The Robustness of Scoring Rules Against Inefficient Manipulation*

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Abstract

It is shown that increasing scoring rules are robust against coalitional manipulations that result in inefficient outcomes. Examples showing that some other (Condorcet consistent) voting rules are subject to such inefficient manipulation are provided.

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1. Introduction

We show that increasing scoring rules are robust against coalitional manipulations that result in Pareto inefficient outcomes. Thus, increasing scoring rules guarantee Pareto efficient outcomes, regardless whether manipulation is attempted or not. Our result mitigates previous results that suggested that scoring rules are highly manipulable (see, e.g., Nitzan (1985) and the references therein).[1] It thus highlights the need to refine the notion of robustness against manipulation to allow for a consideration of the type of manipulation that is attempted. Not all manipulations are equally undesirable from the perspective of the maximization of social welfare. (Counter-) examples showing that other "reasonable" voting rules are subject to such inefficient manipulation are provided. In particular, we show that two Condorcet consistent voting rules – Simpson rule and the top cycle rule – are susceptible to inefficient manipulation.

2. Assumptions and General Setting

We employ the following notation. Let $N = \{1, ..., n\}$ denote a finite set of agents, and A denote a finite set consisting of k distinct alternatives. Denote the utility that any agent i, $i \in N$, derives upon selection of any alternative $a \in A$, by $u^i(a) \in \mathbb{R}$. We assume that the agents' preferences are complete and asymmetric – they are never indifferent between any two alternatives. We also assume that the agents are expected utility maximizers. That is, the utility that any agent i derives from a lottery in which alternative $a_j \in A$ is selected with probability $p_j \geq 0$, $\sum_{j=1}^{k} p_j = 1$, is given by $\sum_{j=1}^{k} p_j u^i(a_j)$.

Increasing scoring rules are defined as follows (see, e.g., Moulin, 1988, p. 231):

Definition: Increasing Scoring Rules. *Fix an increasing sequence of real numbers* $S_0 < S_1 < \cdots < S_{k-1}$.[2] *The agents rank the alternatives, giving a score of* S_0 *to the alternative that is ranked last, a score of* S_1 *to the alternative that is ranked next to last, and so on. The alternative that received the highest total score is selected. In case several alternatives all received the highest total score, an arbitrarily chosen tie-breaking rule may be used to determine the selected alternative from among those that received the maximum total score.*

[1] See Saari (1990) for a comparison of susceptibility to manipulation among scoring rules. See also Lepelley and Mbih (1994).

[2] Increasing scoring rules are a particular case of scoring rules. Scoring rules are more general in that they merely require that $S_0 \leq S_1 \leq \cdots \leq S_{k-1}$ and $S_0 < S_{k-1}$.

Note that we do not impose any restrictions on the information that is available to the agents. In particular, we allow for the extreme case where the entire profile of agents' preferences is commonly known among the agents, which facilitates manipulation.

3. Robustness Against Inefficient Manipulation

A-priori, it is not clear that a manipulation that results in a Pareto inferior outcome is necessarily unattractive to coalitions of agents. Let $a_1, ..., a_k \in A$, and suppose that all the agents (strictly) prefer a_1 to a_2. Suppose that agent 1's preferences are such that she prefers a_1 to a_2, a_2 to a_3, ..., and a_{k-1} to a_k. Suppose further that alternative a_3 is selected by some social choice rule. Agent 1 may not be able to manipulate the social choice rule so that alternative a_1 is chosen instead of a_3, yet, she may be able to manipulate so that alternative a_2, which she prefers but is Pareto dominated, is selected. As the examples below demonstrate, such inefficient manipulation is possible under the Simpson rule and the top cycle rule, both of which are Condorcet consistent.[3]

Example 1: Inefficient Manipulation under Simpson Rule. Simpson rule selects the alternative that has the smallest number of agents against it in bilateral comparisons with other alternatives. It is formally defined as follows (see, e.g., Moulin, 1988, p. 233).

Simpson Rule. *For any two different alternatives $a, b \in A$, let $N(a, b)$ denote the number of agents who prefer alternative a to b. The Simpson score of alternative a is the minimum of $N(a, b)$ over all b, $b \neq a$. The alternative with the highest Simpson score, called the **Simpson winner**, is selected. In case several alternatives all received the highest Simpson score, every such alternative is selected with equal probability.*

Let $a, b, c, d \in A$. Suppose that the preferences of agents 1, 2, and 3 are $d \succ a \succ c \succ b$, $b \succ d \succ a \succ c$, and $c \succ b \succ d \succ a$, respectively. Note that alternative a is Pareto dominated by alternative d. It can be verified that if the agents all report their preferences truthfully, agent 1's expected utility is given by:

$$\frac{1}{3}u^1(b) + \frac{1}{3}u^1(c) + \frac{1}{3}u^1(d).$$

However, agent 1 may increase her expected utility by reporting the following profile of

