Interevent-time statistics for shot-noise-driven self-exciting point processes in photon detection

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Probability densities for interevent time are obtained for a doubly stochastic Poisson point process (DSPP) in the presence of self-excitation. The DSPP is assumed to have a stochastic rate that is a filtered Poisson point process (shot noise). The model of a Poisson process driving another Poisson process produces a pulse-bunching effect. Self-excitation (relative refractoriness) results in a deficit of short time intervals. Both effects are observed in many applications of optical detection. The model is applicable to the detection of fluorescence or scintillation generated by ionizing radiation in a photomultiplier tube. It is also used successfully to fit the maintained discharge interspike-interval histograms recorded by Barlow, Levick, and Yoon [Vision Res. 11, Suppl. 3, 87–101 (1971)] for a cat's on-center retinal ganglion cell in darkness.

1. INTRODUCTION

The Poisson point process and its variations are useful tools for studying discrete phenomena in many scientific fields. One generalization of the simplest case (the homogeneous Poisson point process, HPP) that has been studied extensively is the doubly stochastic Poisson point process (DSPP).[1] This was first examined by Cox,[2] and the designation DSPP was introduced to emphasize that two kinds of randomness take place: randomness associated with the Poisson point process itself and an independent randomness associated with its rate. Much of the recent development of the properties of the DSPP has been in the context of optics.[1,3]

A special case of the DSPP obtains when the stochastic rate is shot noise,[4,5] and this is designated as a shot-noise-driven doubly stochastic Poisson point process (SNDP). Bartlett has shown that this latter process is a particular Neyman–Scott cluster process.[4] We have recently studied this process in detail, obtaining the single and multifold counting and time statistics.[6] The results have been applied to problems in several areas, including scintillation detection, cathodoluminescence, and radiation-induced noise in photomultiplier tubes.[6]

Another useful modification of the HPP is the self-exciting Poisson point process, which permits aftereffects that are triggered by past events.[1,7] In their most general form, the future evolution of self-exciting point processes depends on the occurrence times of all past events as well as on their total number. A special but useful case occurs when the process has limited memory; in particular, the interevent times of a homogeneous self-exciting Poisson point process with a memory that reaches back exactly one pulse form a sequence of statistically independent random variables (renewal process).[1] An extensive body of literature exists on dead-time-modified counters as examples of self-exciting and renewal point processes.[1,8–13] In particular, the interevent-time statistics for a homogeneous one-memory renewal process with gradual recovery have been obtained for a number of cases of interest in visual information processing.[13]

In this paper we combine the above-described modifications of the HPP in one mathematical model and obtain the interevent-time statistics for a doubly stochastic self-exciting Poisson point process. More specifically, we treat the SNDP with one-memory recovery; this is a generalized renewal process. In Section 2 we present an expression for the probability density of the time interval between two consecutive events. This is specialized in Section 3 to the case of exponentially decaying shot-noise pulses and sudden recovery, and the results are related to the detection of ionizing radiation by a scintillation detector affected by fixed dead time. The case of gradual exponential recovery is treated in Section 4, in which we also provide a parametric study of the interevent-time probability density. In Section 5, the theoretical results of Section 4 are applied to the experimental data of Barlow, Levick, amd Yoon[14] for the interspike time-interval histogram recorded from the cat's on-center retinal ganglion cell in darkness.

2. INTEREVENT-TIME STATISTICS

As indicated above, the SNDP is a doubly stochastic Poisson point process whose stochastic rate is shot noise. Self-excitation may be viewed as a feedback process that, whenever an event occurs, modifies the rate of events by a specified time

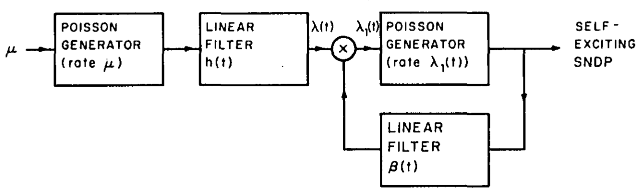

Fig. 1. Schematic representation of the shot-noise-driven doubly stochastic Poisson point process (SNDP) with self-excitation.

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function (e.g., recovery) following the occurrence of that event. The process studied here is assumed to be one-memory, so that the occurrence of an event always resets this function. A schematic representation is provided in Fig. 1.

