2, c_1 = 0.3, \delta = 1, \mu = 0.4, \rho_\eta = 10$.

Note that if $\lambda > 1/\theta_1$, then $U(\lambda) = \mu - c_1 = 0.1$. (See the text).

Compare with the case where delay is not possible, for example because the cost $c_2$ is too high. Because $U(0) < 0$ and $U(1) > 0$, there are two “stable” equilibria with $\lambda = 0$ and $\lambda = 1$.

No investment is an equilibrium in both cases, but when there no delay option, agents may also coordination on the equilibrium with investment, which Pareto dominates the equilibrium with no investment. (more on the efficiency issue). In the numerical example with Gaussian distributions, as in the stylized model of the previous section, the option to delay reduces the possibility of coordination between agents.
4 Heterogeneous private information

In this section, agents have different private informations. It is well known from the literature that the properties of social learning depend on whether the distribution of private beliefs is bounded or unbounded. We first consider the case of bounded beliefs.

4.1 Binary signals

Without loss of generality, we assume that agents receive private binary signals that are independent across individuals and are equal to 1 in state $\theta_i$ with probability $q$. For simplicity, we call “optimists” the agents with a signal 1, with a belief $\mu^+$ and “pessimists” the others, with a belief $\mu^-$. By Bayes’ rule, $\mu^- < \mu < \mu^+$. We also make the assumption that $q > \gamma$. If this inequality does not hold that there is learning only if all the optimists and some of the pessimist invest. That situation is not interesting as it is equivalent to the case of identical agents with a belief $\mu^-$.

The next result extends Proposition 2.

Proposition 6

If $\delta < (1-\mu^+)/\mu^+$, for any value of $\pi$, there is a unique equilibrium. In that equilibrium no agent invests in any of the two periods.

(To be checked)

(Remark on a continuum with a bounded distribution)

4.2 Unbounded private beliefs

In this section, we extend the two-period model of Section 2 with heterogeneous private information: at the beginning of the first period, each agent has a private signal on $\theta$:

$$s_i = \theta + \varepsilon_i$$

where $\varepsilon_i \sim N(0, 1/\rho_\varepsilon)$, with the precision of the signal equal to $\rho_\varepsilon$. The rest of the model is unchanged.

Since we have a binary state and a Gaussian signal, the law that updates the LLR
between $\theta_1$ and $\theta_0$ is
\[
L(s) = L + \rho \varepsilon (\theta_1 - \theta_0) \left( s - \frac{\theta_1 + \theta_0}{2} \right)
\]  \hspace{1cm} (17)
where $L = \log \frac{\mu}{1-\mu}$ and $L(s) = \log \frac{\mu(s)}{1-\mu(s)}$. The belief after the observation of the signal is an increasing function of $s$. More important, the distribution of private beliefs is unbounded.

We proceed as in the benchmark model. Denoting by $\lambda$ and $\hat{\lambda}$ the fraction of agents who invest in the first and the second period, respectively, let $D(s)$ the probability that the regime changes conditional on the private information $s$ of an agent at the beginning of the first period. Since there is no regime change in state $\theta_0$, we have
\[
D(s_i) = \mu(s_i) \Pr \left( \theta_1 \left( \lambda + \hat{\lambda} \right) > 1 \mid \theta_1 \right)
\]  \hspace{1cm} (18)
First we look for equilibria in which the public signal reaches the threshold ($\lambda \theta \geq \gamma$) and then we look for equilibria in which it does not ($\lambda \theta < \gamma$).

We focus on monotone strategies. In the first period, an agent invests if and only if his signal is at least equal to $\bar{s}$. The fraction of agents who invest in the first period is
\[
\lambda_i(\bar{s}) = \Phi(\sqrt{\rho \varepsilon (\theta_i - \bar{s})}),
\]  \hspace{1cm} (19)
and is increasing with $\theta$.

The discussion depends on the information that is revealed in the first period.

