

Multiphoton Stokes-parameter invariant for entangled states

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We consider the Minkowskian norm of the n -photon Stokes tensor, a scalar invariant under the group realized by the transformations of stochastic local quantum operations and classical communications (SLOCC). This invariant is offered as a candidate entanglement measure for n -qubit states and discussed in relation to measures of quantum state entanglement for certain important classes of two-qubit and three-qubit systems. This invariant can be directly estimated via a quantum network, obviating the need to perform laborious quantum state tomography. We also show that this invariant directly captures the extent of entanglement purification due to SLOCC filters.

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Entangled states of photon polarization have been of ongoing interest for their role in probing fundamental aspects of quantum theory, in the coding and manipulation of quantum information and in practical applications of quantum interferometry. In both classical and quantum optics, Stokes parameters have proven to be intuitive and practical tools for characterizing polarization states of light. Here, we examine a group-invariant scalar measure on the space of generalized Stokes parameters. We show that this norm, which is an invariant under transformations of stochastic local quantum operations and classical communications (SLOCC) [1] on n qubits, quantifies entanglement for certain important classes of two-qubit and three-qubit systems, and potentially for similar classes of $n > 3$ qubits. Our results for several photon states, together with their mathematical properties for all values of n , recommend this scalar as a candidate measure of total entanglement for multiparticle pure states in general. This invariant has the valuable property of being directly estimable via a quantum network, in principle obviating the need to perform quantum state tomography, an increasingly laborious task as the number of particles increases, to determine the density matrix first in order to find the degree of entanglement. The invariant allows one to immediately identify the SLOCC filtering transformations as entanglement purifiers and directly captures the amount of purification achieved by these filters or any other process.

I. DEFINITIONS AND THE GENERAL CASE

In classical optics, the four Stokes parameters S_μ , where $\mu = 0, 1, 2, 3$, are known to form a four-vector under the $O_0(1,3)$ group of transformations [2–5]. These four parameters characterize the time-averaged electric-field intensity and the distribution of polarization among three orthogonal polarization directions in the Poincaré sphere. The associated invariant length of the Stokes four-vector is $S^2 \equiv S_0^2 - S_1^2 - S_2^2 - S_3^2$ (Ref. [4]). These transformations can be represented by an ordinary rotation, followed by a hyperbolic rotation, followed by another ordinary rotation [6]. As a practical example, we note that the angles of the ordinary

polarization rotations may parametrize the effect of birefringence during light propagation in optical fiber; those of the hyperbolic polarization rotations may parametrize the effect of dichroism in the fiber [5].

In the quantum case, the Stokes-parameter representation of the single-photon ensemble is formally similar to that of classical polarization optics, and will similarly be seen here to form a four-vector. To address the multiple-photon case, we make use of n -photon generalized Stokes parameters (see, for example, Ref. [7]),

$$S_{i_1 \dots i_n} = \text{Tr}(\rho \sigma_{i_1} \otimes \dots \otimes \sigma_{i_n}), \quad i_1, \dots, i_n = 0, 1, 2, 3, \quad (1)$$

where $\sigma_\mu^2 = 1$, and $\mu = 0, 1, 2, 3$ are the three Pauli matrices together with the identity $\sigma_0 = I_{2 \times 2}$, and $\frac{1}{2} \text{Tr}(\sigma_\mu \sigma_\nu) = \delta_{\mu\nu}$. These parameters form a full set of n -photon generalized Stokes tensors $\{S_{i_1 \dots i_n}\}$ that can be used to describe coherence and entanglement properties of photon-number states. The n -photon polarization density matrix can also be conveniently written in terms of these generalized Stokes parameters:

$$\rho = \frac{1}{2^n} \sum_{i_1, \dots, i_n=0}^3 S_{i_1 \dots i_n} \sigma_{i_1} \otimes \dots \otimes \sigma_{i_n}, \quad i_1, \dots, i_n = 0, 1, 2, 3. \quad (2)$$

Under SLOCC, the initial system density matrix undergoes transformations of the group $SL(2, C)$, while the multiphoton Stokes parameters similarly undergo transformations of the group $O_0(1,3)$, which we notate $S_{i_1 \dots i_n} \rightarrow S'_{i_1 \dots i_n}$. The unitary subgroup of $[SU(2)]$ transformations of the two-qubit and three-qubit density matrices have been carefully studied (see, for example, Ref. [8]). These correspond to subgroup of ordinary $[SO(3)]$ rotations of the quantum Stokes tensor. The set of nonunitary $[SL(2, C)/SU(2)]$ transformations of the density matrix have largely been overlooked in the investigation of entanglement (with a few notable exceptions [9–11]). For the tensor of Stokes parameters, these lat-

ter transformations [O₀(1,3)/SO(3)] involve hyperbolic rotation, corresponding physically to polarization-dependent loss and intensity reduction: $S'_0 \dots_0 < 1$. For these transformations, the Stokes parameters $S'_{i_1 \dots i_n}$ must be renormalized due to the associated removal of a portion of the original quantum ensemble, resulting in the renormalized, physical values $S''_{i_1 \dots i_n} = S'_{i_1 \dots i_n} / S'_0 \dots_0$. This will later allow us to identify a class of filters that purify entanglement.