A shot-noise process $\lambda(t)$, obtained by passing an HPP of rate μ through a linear filter with a nonnegative impulse-response function $h(t)$, provides the stochastic rate that drives a second Poisson point process. This second process is one-memory self-exciting, so that, whenever an event occurs at time t_i, the rate of the process is multiplied by a function $\beta(t - t_i)$ for all $t > t_i$. If the function $\beta(t)$ increases from 0 to 1 in a time that is short in comparison with the average interevent time, the effect of a pulse occurrence on subsequent occurrences is not long lasting, and the one-memory restriction is not critical. An important special case is that in which

$$\begin{aligned} \beta(t) &= 0 \qquad t < \tau_d \\ &= 1 \qquad t > \tau_d. \end{aligned} \tag{1}$$

It corresponds to a nonparalyzable dead time τ_d that follows every event and during which no other event can occur.

In the following we shall determine the probability density for the time between two consecutive events, $P(\tau)$, for the SNDP with self-excitation.

For a DSPP of stochastic rate $\lambda(t)$ (in the absence of self-excitation), $P(\tau)$ is given by[15,3]

$$P(\tau) = \left\langle \lambda(0)\lambda(\tau)\exp\left[-\int_0^\tau \lambda(t)\mathrm{d}t\right]\right\rangle \Big/ \langle \lambda(0)\rangle. \tag{2}$$

The numerator represents the joint probability density of the occurrence of an event at $t = 0$, an event at $t = \tau$, and no events between. The denominator represents the probability density of an event occurring at $t = 0$. The operation $\langle \cdot \rangle$ indicates an ensemble average over the stochastic rate $\lambda(t)$.

In the presence of self-excitation described by the function $\beta(t)$, Eq. (2) must be modified by replacing $\lambda(t)$ by $\lambda(t)\beta(t)$ (see Fig. 1). Thus

$$P(\tau) = \left\langle \lambda(0)\lambda(\tau)\beta(\tau)\exp\left[-\int_0^\tau \lambda(t)\beta(t)\mathrm{d}t\right]\right\rangle \Big/ \langle \lambda(0)\rangle, \tag{3}$$

where the numerator and the denominator have the same meaning as in Eq. (2). Now it remains to evaluate Eq. (3) when $\lambda(t)$ is a shot-noise process.

Let us define two additional shot-noise processes,

$$\lambda_1(t) = \lambda(t)\beta(t) \tag{4}$$

and

$$\lambda_2(t) = \int_0^t \lambda(t')\beta(t')\mathrm{d}t'. \tag{5}$$

The three shot-noise processes $\lambda(t)$, $\lambda_1(t)$, and $\lambda_2(t)$ are obtained from the *same* HPP of rate μ by the use of different filters; these are, respectively, a time-invariant linear filter $h(t)$, a time-variant linear filter of impulse response

$$h_1(t,t') = \beta(t)h(t - t'), \tag{6}$$

and a time-variant linear filter of impulse response

$$h_2(t,t') = \int_0^t \beta(x)h(x - t')\mathrm{d}x. \tag{7}$$

We can now write Eq. (3) in the form

$$P(\tau) = \langle \lambda(0)\lambda_1(\tau)\exp[-\lambda_2(\tau)]\rangle/\langle \lambda(0)\rangle. \tag{8}$$

By introducing the moment-generating function (mfg)

$$Q(s,s_1,s_2) = \langle \exp[-s\lambda(0) - s_1\lambda_1(\tau) - s_2\lambda_2(\tau)]\rangle, \tag{9}$$

we can see that

$$P(\tau) = \frac{1}{\langle \lambda(0)\rangle}\left[\frac{\partial}{\partial s}\frac{\partial}{\partial s_1} Q(s,s_1,s_2)\bigg|_{s=s_1=0,s_2=1}\right]. \tag{10}$$

But the mgf of shot noise is known to be[16]

$$Q(s,s_1,s_2) = \exp\left(\mu \int_{-\infty}^{\infty} \{\exp[-sh(-t) - s_1h_1(\tau,t) - s_2h_2(\tau,t)] - 1\}\mathrm{d}t\right), \tag{11}$$

so that it remains to take the derivatives in Eq. (10) and obtain an expression for $P(\tau)$. By the use of Eqs. (6), (7), (10), and (11) and the fact that

$$\langle \lambda(0)\rangle = \mu\alpha, \tag{12}$$

where the multiplication parameter α is represented by the area under the impulse function response

$$\alpha = \int_0^\infty h(t)\mathrm{d}t, \tag{13}$$

we obtain

$$\begin{aligned} P(\tau) = \frac{1}{\alpha}\beta(\tau)\bigg[\int_0^\infty h(t)h(\tau + t)\phi(\tau,t)\mathrm{d}t \\ + \mu\int_0^\infty h(t)\phi(\tau,t)\mathrm{d}t \int_{-\infty}^{\infty} h(\tau + t)\phi(\tau,t)\mathrm{d}t\bigg] \\ \times \exp\left\{\mu\int_{-\infty}^{\infty} [\phi(\tau,t) - 1]\mathrm{d}t\right\}, \end{aligned} \tag{14}$$

with

$$\phi(\tau,t) = \exp[-h_2(\tau, -t)] = \exp\left\{-\int_0^\tau \beta(x)h(x + t)\mathrm{d}x\right\}. \tag{15}$$

This is an explicit formula for $P(\tau)$ as a function of μ, $h(t)$, and $\beta(t)$.

