1. Note that $A$ is a strictly dominant strategy for player 1 and $B$ is 2’s best strategy when 1 plays $A$. So $(A, B)$ is the only Nash equilibrium of the game. For the repeated version, this means that the only subgame perfect equilibrium is that 1 plays $A$ and 2 plays $B$ always (that is, in every subgame).

2. The easiest way to find a subgame perfect equilibrium where both play $A$ in the first period is to suppose that both play $A$ in every subgame. Since this is an equilibrium in the game when it’s played only once, we see that this is an equilibrium in every second period subgame. Since the second period’s payoffs don’t depend on what happens in the first period, it’s an equilibrium for both to play $A$ in the first period. Similar reasoning gives an equilibrium where both play $B$ in both periods.

To find an equilibrium in which one plays $A$ and the other plays $B$ in the first period is a little trickier. Clearly, either player would have an incentive to change strategies in the first period to increase their payoff by 1. So to make it an equilibrium, there has to be some reason not to switch. We can use the two equilibria in the one shot game to do this. To explain this in concrete terms, let’s focus on the case where 1 plays $A$ and 2 plays $B$ in the first period. (Reversing the roles of the players everywhere gives us the other case.) So suppose the strategies are as follows. In the second period subgame where 1 played $A$ and 2 played $B$ in the first period, they both play $A$. In any other second period subgame, both play $B$. These are the two Nash equilibria of the second period subgame, so subgame perfect equilibrium allows this particular specification of which equilibrium happens in which subgame.

Given this, consider the first period. Taking into account how the second period will work out as a function of what happens in the first period, we see that the payoffs for the whole game as a function of first period actions are:

<table>
<thead>
<tr>
<th></th>
<th>$A$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>11, 11</td>
<td>10, 19</td>
</tr>
<tr>
<td>$B$</td>
<td>10, 1</td>
<td>2, 2</td>
</tr>
</tbody>
</table>
To see this: If both play $A$ in the first period, both get 10 in the first period, but the second period, both will get only 1. Similarly, if both play $B$ in the first period, they’ll both get 1 in each period. Similarly, if 1 played $B$ and 2 played $A$ (the reverse of what we’d wanted), the payoffs are $(9,0)$ in the first period and $(1,1)$ in the second, making the totals $(10,1)$ as shown in the matrix. Finally, if the players do what they are “supposed to,” with 1 playing $A$ and 2 playing $B$ in the first period, then payoffs in the first period will be $(0,9)$ and the second period payoffs will be $(10,10)$, adding to $(10,19)$. Given this, it’s an equilibrium in the first period for 1 to play $A$ and 2 to play $B$.

3. (a) Suppose we do have a subgame perfect equilibrium in which $(D,L)$ is played in the first period. We know that the payoffs in every second period subgame must be either $(3,3)$ or $(1,1)$. Player 1’s payoff if he follows the equilibrium can’t be more than $0 + 3 = 3$. His payoff if he deviates to $U$ must be at least $3 + 1 = 4$. Hence it can’t be an equilibrium for him to play $D$.

(b) Take the strategies to be that the players play $(D,L)$ in the first period. If this is done in the first period, they play $(U,L)$ in every later subgame. Otherwise, they play $(D,R)$ in every later subgame. This makes the payoffs as a function of the first period actions

<table>
<thead>
<tr>
<th></th>
<th>$L$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$</td>
<td>5,5</td>
<td>2,2</td>
</tr>
<tr>
<td>$D$</td>
<td>6,6</td>
<td>3,3</td>
</tr>
</tbody>
</table>

It is easy to see that the only equilibrium with this payoff matrix is $(D,L)$.

4. (a) As discussed in class, the only subgame perfect equilibrium is where both choose $D$ in both periods. We know this must happen in any second-period subgame. Hence actions in the first period don’t affect what happens in the second, so both will defect in the first period too.

(b) Notice that this is not the same as the case we considered in class where they might play twice but might only play once. In the case we analyzed in class, whether or not they play twice is completely unrelated to what they do in the first period. Here whether or not they play the second time is completely determined by what they do in the first period.

The easiest way to solve this one is by backward induction. We know that if we get to the second period, both players will defect then and so both will get 2. But remember that they only get to the second period if they both cooperate in the first. So this says that the overall payoffs as a function of the first period actions are

<table>
<thead>
<tr>
<th></th>
<th>$C$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>6,6</td>
<td>0,5</td>
</tr>
<tr>
<td>$D$</td>
<td>5,0</td>
<td>2,2</td>
</tr>
</tbody>
</table>

2
That is, if they both cooperate in the first period, they each get 4 in the first period plus 2 in the second. In any other situation, they get zero in the second period and so the total payoff is just the first period payoff. So we see that there are two subgame perfect equilibria: one where both cooperate in the first period and defect in the second (since \((C,C)\) is a Nash equilibrium of the game above) and one where both defect in the first period and so never get to the second (since \((D,D)\) is also a Nash equilibrium of the game above).

