Choosing the Rules for Consensus Standardization: Supplemental Proofs and Calculations

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Proofs

Lemma 1 *When qualities are independently distributed, every rationalizable Bayesian strategy is weakly increasing.*

Proof: Consider two possible types of player 1, $q_L < q_H$. Suppose that q_L puts positive probability weight on conceding at time t_L, and q_H puts positive weight on conceding at time t_H. Let Γ be player 1's (perceived) distribution function of t_2, the time when player 2 concedes (if player 1 does not previously do so). Let $\mathrm{E}[q|t] \equiv \mathrm{E}[q_2|t_2 = t]$ be player 1's expected value of player 2's quality q_2 given that player 2 concedes at t. (This conditional expectation does not depend on player 1's own type by assumption of independence.) We now state player 1's incentive compatibility constraints. If type q_L is willing to concede at t_L, we have:

$$Wq_L \int_0^{t_L} e^{-rt}\,\mathrm{d}\Gamma(t) + Le^{-rt_L} \int_{t_L}^{\infty} \mathrm{E}[q|t]\,\mathrm{d}\Gamma(t) \geq Wq_L \int_0^{t_H} e^{-rt}\,\mathrm{d}\Gamma(t) + Le^{-rt_H} \int_{t_H}^{\infty} \mathrm{E}[q|t]\,\mathrm{d}\Gamma(t).$$

The same argument leads to an incentive compatibility constraint for type q_H (replace q_L with q_H and reverse the inequality in the previous expression). Adding these two inequalities yields:

$$\int_0^{t_L} e^{-rt}\,\mathrm{d}\Gamma(t) \leq \int_0^{t_H} e^{-rt}\,\mathrm{d}\Gamma(t)$$

Thus, if $t_L > t_H$, there is zero concession probability for player 2 between t_H and t_L. But then no type of player 1 should concede at t_L (it would be better to concede at or just after t_H), contradicting our assumption that q_L can optimally concede at that time. Thus, we conclude that $t_L \leq t_H$. $\square$

Proposition 1 *When qualities are independently and continuously distributed, the symmetric equilibrium of an uninterrupted war of attrition selects the better proposal.*

Proof: Suppose there is an atom in the distribution of concession times at T, and let $\bar{q}$ ($\underline{q}$) be the supremum (infimum) of the set of types who can optimally concede at that time. For a given strategy, a player's expected payoff is continuous in its type. So type $\bar{q}$ can also optimally concede at T.

If $\bar{q} < \infty$, then by waiting until just after T, type $\bar{q}$ would (strictly) increase its own probability of winning without reducing the expected quality of the system that emerges. This contradicts the statement that type $\bar{q}$ can optimally concede at T. If $\bar{q} = \infty$, all types $q \in (\underline{q}, \infty)$ concede at T (by Lemma 1) and receive an expected payoff $\frac{1}{2}(Wq + LG(\underline{q}))$. This is also a contradiction, as types $q > \frac{LG(\underline{q})}{W}$ could do strictly better by waiting an instant to get Wq for sure.

Now suppose there is a gap in the distribution of concession times between t_L and t_H. This also leads to a contradiction: the lowest type that can optimally concede at t_H would do better to concede just after t_L. Thus, if quality is continuously distributed, the symmetric equilibrium strategies in an uninterrupted war of attrition must be one-to-one. The better system wins because concession times are strictly increasing in q. □

Lemma 3 *Suppose the equilibrium concession strategies of proponents who expect a random choice at T remain unchanged (i.e., as if $T = \infty$). Then, if either $f(q)$ or $1 - F(q)$ is log-concave, there is a unique time $T^* \geq 0$ such that a neutral player prefers screening via the war of attrition at all $t < T^*$ and immediate random choice thereafter.*

Proof: There are several cases to consider. First, if $G(q_{min}) = \mu < (1 + v)q_{min}$ then (7) implies that a neutral player prefers immediate random choice at $t = 0$. Because log-concavity implies that $G(q) - (1 + v)q$ is monotone decreasing, the neutral player will also prefer a random intervention at all $t > 0$; so $T^* = 0$.

If $\mu > (1+v)q_{min}$ then screening is initially worthwhile. Suppose the support of $F(\cdot)$ is $[q_{min}, \infty)$. Log concavity implies that $G(q) - q$ is bounded above by $\mu - q_{min}$. Because vq grows without bound as $q \to \infty$, continuity and the intermediate value theorem imply the existence of a unique q^* such that $G(q^*) - q^* = vq^*$. For all $t > t(q^*)$ the neutral player prefers random intervention.