We note that $P(\tau)$ contains a multiplicative factor $\beta(\tau)$, so that, in the region in which $\beta(\tau)$ is zero, $P(\tau)$ must also be zero. This is, of course, to be expected. We note also that, for $\beta(\tau) = 1$, the expression for $P(\tau)$ reduces to that previously derived for the (non-self-exciting) SNDP.[6]

3. INTEREVENT-TIME STATISTICS FOR EXPONENTIAL SHOT-NOISE PULSES AND FIXED-DEAD-TIME RECOVERY

We consider an example in which the primary Poisson pulses are converted to exponentially decaying shot-noise pulses, which in turn drive the second Poisson generator (see Fig. 1). In this case, the linear-filter impulse-response function $h(t)$ is described by

$$\begin{aligned} h(t) &= (\alpha/\tau_p)\exp(-t/\tau_p) \qquad t > 0 \\ &= 0 \qquad t < 0, \end{aligned} \tag{16}$$

so that τ_p is the lifetime of the primary shot-noise pulses. The system is assumed to recovery instantaneously *after* a fixed dead-time period τ_d so that Eq. (1) for $\beta(t)$ is satisfied.

This model can be used to describe the interevent-time density function for optical fluorescence or scintillation[17] generated by ionizing radiation in a photomultiplier tube (in conjunction with its associated electronics and overall system dead time τ_d). Conditions for the validity of the model are that the incident primary ionizing particles (e.g., gamma rays) be represented as an HPP, that each event produce an exponentially decaying impulse-response function[18] that governs the rate of production of Poisson optical photons, and that the system have a rectangular recovery with fixed dead time.

In certain applications in which we observe the interevent statistics of an optical signal by using a photomultiplier tube, e.g., in high-altitude astronomy or in space, the distributions discussed above may be characteristic of the noise rather than of the signal. Viehmann and Eubanks[19,20] have discussed sources of noise in photomultiplier tubes in the radiation environment of space. Pulse-counting distributions when dead time is negligible have been discussed previously.[6,21]

The above-described self-exciting point process is characterized completely by four parameters: μ, the rate of the primary point process; α, the multiplication parameter; τ_d, the dead time; and τ_p, the lifetime of the primary shot-noise pulses. In order to understand the dependence of the density $P(\tau)$ on these parameters, we introduce the parameter

$$\langle\tau\rangle = 1/\mu\alpha, \tag{17}$$

which represents the mean interevent time in the absence of self-excitation. There is also no loss of generality in assuming that

$$\langle\tau\rangle = 1, \tag{18}$$

in which case τ_d, τ_p, and τ are measured in units of $\langle\tau\rangle$. We now obtain the dependence of $P(\tau)$ on τ_d, τ_p, and α. Substituting Eqs. (1), (16), and (17) into Eqs. (14) and (15) leads to an expression of the form

$$P(\tau) = \beta(\tau)[C(\tau) + D(\tau)E(\tau)]\exp[F(\tau)], \tag{19}$$

where

$$\begin{aligned}\beta(\tau) &= 0 \qquad \tau < \tau_d\\ &= 1 \qquad \tau > \tau_d,\end{aligned} \tag{20}$$

$$C(\tau) = \frac{\alpha}{\tau_p} r e^{-\tau'} \sum_{k=0}^{\infty} (-1)^{k+1}\alpha^k r^k (1 - e^{-\tau'})^k / k!(k+2), \tag{21}$$

$$D(\tau) = \exp[\alpha r(1 - e^{-\tau'})]/[\alpha r(1 - e^{-\tau'})], \tag{22}$$

$$E(\tau) = \frac{1}{\alpha}\exp(-\alpha e^{-\tau'}) - \exp[\alpha(1 - e^{-\tau'})]/[\alpha(1 - e^{-\tau'})], \tag{23}$$

$$F(\tau) = \frac{\tau_p}{\alpha}\left[(1 - e^{-\alpha})\tau' + (1 + e^{-\alpha}) \sum_{k=1}^{\infty} \frac{\alpha^k}{k!k}(1 - e^{-k\tau'})\right], \tag{24}$$

