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LETTER TO THE EDITOR

Quantum theory of femtosecond solitons in optical fibres

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Abstract. We use the time-dependent Hartree approximation to obtain the solution to the quantum higher-order non-linear Schrödinger equation. This equation describes femtosecond pulses propagating in non-linear optical fibres and can have soliton solutions. These solitons travel at velocities that differ from the picosecond solitons obtained from the standard quantum non-linear Schrödinger equation. We find that these femtosecond solitons cannot propagate in graded-index fibres; rather, they require quadruple-clad fibres. This is the first investigation of quantum effects in femtosecond solitons to our knowledge.

There is considerable interest in the non-linear Schrödinger equation (NLS) in terms of both classical and quantum phenomena [1–6]. In particular it has been used extensively to model the propagation of pulses in non-linear optical fibres; however, the NLS is generally not valid for pulses with durations shorter than the picosecond time scale. Yet the recent development of optical sources that generate pulses in the femtosecond domain makes possible the exploration of many new phenomena. Therefore the investigation of solitons arising from the higher-order NLS (HNLS), which can be used in the femtosecond time domain, is of interest.

One of the simpletst HNLS is [7]

$$\mathrm{i}\frac{\partial\phi}{\partial t}+\frac{\partial^2\phi}{\partial x^2}+2C|\phi|^2\phi+\mathrm{i}d\frac{\partial^3\phi}{\partial x^3}+\mathrm{i}pC|\phi|^2\frac{\partial\phi}{\partial x}=0 \tag{1}$$

where C, d and p are constants. We follow the conventional notation in the mathematical literature, which uses t and x to represent normalized *space* and *time*, respectively. This equation gives rise to soliton solutions when $p=6d$ [7]. Equation (1) reduces to the NLS for $p=d=0$.

In certain circumstances the HNLS can be used to describe femtosecond pulses propagating in optical fibres; these are outlined in [8] and described in detail by us in [9]. Using experimental fibre parameters to evaluate the physical parameters in equation (1) we find that the pulse width must be below 200 fs for wavelengths in the 1.48–1.57 μm region in order for d and p to become significant. In addition, the dispersion parameters, β_2 and β_3, given by the second and third derivatives of the propagation constant respectively, evaluated at the carrier frequency ω_0, must be negative. This necessitates a quadruple-clad fibre rather than the typical graded-index fibres used in calculations and experiments to date. This is a significant feature of our results [9]. The soliton self-frequency shift (SSFS) [10, 11] may be an important effect when considering femtosecond solitons. However, we use the numerical-beam propagation method to show that at distances required for the quantum effect to be observed the effect of the SSFS on the soliton described by equation (1) can be neglected [9].

In the case of optical solitons, ϕ represents the normalized envelope of the electromagnetic field. The quantities C and d are given by

$$C \equiv \frac{n_2 \omega_0 \sigma^2 I^2}{c|\beta_2|} \qquad d \equiv \frac{|\beta_3|}{3\sigma|\beta_2|} \tag{2}$$

and p is a parameter involving the frequency-dependent index of refraction and the frequency-dependent radius of the mode of the fibre [8]. n_2 is the non-linear index of refraction, σ is the $1/e$-width of the pulse intensity, I is the peak amplitude of the pulse and c is the speed of light.

The general solution of equation (1) has the form [7]

$$\phi = \phi_0 \operatorname{sech}[\varepsilon(x-x_0)+\beta t] \exp\{\mathrm{i}[\gamma(x-x_0)+\delta t]\} \tag{3}$$

where ε, β, γ and δ are constants and x_0 is the zero of time. Substituting this in equation (1) yields the following relations

$$|\phi_0|^2 = \varepsilon^2/C \qquad \delta = \varepsilon^2 - \gamma^2 - 3d\gamma\varepsilon^2 + d\gamma^3 \qquad \beta = \varepsilon(2\gamma + d\varepsilon^2 - 3d\gamma^2). \tag{4}$$

We proceed by considering the quantum version of equation (1) from a mathematical point of view. In [9] we examine the physical aspects of this problem in detail and describe the role played by other effects such as the SSFS. The initial portion of our analysis closely follows that of Lai and Haus [5] for the NLS. To obtain the quantum version of equation (1), the quantities $\phi(t,x)$ and $\phi^*(t,x)$ are replaced by the field operators $\hat{\phi}(t,x)$ and $\hat{\phi}^+(t,x)$, which satisfy the boson commutation relations

$$[\hat{\phi}(t,x'),\hat{\phi}^+(t,x)] = \delta(x-x') \qquad [\hat{\phi}(t,x'),\hat{\phi}(t,x)] = [\hat{\phi}^+(t,x'),\hat{\phi}^+(t,x)] = 0 \tag{5}$$

where $\hat{\phi}(t,x)$ and $\hat{\phi}^+(t,x)$ are the photon annihilation and creation operators, respectively, at t and x.