(c) The analysis is the same as above up to the point of writing down the payoffs as a function of the first period actions. In this case, they are

\[
\begin{array}{cc}
C & D \\
C & 5, 5 & 0, 6 \\
D & 6, 0 & 2, 2 \\
\end{array}
\]

so the unique subgame perfect is for both players to defect in the first period.

5. (a) Suppose the strategy of each player is to play \(A\) as long as the opponent does and to switch to \(B\) forever if the opponent ever deviates from this. What is the best strategy to use if the opponent is following this strategy? If you play \(A\) always, the payoff is \(3/(1-\delta)\). If you play \(B\) once, you should play \(A\) from the following period onward since the opponent is going to be playing \(B\). The payoff to this is \(5 + \delta/(1-\delta)\). So it’s better to always play \(A\) if

\[3 \geq 5(1-\delta) + \delta\]

or \(\delta \geq 1/2\). Hence for \(\delta \geq 1/2\), if your opponent plays the strategy above, it is an optimal strategy for you as well. Hence these strategies against one another are a Nash equilibrium.

(b) There’s (at least) two approaches you could use. I start with a more complicated approach since it’s more similar to what we’ve usually done. Suppose the strategy of each player is to follow this pattern as long as the opponent does and to switch to \(B\) forever if the opponent ever deviates from this. What is the best strategy to use if the opponent is following this strategy? Intuitively, it’s clear that you should follow the pattern. Suppose you deviate from it in a period where you were supposed to get 5. Then you only get 3 in that period. In the future, you would have gotten 1, then 5, then 1, etc. Instead, because you deviated, you’ll get 1 every period. Clearly, deviating makes you worse off. So for any \(\delta\), you should follow this strategy.

I’ll give a more precise calculation for one case to make this more concrete. Consider player 1 and suppose he considers deviating from the pattern in the very first period. If he does not deviate, his expected payoff is

\[
1 + 5\delta + \delta^2 + 5\delta^3 + \ldots = (1 + 5\delta)(1 + \delta^2 + \delta^4 + \delta^6 + \ldots) = \frac{1 + 5\delta}{1 - \delta^2}.
\]
Suppose player 1 deviates in the first period from this. If he deviates, we know that the opponent will play B from then on, so 1 should play A from then on. So if he deviates in the first period, his payoff will be

$$0 + \delta + \delta^2 + \delta^3 + \ldots = \frac{\delta}{1 - \delta}.$$  

So sticking to the pattern is better than deviating if

$$1 + 5\delta \geq \frac{\delta}{1 - \delta}.$$  

Multiply both sides by $1 - \delta^2$:

$$1 + 5\delta \geq \frac{\delta(1 - \delta^2)}{1 - \delta}.$$  

But $1 - \delta^2 = (1 - \delta)(1 + \delta)$, so this is

$$1 + 5\delta \geq \delta(1 + \delta) = \delta + \delta^2$$

or $1 - \delta^2 + 4\delta \geq 0$. Since $\delta$ is between 0 and 1, $1 - \delta^2 \geq 0$ and $4\delta \geq 0$, so this is always true.

To complete the argument, we should check whether player 1 wants to deviate in the second period instead. Then we should consider the (entirely symmetric) situation of whether player 2 wants to deviate in either the first or second period.

A simpler approach: Take the strategies to be to follow this pattern *no matter what the opponent does*. In other words, 1’s strategy is to alternate between A and B forever starting with A, while 2’s strategy is to alternate between A and B forever starting with B. Given that you can’t affect the opponent’s actions, you simply want to choose the best response period by period. Thus when he’s playing A, you want to play B and vice versa. Hence these strategies are best replies to one another and so form a Nash equilibrium for any $\delta$.

6. Suppose the two firms both set the monopoly price and so split the demand at that price. The monopoly price would be the price which maximizes $pQ = p(1 - p)$. The first order condition is $1 - 2p = 0$, so $p = 1/2$. If both firms set their price equal to 1/2, total demand will be $1 - (1/2) = 1/2$, so each firm will get half of this or 1/4. Hence each firm’s profit will be $(1/2)(1/4) = 1/8$.

How can we make this into an equilibrium? Suppose the strategy of each player is to set a price of 1/2 as long as both have always done so in the past and to set a price of 0 otherwise. (Since cost is zero here, this is the one–shot Bertrand price.) What would the best way to deviate be if you were to deviate? Clearly, you’d charge a price just a tiny bit less than 1/2 in the period you deviate, giving you all the sales and hence profits
very close to $1/4$. I.e., you’d effectively steal the other guy’s half of monopoly profits. However, no matter what you do after that, you can’t get profits greater than 0. So it would be optimal not to deviate if

$$\frac{1}{4} \leq \left(\frac{1}{8}\right) \frac{1}{1 - \delta}$$

or $\delta \geq 1/2$. 