For each finite number n of entangled photons, let us examine the O₀(1,3) group-invariant length, namely, the Minkowskian squared norm of the Stokes tensor $\{S_{i_1 \dots i_n}\}$, which we refer to as the ‘‘Stokes scalar.’’ For reasons of convenience, we choose to normalize this quantity by the factor 2^{-n} :

$$S_{(n)}^2 \equiv \frac{1}{2^n} \left\{ (S_{0 \dots 0})^2 - \sum_{k=1}^n \sum_{i_k=1}^3 (S_{0 \dots i_k \dots 0})^2 + \sum_{k,l=1}^n \sum_{i_k, i_l=1}^3 (S_{0 \dots i_k \dots i_l \dots 0})^2 - \dots + (-1)^n \sum_{i_1, \dots, i_n=1}^3 (S_{i_1 \dots i_n})^2 \right\}. \quad (3)$$

Here, we show this scalar to be useful for understanding state purity and entanglement properties of multiphoton systems. We note immediately that these satisfy the fundamental requirement of entanglement measures that they should be invariant under local unitary transformations of SU(2), since these are a subgroup of the O₀(1,3) transformations of SLOCC, under which they are invariant, and since no renormalization is required after their action. We see that the quantum state purity for a general n -photon state can be written simply in terms of the multiphoton Stokes parameters as

$$\frac{1}{2^n} \sum_{i_1, \dots, i_n=0}^3 S_{i_1 \dots i_n}^2 = \text{Tr} \rho^2, \quad (4)$$

which is seen to be the corresponding Euclidean norm of the multiphoton Stokes tensor. More importantly, we see that the Stokes scalar can be expressed in terms of density matrices as

$$S_{(n)}^2 = \text{Tr}(\rho_{12 \dots n} \tilde{\rho}_{12 \dots n}), \quad (5)$$

where $\tilde{\rho}_{1 \dots n} = (\sigma_2 \otimes \sigma_2 \otimes \dots \otimes \sigma_2) \rho_{1 \dots n}^* (\sigma_2 \otimes \sigma_2 \otimes \dots \otimes \sigma_2)$ is the ‘‘spin-flipped’’ density matrix. As we will show, $S_{(2)}^2$ captures the entanglement of important classes of multiple-photon states. This relation makes the Stokes scalar of exceptional interest as a candidate entanglement measure since, as shown recently, functionals of the form $\text{Tr}(\rho_a \rho_b)$ are directly estimable through the visibility of interference arising in an appropriate quantum network [12], in addition to being indirectly measurable via the quantum tomography approach [7]. We note that $S_{(n)}^2$ is of the required form, for example, $\rho_a = \rho_{12}$ and $\rho_b = \tilde{\rho}_{12}$ in the case $n = 2$.

The connection between our invariant and the basic Stokes parameters in the $n = 1$ case is simple. In that case, the Stokes parameters form a vector of elements $S_\mu = \text{Tr}(\rho \sigma_\mu)$, $\mu = 0, 1, 2, 3$, and $S_{(1)}^2 = S_0^2 - S_1^2 - S_2^2 - S_3^2$, similar to the classical case. In this case, the only relevant quantity is $S_{(1)}^2$. One can relate the single-photon state purity to the Stokes scalar as

$$\text{Tr}(\rho^2) = 1 - S_{(1)}^2, \quad (6)$$

while $P^2 = S_1^2 + S_2^2 + S_3^2 = 1 - 2S_{(1)}^2$ is the well-known degree of photon polarization. Equivalently, we see that $S_{(1)}^2$ is the single-photon linearized entropy,