$$\tau' = (\tau - \tau_d)/\tau_p, \tag{25}$$

$$r = \exp(-\tau_d/\tau_p). \tag{26}$$

4. INTEREVENT-TIME STATISTICS FOR EXPONENTIAL SHOT-NOISE PULSES AND GRADUAL EXPONENTIAL RECOVERY

We again consider an example in which the shot-noise pulses are time-decaying exponentials as described by Eq. (16), but we now choose the recovery function $\beta(t)$ to be exponentially increasing in accordance with

$$\begin{aligned}\beta(t) &= 1 - \exp(-t/\tau_d) \qquad t > 0\\ &= 0 \qquad\qquad\qquad\qquad t < 0.\end{aligned} \tag{27}$$

The resulting interevent-time probability density is again completely characterized by the four parameters τ_d, τ_p, $\langle\tau\rangle$, and α, but τ_d now represents the recovery (or sick) time rather than the dead time as in the previous section. Substituting Eqs. (16), (17), and (27) into Eqs. (14) and (15) leads again to an expression of the form

$$P(\tau) = \beta(\tau)[C(\tau) + D(\tau)E(\tau)]\exp[F(\tau)], \tag{28}$$

but where now

$$\beta(\tau) = 1 - \exp(-\tau/\tau_d), \tag{29}$$

and

$$C(\tau) = \frac{\alpha}{\tau_p} e^{-\tau/\tau_p} \sum_{k=0}^{\infty} \frac{(-1)^k[A(\tau)]^k}{k!(k+2)}, \tag{30}$$

$$D(\tau) = [1 - e^{-A(\tau)}]/A(\tau), \tag{31}$$

$$E(\tau) = D(\tau)e^{-\tau/\tau_p} + \frac{e^{-\alpha}}{\tau_p} e^{-\tau/\tau_p} \int_0^{\tau} \exp\left[\frac{t}{\tau_p} + G(t)\right] dt, \tag{32}$$

$$F(\tau) = \frac{\tau_p}{\alpha} \sum_{k=1}^{\infty} \frac{(-1)^k[A(\tau)]^k}{k!k} - \frac{\tau}{\alpha} + \frac{e^{-\alpha}}{\alpha} \int_0^{\tau} \exp[G(t)] dt, \tag{33}$$

$$G(t) = \frac{\alpha\tau_p}{\tau_p + \tau_d} e^{-t/\tau_d} + \left[\frac{\alpha\tau_p}{\tau_p + \tau_d} - A(\tau)\right] e^{t/\tau_p}, \tag{34}$$

$$A(\tau) = \alpha\left\{\frac{\tau_d}{\tau_p + \tau_d}\left[e^{-\tau(\tau_p+\tau_d)/\tau_p\tau_d} - 1\right] + 1 - e^{-\tau/\tau_p}\right\}. \tag{35}$$

In Fig. 2 we provide a graphical parametric study of the normalized probability density $\langle\tau\rangle P(\tau)$ versus the normalized interevent time $\tau/\langle\tau\rangle$. Three curves are presented in each of Figs. 2(a)–(c) with α fixed at values 0.1, 2, and 10, respectively, and τ_p fixed at 1 in all cases; the normalized recovery time τ_d varies in each figure between 0.01 and 0.25. By examining Figs. 2(a), 2(b), and 2(c), it is evident that increasing τ_d decreases the initial slope and the peak height of the density function, thereby providing for a slower decay of the tail. The effect of an increase in the multiplication parameter α is evident in the progression of the shape of the densities from Fig. 2(a) to Fig. 2(c). The initial slope and peak height increase with increasing α, and the deviation from exponential form (which is a straight line on this logarithmic plot) becomes more pronounced. (For sufficiently large τ, however, all the probability densities have an exponential tail.)

The effect of the lifetime of the primary shot-noise pulses, τ_p, is illustrated in Fig. 2(d), in which α and τ_d are fixed at 2 and 0.1, respectively, and the normalized lifetime τ_p varies between 0.1 and 1. An increase in τ_p results in a decrease of the peak height. With other parameters fixed, a smaller value of τ_p corresponds to a sharper deviation from exponential,