**Lemma 2**

*If the minimum level of no delaying agents, $\bar{s}$ is such that $\bar{s} \leq s^*$ with $\lambda_1(s^*)\theta_1 = \gamma$, there is perfect information on the state $\theta$ at the end of period 1. Otherwise there is no information at the end of that period.*

Suppose that in equilibrium agents with a signal greater than $\bar{s}$ do not delay and that $\bar{s} > s^*$. There is no information at the end of period 1, then the public belief in period 2 is unchanged at $\mu$. Suppose that agents with a signal in $[\bar{s}', \bar{s}]$ invest in period 2. Such agents invest because they have a positive expected payoff. But there has been no new information after period 1. These agents have perfect foresight on the game that is played in period 2. In that case, they should invest in period 1 when the cost is lower.
Therefore, in an equilibrium, there cannot be investment in period 2. But then, the mass of investment in period 1 is less than 1 and the net payoff of investment in that period is negative. The agents with signal higher than $\bar{s}$ do not invest and if $\bar{s} > s^*$, then $\bar{s} = \infty$. No agent invests in the first period.

Assume now that $\bar{s} \leq s^*$. There is perfect information at the end of the first period. In the low state there is no investment. In the high state, there are two equilibria as in Section 2. Suppose that in the high state, all remaining players invest. Let $\mu(s)$ the belief of an agent with signal $s$. His payoff of no delay is $\mu(s) - c_1$. If he delays, his payoff is $\mu(s)(1 - c_2)$. The first is greater than the second if and only if $s > \bar{s}$ with

$$L(\bar{s}) = -\log(\delta). \quad (20)$$

Since $L$ is increasing in $s$, condition $\bar{s} \leq s^*$ is equivalent to $-\log(\delta) < L(s^*)$, or

$$\delta > \delta^{**}, \quad \text{with} \quad \log(\delta^{**}) = -L(s^*). \quad (21)$$

Using

$$-L(s^*) = -L - \rho_c(\theta_1 - \theta_0)(s^* - \theta_1 + \theta_0/2),$$

we have

$$\log(\delta^{**}) = \log(\delta^*) - \rho_c(\theta_1 - \theta_0)(s^* - \theta_1/2),$$

with

$$s^* = \theta_1 - \frac{1}{\sqrt{\rho_c}} \Phi^{-1}\left(\frac{1}{\theta_1}\right),$$

or

$$\log(\delta^{**}) = \log(\delta^*) - \rho_c \frac{(\theta_1 - \theta_0)^2}{2} + \sqrt{\rho_c}(\theta_1 - \theta_0)\Phi^{-1}\left(\frac{1}{\theta_1}\right). \quad (22)$$

We have proven the following result.
Proposition 7

If $\delta > \delta^{**}$ defined in (22), then the unique equilibrium is the one where no agent invests. If $\delta \leq \delta^{**}$, there is in addition another equilibrium where some agents invest in the first period, the state is revealed at the end of that period and all remaining players invest in the good state in period 2.

Intuitive discussion to be added

Proposition 8

The minimum value $\delta^{**}$ for a coordination with investment in the good state is a function of the precision $\rho_\epsilon$ of the individual signal. It is increasing on an interval $(0, \hat{\rho})$ and decreasing when $\rho_\epsilon > \hat{\rho}$ with

$$\hat{\rho} = \frac{1}{2(\theta_1 - \theta_0)} \Phi^{-1}\left(\frac{1}{\theta_1}\right).$$

When agents get nearly perfect information and $\rho_\epsilon$ tends to infinity, $\delta^{**}$ tends to 0. Asymptotically, agents can coordinate on the equilibrium with investment in the good state.
Assume that the state is $\theta_1$. Consider the contingent rule that if $\lambda \geq \gamma/\theta$ and if the high state is revealed, all remaining players, if any, invest in the second period. (Note that all players investing in the second period is compatible with both $\lambda \geq 1/\theta_1$ and $\lambda < 1/\theta_1$). Due to the fact that all players invest in the second period and that in the first period players can choose between Invest/Delay (Not Invest is not an option) we have $D(s_i) = \mu_1(s_i)$.