$$S_{(1)}^2 = 1 - \text{Tr} \rho^2 \quad (7)$$

(see, for example, Ref. [13]). $S_{(1)}^2$ allows us to understand the effect of O₀(1,3) transformations on state purity. Under ordinary polarization rotations, S_0 itself remains unchanged, so the purity is unchanged. However, the hyperbolic polarization rotations filter the beam in a basis-dependent way, reducing the quantum ensemble and diminishing the intensity to $S'_0 < 1$. We notate the Stokes vector transformation under an element of O₀(1,3) as $S_\mu \rightarrow S'_\mu$. Recalling that it is therefore necessary to renormalize the state, $S''_\mu \rightarrow S'_\mu = S'_\mu / S'_0$, we see that these filtering transformations effectively increase the Stokes scalar in the single-photon case: $S''_{(1)}^2 > S_{(1)}^2$, since $S''_{(1)}^2 \equiv S'^2_{(1)} / S'^2_0$ with $S'^2_{(1)} = S_{(1)}^2$ due to the invariance of $S_{(1)}^2$ under O₀(1,3). In this way, these filtering transformations are seen to decrease the purity and increase the linearized entropy of single-photon polarization states, just as it does in the classical case.

II. THE CASE $n = 2$

Our central interest here, however, is that of two or more entangled photons. The generalized Stokes parameters that characterize the two-photon polarization quantum ensemble are

$$S_{\mu\nu} = \text{Tr}(\rho \sigma_\mu \otimes \sigma_\nu), \quad (8)$$

where $\mu, \nu = 0, 1, 2, 3$ (Refs. [7,14,15]). This collection of Stokes parameters has 16 elements, which are systematically measurable, as is done in quantum state tomography to determine the density matrix, and these elements capture all the polarization correlations present in a photon pair as well as single-photon polarization and beam intensity information [15]. Consider the scalar invariant for the case $n = 2$,

$$S_{(2)}^2 = \frac{1}{4} \left\{ (S_{00})^2 - \sum_{i=1}^3 (S_{i0})^2 - \sum_{j=1}^3 (S_{0j})^2 + \sum_{i=1}^3 \sum_{j=1}^3 (S_{ij})^2 \right\}. \quad (9)$$

Note that the second and third terms of the right-hand side of Eq. (9) pertain only to the one-photon subsystems, being the squares of the polarizations of the individual particles 1 and 2, $P_1^2 = S_{10}^2 + S_{20}^2 + S_{30}^2$ and $P_2^2 = S_{01}^2 + S_{02}^2 + S_{03}^2$, while the final term refers only to the two-photon composite system

[16]. Again, the state purity can be simply related to the Stokes scalar $S_{(2)}^2$:

$$\text{Tr}(\rho^2) = \bar{P}^2 + S_{(2)}^2, \quad (10)$$

where $\bar{P}^2 \equiv \frac{1}{2}(P_1^2 + P_2^2)$ is the average of the squares of the single-photon polarizations. More important, however, is the fact that $S_{(2)}^2$ can be seen to be a measure of entanglement for two-photon pure states. In fact, Eq. (9) and some algebra show that in the case of pure states this scalar coincides with the concurrence squared, expressed in terms of two-photon Stokes parameters (cf. Ref. [15])—that is, the tangle τ [17, 18]:

$$\tau = S_{(2)}^2, \quad (11)$$

so the entanglement of formation is seen to be $h(\frac{1}{2}[1 + \sqrt{1 - S_{(2)}^2}])$, where $h(x) \equiv -x \log_2 x - (1-x) \log_2 (1-x)$. However, the valuable property of $S_{(2)}^2$, beyond its being equal to the tangle for two-photon pure states, appears in its application to *mixed* states, which we consider next, where it is not equal to the square of the concurrence.

Consider now the class of mixed states that describe two photons of a three-photon system in a pure quantum state. For two-photon mixed states, $S_{(2)}^2$ is different from the tangle. For this important class of states, we find $S_{(2)}^2$ to be a specific sum of entanglement measures over the pertinent subsystem and the larger, three-particle system. To see this, recall that any three-photon state can be written as

$$|\Psi\rangle = \sum_{ijk} a_{ijk} |i\rangle_A |j\rangle_B |k\rangle_C. \quad (12)$$