The quantized equation can be written as

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \hat{\phi}(t,x) = [\hat{\phi}(t,x), \hat{H}] \tag{6}$$

with

$$\hat{H} = \hbar\Bigg[\int \hat{\phi}_x^+(t,x)\hat{\phi}_x(t,x)\,\mathrm{d}x - C\int \hat{\phi}^+(t,x)\hat{\phi}^+(t,x)\hat{\phi}(t,x)\hat{\phi}(t,x)\,\mathrm{d}x$$
$$+\mathrm{i}d\left(\int \hat{\phi}_x(t,x)\hat{\phi}_{xx}(t,x)\,\mathrm{d}x - 3C\int \hat{\phi}^+(t,x)\hat{\phi}^+(t,x)\hat{\phi}(t,x)\hat{\phi}_x(t,x)\,\mathrm{d}x\right)\Bigg] \tag{7}$$

where the subscripts x and xx signify differentiation and double differentiation respectively.

In the Schrödinger picture, the state of the system $|\psi\rangle$ evolves according to

$$\mathrm{i}\hbar \frac{\partial}{\partial t}|\psi\rangle = \hat{H}_s|\psi\rangle \tag{8}$$

where

$$\hat{H}_s=\hbar\left[\int \hat{\phi}_x^+(x)\hat{\phi}_x(x)\,\mathrm{d}x-C\int \hat{\phi}^+(x)\hat{\phi}^+(x)\hat{\phi}(x)\hat{\phi}(x)\,\mathrm{d}x\right.$$
$$\left.+\mathrm{i}d\left(\int \hat{\phi}_x(x)\hat{\phi}_{xx}(x)\,\mathrm{d}x-3C\int \hat{\phi}^+(x)\hat{\phi}^+(x)\hat{\phi}(x)\hat{\phi}_x(x)\,\mathrm{d}x\right)\right]. \tag{9}$$

In general, any state of this system can be expanded in Fock space as

$$|\psi\rangle=\sum_n a_n\int \frac{1}{\sqrt{n!}}f_n(x_1,\ldots,x_n,t)\hat{\phi}^+(x_1)\cdots\hat{\phi}^+(x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n|0\rangle. \tag{10}$$

The quantity $|a_n|^2$ is the probability of finding n photons in the field and we require

$$\sum_n |a_n|^2=1; \tag{11}$$

f_n obeys the normalization condition

$$\int |f_n(x_1,\ldots,x_n,t)|\,\mathrm{d}x_1\cdots\mathrm{d}x_n=1. \tag{12}$$

Substituting equations (9) and (10) into (8) we obtain

$$\mathrm{i}\frac{\partial}{\partial t}f_n(x_1,\ldots,x_n,t)=\left(-\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}-2C\sum_{1\leq i<j\leq n}\delta(x_j-x_i)\right.$$
$$\left.-\mathrm{i}d\sum_{j=1}^{n}\frac{\partial^3}{\partial x_j^3}-6\mathrm{i}Cd\sum_{1\leq i<j\leq n}\delta(x_j-x_i)\frac{\partial}{\partial x_j}\right)f_n(x_1,\ldots,x_n,t). \tag{13}$$

We solve equation (13) using the time-dependent Hartree approximation [12]. We define a Hartree wavefunction

$$f_n^{(\mathrm{H})}(x_1,\ldots,x_n,t)=\prod_{j=1}^{n}\Phi_n(x_j,t) \tag{14}$$

where Φ_n has the normalization

$$\int |\Phi_n(x,t)|^2\,\mathrm{d}x=1. \tag{15}$$

The functions Φ_n are determined by minimizing the functional

$$I=\int f_n^{*(\mathrm{H})}(x_1,\ldots,x_n,t)\left[\mathrm{i}\frac{\partial}{\partial t}+\sum_{j=1}^{n}\left(\frac{\partial^2}{\partial x_j^2}+\mathrm{i}d\frac{\partial^3}{\partial x_j^3}\right)\right.$$
$$\left.+\sum_{1\leq i<j\leq n}\delta(x_j-x_i)\left(2C+6\mathrm{i}Cd\frac{\partial}{\partial x_j}\right)\right]f_n^{(\mathrm{H})}(x_1,\ldots,x_n,t)\,\mathrm{d}x_1\cdots\mathrm{d}x_n \tag{16}$$

which provides

$$\mathrm{i}\frac{\partial\Phi_n}{\partial t}+\frac{\partial^2\Phi_n}{\partial x^2}+2C(n-1)|\Phi_n|^2\Phi_n+\mathrm{i}d\frac{\partial^3\Phi_n}{\partial x^3}+6\mathrm{i}Cd(n-1)|\Phi_n|^2\frac{\partial\Phi_n}{\partial x}=0. \tag{17}$$

This is identical to the classical HNLS given in equation (1), with C replaced by $C(n-1)$. Thus the solution to the quantized femtosecond soliton equation is obtained directly from equations (3) and (4):

$$\Phi_n(x,t)=[C(n-1)]^{-1/2}\,\varepsilon\,\mathrm{sech}\{\varepsilon[(x-x_0)+(-3d\gamma^2+d\varepsilon^2+2\gamma)t]\}$$
$$\times\exp[-\mathrm{i}(d\gamma^3+3d\gamma\varepsilon^2+\gamma^2-\varepsilon^2)t+\mathrm{i}\gamma(x-x_0)]. \tag{18}$$