The condition for the delay of an agent in period 1 depends on his private signal $s_i$. The condition (??) takes the form

$$\mu_1(s_i) (1 - c_2) \geq \mu_1(s_i) - c_1,$$

which is equivalent to

$$L_s \leq -\log(\delta).$$

Using (17), this condition is equivalent to

$$s_i < s^* = \frac{\theta_1 + \theta_0}{2} - \frac{\log(\delta) + L}{\rho_c(\theta_1 - \theta_0)}.$$

The fraction of agents that observe a signal smaller than $s^*$ delay their investment decision, those that observe a signal higher than $s^*$ invest in the first period. Hence in state $j$,

$$\lambda_j = 1 - \Phi \left( \sqrt{\rho_c} (s^* - \theta_j) \right).$$

where $\Phi$ is the c.d.f. of the standardized normal distribution.

The condition $\lambda_j \geq \gamma/\theta_j$ is equivalent to

$$1 - \Phi \left( \sqrt{\rho_c} (s^* - \theta_j) \right) \geq \gamma/\theta_j,$$

or

$$s^* \leq \theta_j + \frac{1}{\sqrt{\rho_c}} \Phi^{-1} \left( 1 - \frac{\gamma}{\theta_j} \right).$$

Now, consider the contingent rule that if $\lambda \geq \gamma/\theta$ and if the high state is revealed, all remaining players, if any, do not invest in the second period (Note that all players not investing in the second period is compatible only with $\lambda < 1/\theta_1$). In this case, no agent would want to invest in the first period no matter his signal because he knows $\lambda < 1/\theta_1$. We have a contradiction with the initial assumption.

Assume that the state is $\theta_0$. There is no investment in the second period. In the first period also nobody invest. We have a contradiction with the initial assumption. We proved the following result.
RESULT 1

There is an equilibrium in which in the first period agents that observe a signal $s_i \leq s^*$ delay their investment decision, those whose signal is bigger than this threshold invest. The equilibrium threshold signal must be such that

$$s^* \leq \theta_1 + \frac{1}{\sqrt{\rho_c}} \Phi^{-1} \left( 1 - \frac{\gamma}{\theta_1} \right)$$

so that, when the state is $\theta_1$, the players that delayed their decision in the first period, will observe $\lambda \theta_1 \geq \gamma$ and will all invest in the second period. The regime changes with probability one. A candidate for $s^*$ is

$$s^* = \frac{\theta_1 + \theta_0}{2} - \frac{\text{Log}(\delta) + L}{\rho_c(\theta_1 - \theta_0)}$$

whenever the following inequality is satisfied

$$\frac{\theta_1 + \theta_0}{2} - \frac{\text{Log}(\delta) + L}{\rho_c(\theta_1 - \theta_0)} \leq \theta_1 + \frac{1}{\sqrt{\rho_c}} \Phi^{-1} \left( 1 - \frac{\gamma}{\theta_1} \right)$$

Assuming that $\rho_c \to \infty$ the inequality is satisfied and the equilibrium threshold is

$$\tilde{s}^* = \frac{\theta_1 + \theta_0}{2}$$

[Observation: Take the contingent rule where $\lambda \theta \geq \gamma$, the high state is revealed and all agents invest in the second period. According to the benchmark model, in the first period, either all agents prefer to delay (when $\mu_1 (1 - c_2) > \mu_1 - c_1$) or all agents prefer to invest. When all agents prefer to delay, there is a contradiction with the assumption $\lambda \theta_1 \geq \gamma$. In the model with private signals instead, there will be always a fraction of players that prefer to delay and another fraction that prefer to invest. So we can impose that the fraction of players investing is large enough (larger than $\gamma$) and this translates in a condition on the equilibrium threshold signal.]