Examining the relationship of the entanglement of photon A with a pair of photons B and C , we have from Eq. (5)

$$S_{AB}^2 + S_{AC}^2 = C_{A(BC)}^2, \quad (13)$$

where $C_{A(BC)}^2$ is the concurrence calculated for a bipartite decomposition of ABC into subsystem A and (composite) subsystem BC [17], and where S_{AB}^2 and S_{AC}^2 are the values of $S_{(2)}^2$ for the two-photon subsystems AB and AC . Furthermore, since $C_{A(BC)}^2 = C_{AB}^2 + C_{AC}^2 + \tau_{ABC}$, where τ_{ABC} is a three-particle entanglement measure (the ‘‘three tangle’’) [17,19], we have

$$S_{AB}^2 + S_{AC}^2 = C_{AB}^2 + C_{AC}^2 + \tau_{ABC}. \quad (14)$$

Thus, we have

$$\tau_{ABC} = (S_{AB}^2 - C_{AB}^2) + (S_{AC}^2 - C_{AC}^2), \quad (15)$$

which shows that the sum of the $S_{(2)}^2$ values for the two-photon subsystems AB and AC captures the contribution to the total three-particle state entanglement encoded in these two-particle subsystems, as well as their own internal two-particle entanglements as measured by the concurrence. Similar expressions are obtained when one begins with the

other two bipartite decompositions of ABC . By jointly considering the resulting expressions, one finds that

$$S_{AB}^2 = C_{AB}^2 + \tau_{ABC}/2, \quad (16)$$

and similarly for S_{AC}^2 and S_{BC}^2 .

We see from Eq. (16) that $S_{(2)}^2$ captures its contribution to the three-particle entanglement as well as the two-particle entanglement of the corresponding subsystem. Equation (16) shows it to be an entanglement measure for three-photon pure states, which includes entanglement not present in the concurrence of its two-particle subsystems. We also see from Eq. (15) that, though the photon-pair contributions to the total entanglement may differ as a result of their own internal entanglements, each of the pairings AB and AC can be viewed as also containing the three tangle of the three-photon state $|\Psi\rangle$. This result is analogous to what one finds for the entanglement of single-qubit subsystems of a two-qubit system in a pure state, where the reduced states of the two subsystems encode the tangle of the overall system.

III. FILTERING OPERATIONS

Recall that, in order to be properly interpreted physically after filtering, the n -photon Stokes parameters $\{S_{i_1 \dots i_n}\}$ must be renormalized, $S'_{i_1 \dots i_n} \rightarrow S''_{i_1 \dots i_n}/S_{0 \dots 0}$. After a local filtering operation, one has $S'^2_{(n)} = S''^2_{(n)}/S_{0 \dots 0}^2$. The value of $S'^2_{(2)}$ thus monotonically increases with filtering. For the case $n=2$, this has a clear meaning in terms of entanglement. In that case, $S'_{\mu\nu} \rightarrow S''_{\mu\nu}$: after a filtering operation, one has $S'^2_{(2)} = S_{(2)}^2/S_{00}^2$. The invariance of the scalar under $O_0(1,3)$ transformations $S_{\mu\nu} \rightarrow S'_{\mu\nu}$, that is $S'^2_{(2)} = S_{(2)}^2$, means that the effect of these transformations on the invariant is entirely captured by the total attenuation. The filtering operation thus results in an increase of entanglement, since $S'_{00} < 1$, by virtue of Eqs. (11) and (16). Such local operations can be implemented, for example, by dichroic optical fiber, where they are associated with polarization-dependent losses (see, for example, Ref. [20]).

The Stokes scalar $S_{(2)}^2$ allows us to identify and quantify the beneficial effect of the SLOCC filtering operations on entanglement for the classes considered above. Thus, we see how the attenuating transformations [$O_0(1,3)$ / $SO(3)$], together with the quantum ensemble renormalization they engender, correspond to entanglement purifying transformations (see also Refs. [9,10] and, in particular, Ref. [11]). These operations affect the invariant $S_{(n)}^2$ in exactly the same way for general values of n .

IV. CONCLUSION

In summary, we have introduced a group-invariant Stokes scalar for studying n -photon entangled polarization states. In the case of two-photon pure states, this invariant is equal to the tangle. In the case of three-photon pure states, it mea-

sures entanglement in the total system through its photon-pair subsystems, which are in general described by mixed states. These results allow us to identify sets of optical elements that give rise to polarization-dependent filtering, such as dichroic optical fibers, as entanglement purifiers, and to quantify their entanglement-increasing effect on two-photon and three-photon states. Such local transformations have a similar effect on the invariant in the case of photon numbers n greater than 3.

Because it satisfies the necessary condition of being invariant under local unitary transformations for general values of n , and since it has a clear connection to the accepted entanglement measures of tangle and three tangle characterizing few-photon states, this invariant can be considered a

good candidate to measure entanglement for pure states of n photons. In addition, it can be directly estimated, at least in principle, via a suitable quantum network. Finally, unlike the proposed n -tangle measure [19] for uniquely n -particle entanglement, which is ill-defined for odd values of n , this invariant is well defined for all finite values of n .

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