[3] A rule is Condorcet consistent (see, e.g., Moulin (1988, p. 229)) if it selects the Condorcet winner (an alternative that is preferred to every other alternative by a majority of the agents) when it exists.

preferences: $a \succ d \succ c \succ b$. Such a manipulation results an expected utility of

$$\frac{1}{4}u^1(a) + \frac{1}{4}u^1(b) + \frac{1}{4}u^1(c) + \frac{1}{4}u^1(d)$$

for agent 1. It is straightforward to verify that if $3u^1(a) > u^1(b) + u^1(c) + u^1(d)$, then such manipulation indeed increases agent 1's expected utility. Furthermore, there is no other manipulation that gives agent 1 a higher expected utility. While this outcome might be preferred by agent 1 to the previous one, it allows the choice of alternative a that is Pareto dominated by alternative d.

Moreover, if a non-neutral tie-breaking rule is employed, it may be that the Pareto dominated alternative is chosen with probability one. For example, suppose that ties are decided according to the following priority rule, $a \succ b \succ c \succ d$. Agents' preferences are the same as above. If the agents report their preferences truthfully, alternative b is selected as the Simpson winner. However, agent 1 may manipulate by reporting the following preferences: $a \succ d \succ c \succ b$. Such manipulation results in alternative a being selected. As before, it can be verified that this is the best manipulation that is available to agent 1.

Example 2: Inefficient Manipulation under the Top Cycle Rule. The top cycle rule selects the alternative that is preferred by a majority of the agents to all other alternatives if such an alternative exists. When such a Condorcet winner does not exist, the top cycle rule selects randomly from among the top-cycle (the transitive closure of majority rule). It is formally defined as follows (see, e.g., Moulin, 1988, p. 253).

Top Cycle Rule. *For any two different alternatives $a, b \in A$, let $a \succeq_T b$ if and only if there is an integer q and a sequence $a = a_0, a_1, ..., a_q = b$, such that a_j is preferred to a_{j+1} by at least half of the agents for every $j \in \{0, 1, ..., q-1\}$. The top cycle is defined as the non empty set of maximal elements of $\succeq_T$. Namely, an alternative a belongs to the top cycle if and only if $a \succeq_T b$ for every alternative $b \neq a$. The top cycle rule selects each alternative in the top cycle with equal probability.*

Let $a, b, c, d, e \in A$. There are 9 agents. The preferences of the first four agents are given by: $a \succ b \succ d \succ c \succ e$; of the next three agents by: $c \succ b \succ e \succ a \succ d$; and of the last two agents by: $e \succ a \succ c \succ d \succ b$. Note that alternative d is Pareto dominated by alternative a. If all the agents report their preferences truthfully, then the top cycle $(a \succeq_T c \succeq_T b \succeq_T e \succeq_T a)$ rule generates an expected utility of:

$$\frac{1}{4}u^i(a) + \frac{1}{4}u^i(b) + \frac{1}{4}u^i(c) + \frac{1}{4}u^i(e)$$

for the first four agents. However, the coalition of the first four agents might prefer to manipulate by reporting the following preferences: $a \succ d \succ b \succ c \succ e$. In this case the top-cycle consists of alternatives a, b, c, d, and e, $(a \succeq_T c \succeq_T d \succeq_T b \succeq_T e \succeq_T a)$ and applying the top cycle rule generates an expected utility of:

$$\frac{1}{5}u^i(a) + \frac{1}{5}u^i(b) + \frac{1}{5}u^i(c) + \frac{1}{5}u^i(d) + \frac{1}{5}u^i(e)$$

for the first four agents. It is straightforward to verify that if $4u^i(d) > u^i(a) + u^i(b) + u^i(c) + u^i(e)$ for every one of the first four agents, then these four agents increase their expected utility through this manipulation. Furthermore, it can also be verified that this is the best manipulation available for these first four agents. Under this manipulation, the Pareto inferior alternative d, might be selected.

As in the previous example, employing a non-neutral tie-breaking rule may result in the Pareto dominated alternative being chosen with probability one. Suppose that ties are decided according to the priority rule: $d \succ e \succ c \succ b \succ a$. Suppose that the nine agents above retain their preferences. With the new tie-breaking rule, if all the agents report their preferences truthfully, then alternative e is selected. However, the first four agents can manipulate as before and guarantee the choice of alternative d. It is readily verified that this is the best manipulation available for these agents.

We now show that inefficient manipulation by a coalition of rational agents is impossible under increasing scoring rules.