The second period signal does not reach the threshold: $\lambda \theta < \gamma$

All players in the first period use their private signal $s_i$ to update their prior, as we have shown. Since $\lambda \theta < \gamma$, the players in the second period, if any, do not know if the
state is $\theta_1$ or $\theta_0$. We look for an equilibrium in which agents use switching strategies both in the first and second period. In particular, given two thresholds $\bar{s}^* > \bar{s}^{**}$, in the first period all players that observe a signal $s_i \leq \bar{s}^*$ delay their decision (invest otherwise), in the second period players that delayed will invest if the signal is bigger than $\bar{s}^{**}$. Given these thresholds the investment in the first and in the second period is respectively

$$
\lambda_j = \Pr (s_i > \bar{s}^*) = 1 - \Phi (\sqrt{\rho_e} (\bar{s}^* - \theta_j))
$$

$$
\hat{\lambda}_j = \Pr (\bar{s}^{**} \leq s_i \leq \bar{s}^*)
$$

$$
= \{ \Phi (\sqrt{\rho_e} (\bar{s}^* - \theta_j)) - \Phi (\sqrt{\rho_e} (\bar{s}^{**} - \theta_j)) \}
$$

In what follows, we find a restriction on the thresholds $\bar{s}^{**}$ such that the regime change condition is satisfied given $\theta_1$ and given $\theta_0$. If we assume that the state is high ($j = 1$), the regime changes whenever

$$
(\lambda_j + \hat{\lambda}_j) \geq \frac{1}{\theta_j}
$$

$$
1 - \Phi (\sqrt{\rho_e} (\bar{s}^* - \theta_j)) + \Phi (\sqrt{\rho_e} (\bar{s}^{**} - \theta_j)) - \Phi (\sqrt{\rho_e} (\bar{s}^{**} - \theta_j)) \geq \frac{1}{\theta_j}
$$

$$
1 - \Phi (\sqrt{\rho_e} (\bar{s}^{**} - \theta_j)) \geq \frac{1}{\theta_j}
$$

$$
\bar{s}^{**} \leq \theta_j + \frac{1}{\sqrt{\rho_e}} \Phi^{-1} \left( 1 - \frac{1}{\theta_j} \right)
$$

hence the threshold signal must be small enough (meaning that a big enough mass of agents have to invest in the second period). When the state is low, even if the aggregate investment is one the regime does not change. We first look for equilibria such that there is a regime change and then for equilibria without regime change.

**Equilibria with regime change.** In the second period an agent prefers to invest if the expected payoff of investing, $D - c_2$, is bigger or equal to zero

$$
D (s_i) - c_2 \geq 0
$$

$$
\mu_1 (s_i) \Pr \left( \theta_1 \left( \lambda + \hat{\lambda} \right) \geq 1 \mid \theta_1 \right) \geq c_2
$$

The previous condition is equivalent to $\mu_1 (s) \geq \frac{c_1 (1 + \delta)}{\Pr (\theta_1 (\lambda + \hat{\lambda}) \geq 1 \mid \theta_1)}$ and assuming that the constraint on $\bar{s}^{**}$ is satisfied we have that $\Pr \left( \theta_1 \left( \lambda + \hat{\lambda} \right) \geq 1 \mid \theta_1 \right) = 1$, so $\mu_1 (s_i) \geq c_1 (1 + \delta)$ which is equivalent to

$$
L_s \geq \log \left( \frac{c_1 (1 + \delta)}{1 - c_1 (1 + \delta)} \right)
$$

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using (17) this condition is equivalent to

\[ L + \rho \varepsilon (\theta_1 - \theta_0) \left( s_i - \frac{\theta_1 + \theta_0}{2} \right) \geq \log \left( \frac{c_1 (1 + \delta)}{1 - c_1 (1 + \delta)} \right) \]

\[ s_i \geq \bar{s}^{**} = \left( \log \left( \frac{c_1 (1 + \delta)}{1 - c_1 (1 + \delta)} \right) - L \right) \left( \frac{1}{\rho \varepsilon (\theta_1 - \theta_0)} \right) + \frac{\theta_1 + \theta_0}{2} \]

\[ \bar{s}^{**} = \log \left( \frac{(1 - \mu_1) (1 + c_1 \delta)}{\mu_1 (1 - c_1 - c_1 \delta)} \right) \frac{1}{\rho \varepsilon (\theta_1 - \theta_0)} + \frac{\theta_1 + \theta_0}{2} \]