If the support of $F(\cdot)$ is $[q_{min}, \overline{q}]$, then $G(q)$ must converge to $\overline{q}$ as q approaches the upper bound. Because $v\overline{q} > 0$, continuity and the intermediate value theorem imply a unique $q^* < \overline{q}$. □

Proposition 3 *The following strategies are a perfect Bayesian equilibrium: Sponsor-types $q < q^*$ use the concession strategy derived in Section 2.1. Types $q > q^*$ never concede. The neutral player waits until T^* before making a random choice (and would also intervene at all times $t > T^*$).*

Proof: To begin, note that Lemma 3 implies $Wq^* = LG(q^*)$, so the last type to concede before a random intervention must be indifferent between immediate victory and concession. We need to show that two kinds of deviations are unrewarding for the players. First, in the candidate equilibrium, a sponsor with type $q > q^*$ will wait until its rival concedes or the random choice is made; might this player deviate by conceding early, before T^*? Second, a sponsor with type $q < q^*$ is meant to concede before the random choice at T^*; might it deviate by waiting till T^* and having a chance of winning?

First, by Lemma 3, at T^* a sponsor with $q > q^*$ prefers random choice to concession. Since the type q^* sponsor at least weakly prefers concession at T^* to concession at any $t < T^*$, Lemma 3 implies that the same is true of any higher type. Thus, types $q > q^*$ do not have an incentive to deviate from equilibrium play. We must also check that they would not deviate from the prescribed out-of-equilibrium play. That is, if at $t > T^*$ there has still been no random choice, but they expect one imminently, would a high-type try to concede? No: by Lemma 3, they would prefer an immediate random choice.[1]

Second, a player with $q < q^*$ could, by waiting till T^*, obtain a strictly positive chance of winning. Couldn't that be a tempting deviation? No—by Lemma 3, for a low quality sponsor, concession just before T^* dominates random choice at $t \geq T^*$: the expected benefits of choosing a rival system exceed those of being the winner.

Finally, consider the neutral player's random intervention at T^*. By the definition of T^*, this player prefers to allow screening at all $t < T^*$ but wants to intervene then. Because the out-of-equilibrium expectations are that no vendors will concede at any $t > T^*$, if no actual concession has happened by such a time, the same calculation says that the neutral player wants to intervene now. Of course, if a concession has happened, there is no incentive question to resolve. □

Lemma 4: *When proponents expect a random intervention at $T < \infty$, their symmetric equilibrium concession strategies have thresholds $\underline{q} \leq \overline{q}$, such that types $q < \underline{q}$ concede according to $t(q)$; types $q \in (\underline{q}, \overline{q})$ concede at T; and types $q > \overline{q}$ wait for the random intervention.*

Proof: Suppose $\underline{q}$ is the supremum of the set of types that concede strictly before T. As $t(q)$ is weakly increasing, types $q < \underline{q}$ will also concede before T. And because the intervention at T has no impact on the marginal costs or benefits of delay for types below $\underline{q}$, they will concede according to their first-order conditions. To see that $\underline{q}$ is finite, note that $t(q) \to \infty$ as q approaches ∞ (or any finite upper bound). Moreover, because $F(\cdot)$ and the payoffs are continuous in q, type $\underline{q}$ will be indifferent between conceding at $t(\underline{q})$ and T.

The remaining types $q > \underline{q}$ have two options; they can concede at T — and win with probability $\frac{1}{2}$ if the other player also concedes at that time — or "wait" (an instant) for the random intervention. Lemma 1 says that if $\overline{q}$ is the supremum in the set of types that can optimally concede at T, then any proposals between $\underline{q}$ and $\overline{q}$ must also concede at that time. Types greater than $\overline{q}$ wait for the random choice by construction. □

Proposition 4: *If the marginal benefit of unobserved ex ante R&D is constant in q, a predetermined*

[1] This would break down if the sponsors believe that types $q \in (q^*, Q)$ would have already conceded by now (after T^*), so that concession looks better (relative to random choice) because the opponent's expected quality is now $G(Q) > G(q^*)$. But this is not an issue, as we are specifying that each believes that the other is playing "never concede after T^*," and there can be no evidence to falsify that belief: the only available evidence would be concession, after which the question doesn't arise.

standard setter has the strongest incentive to innovate, followed by firms facing an immediate random choice. Firms that anticipate a random intervention at $T^* > 0$ have weaker incentives than under immediate random choice, but stronger than under an uninterrupted war of attrition: $I^{\mathrm{PS}} = W > I^{\mathrm{RC}} = \frac{1}{2}W \geq I^{\mathrm{RI}} \geq I^{\mathrm{WOA}}$.

Proof: Because $I^{\mathrm{WOA}} \equiv E[u'(q)]$, equation (10) implies that

$$I^{\mathrm{WOA}} = W \int_{q_{min}}^{\infty} \int_{q_{min}}^{x} \delta(y)\,\mathrm{d}F(y)\,\mathrm{d}F(x) \leq W \int_{q_{min}}^{\infty} \int_{q_{min}}^{x} \mathrm{d}F(y)\,\mathrm{d}F(x) = \frac{W}{2} = I^{\mathrm{RC}}$$

and the inequality is strict for $v > 0$, because $\delta(y) < 1$ for all $y > q_{min}$ (by Proposition 1).