The normalization condition, equation (11), gives

$$\varepsilon=\tfrac{1}{2}(n-1)C. \tag{19}$$

Substituting equation (19) into (18) leads to

$$\Phi_{n\gamma}(x,t)=\tfrac{1}{2}(n-1)^{1/2}C^{1/2}\,\mathrm{sech}\{\tfrac{1}{2}(n-1)C[(x-x_0)+(-3d\gamma^2+\tfrac{1}{4}d(n-1)^2C^2+2\gamma)t]\}$$
$$\times\exp\{[\mathrm{i}d\gamma^3-\tfrac{3}{4}\mathrm{i}d\gamma(n-1)^2C^2-\mathrm{i}\gamma^2+\tfrac{1}{4}\mathrm{i}(n-1)^2C^2]t+\mathrm{i}\gamma(x-x_0)\}. \tag{20}$$

The Hartree product eigenstates are, using equations (10) and (14),

$$|n,\gamma,t\rangle_{\mathrm{H}}=\frac{1}{\sqrt{n!}}\left[\int\Phi_{n\gamma}(x,t)\hat{\phi}^{+}(x)\,\mathrm{d}x\right]^n|0\rangle. \tag{21}$$

A superposition of these states, using a Poissonian distribution of n for a coherent-state pulse, gives

$$|\psi_{\mathrm{s}}\rangle_{\mathrm{H}}=\sum_n\frac{\alpha_0^n}{n!}\mathrm{e}^{-|\alpha_0|^2/2}\left(\int\Phi_{n\gamma}(x,t)\hat{\phi}^{+}(x)\,\mathrm{d}x\right)^n|0\rangle \tag{22}$$

where $|\alpha_0|^2=n_0$ is the mean photon number.

The quasiprobability density for the amplitude of the envelope of the field is defined as

$$Q(\alpha,x,t)\equiv|\langle\alpha,x|\psi_{\mathrm{s}}\rangle|^2 \tag{23}$$

where

$$|\alpha,x\rangle\equiv\mathrm{e}^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}[\hat{\phi}^{+}(x)]^n|0\rangle \tag{24}$$

is a local coherent state at the time x. Substituting equation (22), with (20) and (24), into (23) gives

$$Q(\alpha,x,t)=\mathrm{e}^{-|\alpha|^2-|\alpha_0|^2}$$
$$\times\Bigg|\sum_{n=0}^{\infty}\frac{(\alpha^*\alpha_0)^n}{n!}\Bigg(\frac{(n-1)^{1/2}}{2}C^{1/2}$$
$$\times\,\mathrm{sech}\left\{\frac{n-1}{2}C\left[(x-x_0)+\left(-3d\gamma^2+d\frac{(n-1)^2}{4}C^2+2\gamma\right)t\right]\right\}\Bigg)^n$$
$$\times\exp\Bigg[\left(\mathrm{i}nd\gamma^3-\mathrm{i}n(n-1)^2\tfrac{3}{4}d\gamma C^2-\mathrm{i}n\gamma^2+\mathrm{i}n\frac{(n-1)^2}{4}C^2\right)t$$
$$+\mathrm{i}\gamma n(x-x_0)\Bigg]\Bigg|^2. \tag{25}$$

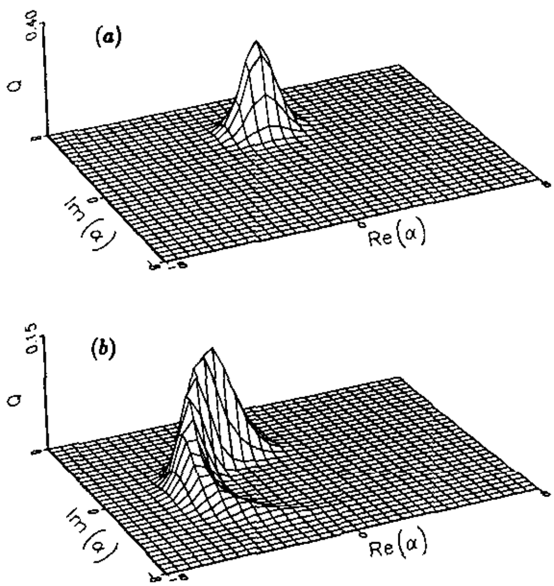

Figure 1. Plots of the quasiprobability density $Q(\alpha, x, t)$ against the real and imaginary parts of α for $\alpha_0 = 4$, $C = 0.25$, $d = 0.25$, $\gamma = 0$, $(x - x_o) = 0$ and (*a*) $t = 0$ and (*b*) $t = 0.1$.

In figure 1 we illustrate how this quantity changes as the soliton propagates in space. We have ignored the n dependence of the amplitude and kept it in the phase. We observe phase spreading similar to that in the NLS case [5, 6].

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