Since the belief after the observation is increasing in \( s_i \), there will be an \( \bar{s}^{**} \) such that an agent invests whenever \( s_i \geq \bar{s}^{**} \) and does not otherwise. In the first period players, given the constraint on \( \bar{s}^{**} \) know that if the state is high there will be a regime change, they compare the expected payoff of investing in the first period

\[ D(s) - c_1 = \mu_1 (s) \Pr \left( \theta_1 \left( \lambda + \hat{\lambda} \right) \geq 1 \mid \theta_1 \right) - \frac{c_2}{1 + \delta} = \mu_1 (s) - \frac{c_2}{1 + \delta} \]

with the equilibrium continuation payoff of postponing their decision and acting optimally in the second period. Since agents know \( \bar{s}^{**} \) in equilibrium, those whose signal is bigger than this threshold, in the first period delay if

\[ \mu_1 (s) - c_2 \geq \mu_1 (s) - c_1 \]

\[ c_1 \geq c_2 \]

this is never true, so this fraction always invests. Those whose signal is smaller than \( \bar{s}^{**} \), in the first period delay if

\[ 0 \geq \mu_1 (s) - c_1 \]

\[ \mu_1(s) \leq c_1 \]

\[ L_s \leq \log \left( \frac{c_1}{1 - c_1} \right) \]

\[ L + \rho \varepsilon (\theta_1 - \theta_0) \left( s_i - \frac{\theta_1 + \theta_0}{2} \right) \leq \tilde{L} \]

\[ s^* = \frac{\tilde{L} - L}{\rho \varepsilon (\theta_1 - \theta_0)} + \frac{\theta_1 + \theta_0}{2} \]
So agents in the first period delay only if $s_i \leq s^*$ otherwise they invest. This threshold must satisfy the condition on the threshold $\lambda_j < \gamma / \theta_j$, i.e.

$$s^* > \theta_j + \frac{1}{\sqrt{\rho \epsilon}} \Phi^{-1} \left( 1 - \frac{\gamma}{\theta_j} \right)$$

hence we have as $\rho \epsilon \to \infty$

$$\frac{\theta_1 + \theta_0}{2} > \theta_1$$

which is never true. Hence both groups invest in period 1 ($\tilde{s}^* = -\infty$) and this contradicts the initial assumption since $\lambda \theta_j < \gamma$ which would require

$$[1 - \Phi (\sqrt{\rho \epsilon} (\tilde{s}^* - \theta_j))] \theta_j < \gamma$$

$$\tilde{s}^* > \theta_j + \frac{1}{\sqrt{\rho \epsilon}} \Phi^{-1} \left( 1 - \frac{\gamma}{\theta_j} \right)$$

**Equilibria without regime change.** In the second period, agents prefer to invest if

$$D(s_i) - c_2 \geq 0$$

$$\mu'(s_i) \Pr(\theta_1 (\lambda + \hat{\lambda}) \geq 1 | \theta_1) \geq c_2$$

Note that given the assumption $\tilde{s}^{**} > \theta_j + \frac{1}{\sqrt{\rho \epsilon}} \Phi^{-1} \left( 1 - \frac{1}{\theta_j} \right)$ the regime does not change, hence

$$0 \geq c_2$$

so players do not invest in the second period ($\tilde{s}^{**} = \infty$). In the first period agents prefer not to invest since $\lambda \theta_j < \gamma$. Hence we have the following result.

**RESULT 2**

*There is an equilibrium in which no investment takes place in any period.*

From Result 1 and Result 2 we get the following proposition:

**PROPOSITION (...)**
When the state is $\theta_0$ the only equilibrium is one in which no investment takes place in any period. When the state is $\theta_1$ there are two equilibria: in the first one, no investment takes place in any period; in the second one some agents invest in the first period and the remaining in the second period (as specified in Result 1)

5 Heterogeneous agents and aggregate noise

to be continued
REFERENCES