Random intervention is equivalent to an immediate random choice as of T^*, and the incentive comparison is no different, as *both* proposals must be better than q^* for random intervention to make any difference in outcomes. Formally, we have

$$I^{\mathrm{RI}} - I^{\mathrm{WOA}} = \int_{q^*}^{\infty} \left(\frac{1}{2} W \delta(q^*) - W \int_{q^*}^{x} \delta(y)\,\mathrm{d}F(y) \right) \mathrm{d}F(x)$$

Because $\delta(y) < \delta(q^*)$ for all $y > q^*$, we can factor out $\delta(q^*)$ and integrate the previous expression to show that $I^{\mathrm{RI}} - I^{\mathrm{WOA}} > \frac{1}{2} W \delta(q^*) F(q^*) \left[1 - F(q^*)\right] > 0$ □

Calculations

All examples are based on the Pareto distribution: $F(x) = 1 - x^{-(1+a)}$ for $x \geq 1$ and $a > 0$.

Example 1: *Suppose quality has the Pareto distribution, and that the equilibrium concession strategies of proponents who expect a random choice at T remain unchanged (i.e., as if $T = \infty$). Then a neutral player makes an immediate random choice if $av > 1$ and allows an uninterrupted war of attrition if $av < 1$.*

$$G(x) = \frac{\int_x^{\infty} s f(s)\,\mathrm{d}s}{1 - F(x)} = \frac{-(1+a)\int_x^{\infty} s^{-(1+a)}\,\mathrm{d}s}{x^{-(1+a)}} = \frac{-(1+a)[s^{-a}]_x^{\infty}}{a x^{-(1+a)}} = \frac{(a+1)x}{a}$$

The results follows immediately from substituting $G(x)$ into equation (7).

Example 2 *In the uninterrupted war of attrition with a Pareto distribution, $I^{\mathrm{WOA}} = E[u'(q)]$ is increasing in W if and only if $k < \frac{a+2}{a(v+1)^2}$.*

To find $E[u'(q)]$, start with $u'(q)$ from equation (10). From the calculations for Example 1, we know

$\mu = \frac{(a+1)}{a}$ and $K(x) = \frac{(a+1)x^{-a}}{a}$. Thus, we have

$$u'(q) = W \int_1^q \left[\frac{K(s)}{\mu}\right]^v \mathrm{d}F(s) = W(1+a)\int_1^q s^{-a(v+1)-2}\,\mathrm{d}s = \frac{W(1+a)\left[1-q^{-a(v+1)-1}\right]}{a(v+1)+1}$$

and taking expectations yields

$$\begin{aligned} E[u'(q)] &= \frac{W(1+a)}{a(v+1)+1}\int_1^\infty \left[1-s^{-a(v+1)-1}\right](1+a)s^{-a-2}\,\mathrm{d}s \\ &= \frac{W(1+a)}{a(v+1)+1}\left(1-\frac{(a+1)}{a(v+2)+2}\right) = \frac{LW(a+1)}{Wa+L(a+2)} \end{aligned}$$

where the final equality is found by substituting $v = \frac{W-L}{L}$ and simplifying. In Example 2 we differentiate this expression with respect to w where (by assumption) $\frac{\mathrm{d}W}{\mathrm{d}w} = 1$ and $\frac{\mathrm{d}L}{\mathrm{d}w} = -k$, and ask whether the result is greater than zero. This yields

$$\frac{\mathrm{d}I^{WOA}}{\mathrm{d}w} = \frac{(a+1)[L^2(a+2)-kaW]}{(Wa+L(a+2))^2}$$

and the numerator is positive if and only if $k < \frac{L^2(a+2)}{W^2a} = \frac{a+2}{a(v+1)^2}$.

Example 3: $E[s'(q)] < E[u'(q)]$ *if and only if* $v(v+2)(a+1) > 1$*: private incentives to improve quality are too high if there is strong vested interest.*

Integrating equation (11) yields $E[s'(q)]$ as a function of $E[u'(q)]$. Thus, using $E[u'(q)]$ from Example 2, we have

$$E[u'(q)] < E[s'(q)] \Leftrightarrow \frac{L}{W}E[u'(q)] > (W+L)E[qf(q)v\delta(q)] \Leftrightarrow \frac{L(a+1)}{a(v+2)+2} > (W+L)E[qf(q)v\delta(q)]$$

Further calculations show that

$$E[qf(q)v\delta(q)] = v(1+a)^2\int_1^\infty s^{-a(2+v)-3}\,\mathrm{d}s = \frac{v(1+a)^2}{a(v+2)+2}$$

and substituting this expression into the previous inequality, yields the result stated in the text.

$$E[u'(q)] < E[s'(q)] \Leftrightarrow \frac{L(a+1)}{a(v+2)+2} > \frac{(W+L)v(1+a)^2}{a(v+2)+2} \Leftrightarrow 1 > v(v+2)(a+1)$$