Proposition. *Suppose that all agents are rational – among several possible manipulations, they always use the one that maximizes their expected utility under the assumption that all other agents report truthfully. Then, a manipulation by a coalition of agents that results in the selection of a Pareto inferior alternative is impossible under any increasing scoring rule.*

Proof: Fix a profile of utility functions $u \in \mathbb{R}^{kn}$. For every agent $i \in N$, let $s^i_{u^i}(a) : A \mapsto \{S_0, ..., S_{k-1}\}$, denote the true score given by agent i with preferences u^i to the alternative $a \in A$.

For any two different alternatives $a, b \in A$, and a (non empty) coalition of agents $M \subseteq N$, let

$$D^M_u(a,b) = \sum_{i \in M} \left(s^i_{u^i}(a) - s^i_{u^i}(b) \right)$$

denote the difference between the total score of alternatives a and b for the agents in the coalition M.

Suppose that alternative a Pareto dominates alternative b. That is, $s^i_{u^i}(a) > s^i_{u^i}(b)$ for every agent i. It follows that,

$$D^M_u(a,b) > 0$$

for every non empty coalition $M \subseteq N$.

Suppose that some alternative c is receiving the highest total score, and that every member in some non empty coalition of agents, $M \subseteq N$, prefers alternative a to b to c. Otherwise, the coalition would surely not want to manipulate in favor of b against c. Our assumption about the rationality of the agents implies that we may assume that the coalition M cannot manipulate such that alternative a is selected instead of alternative c. (If the coalition members' could manipulate in favor of alternative a in such a way, they would, but this would *not* result in a Pareto inferior alternative being chosen with a positive probability.) If the coalition consists of $1 \leq m \leq n$ agents, this implies that:

$$D^{N\backslash M}_u(c,a) \geq m\left(S_{k-1} - S_0\right).$$

By definition of D,

$$\begin{aligned} D^{N\backslash M}_u(c,b) &= D^{N\backslash M}_u(c,a) + D^{N\backslash M}_u(a,b) \\ &> m\left(S_{k-1} - S_0\right). \end{aligned}$$

So the coalition M cannot manipulate so that alternative b is chosen with a positive probability either. ■

We conclude with two remarks. First, in the proof, we consider only the possibility of unilateral (coalitional) manipulations. However, our result also applies when several coalitions may independently attempt to manipulate simultaneously. The reason is that if several simultaneous independent manipulations can induce the choice of a Pareto dominated alternative, then the selection of an alternative that Pareto dominates it and is efficient can be achieved as well.

Second, the proposition requires scoring rules to be increasing. The next two examples demonstrate that our result cannot be generalized to the class of all (not necessarily increasing) scoring rules.

Example 3: Inefficient Manipulation under Plurality rule with a non-neutral tie-breaking rule. Plurality rule selects the alternative that is ranked the highest among the largest set of agents. Ties are decided according to some prespecified tie-breaking rule.

Let $a, b, c, d \in A$. There are 6 agents. The preferences of the first two agents are given by: $a \succ b \succ d \succ c$; of the second two agents by: $c \succ a \succ b \succ d$; and of the last two agents by: $d \succ c \succ a \succ b$. Ties are decided according to the non-neutral priority rule: $b \succ c \succ a \succ d$. Note that alternative b is Pareto dominated by alternative a. If all agents report their preferences truthfully, plurality rule selects alternative c. However, agents 1 and 2 may manipulate so that alternative b is chosen by reporting the preferences: $b \succ a \succ d \succ c$. It can be verified that this is the best manipulation that is available to agents 1 and 2.

Finally, the next example shows that even if we restrict our attention to neutral tie-breaking rules, the proposition still cannot be generalized to the class of all scoring rules.

Example 4: Inefficient Manipulation under a non increasing scoring rule with a neutral tie-breaking rule. Let $a, b, c, d, e \in A$. The scores are given by: $S_0 = 0$, $S_1 = S_2 = 6$, and $S_3 = S_4 = 10$. Ties are decided according to a neutral tie-breaking rule. That is, in case several alternatives all received the highest total score, every such alternative is selected with equal probability. There are 14 agents. The preferences of the first five agents are given by: $c \succ d \succ a \succ b \succ e$; of the next five agents by: $c \succ e \succ a \succ b \succ d$; and of the last four agents by: $a \succ e \succ b \succ d \succ c$. Note that alternative b is Pareto dominated by alternative a. If all the agents report their preferences truthfully, alternatives a and c tie with a score of a 100 each, and are each selected with probability $\frac{1}{2}$. However, the coalition of four agents may manipulate so that alternatives a, c, and b, are chosen with a probability $\frac{1}{3}$ each, by reporting the preferences: $a \succ b \succ e \succ d \succ c$.

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