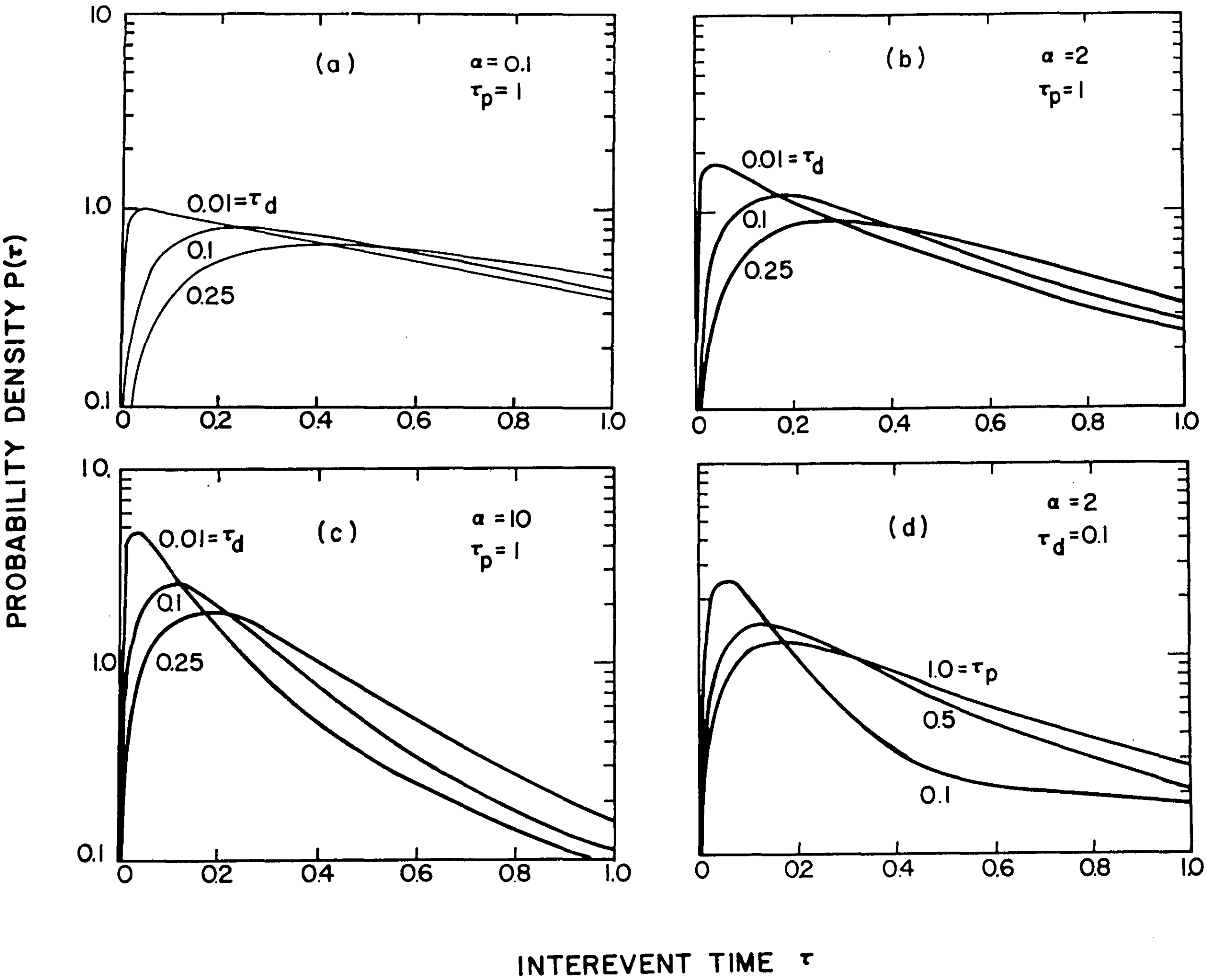

Fig. 2. (a) Interevent-time (pulse-interval) probability density functions for an SNDP with self-excitation. The impulse response of the shot-noise filter is exponential with time constant $\tau_p = 1$ and the multiplication parameter $\alpha = 0.1$. The self-excitation recovery function is exponential with time constants $\tau_d = 0.01$, 0.1, and 0.25, as indicated. The mean interevent time in the absence of self-excitation is $\langle \tau \rangle = 1$. (b) As in (a) with $\alpha = 2$. (c) As in (a) and (b) with $\alpha = 10$. (d) As in (a)–(c) with $\alpha = 2$, $\tau_d = 0.1$, and $\tau_p = 0.1$, 0.5, and 1.0, as indicated.

representing enhanced clustering or bunching of events.

It is evident that there is substantial interplay of the various parameters in forging the ultimate shape of the interevent-time density function. (It must be kept in mind that all the curves presented in Fig. 2 have a mean interevent time in the absence of self-excitation that is fixed at $\langle \tau \rangle = 1$ so that, e.g., an increase of α means a decrease of μ.) Nevertheless, the parametric study presented here can provide guidance in choosing parameters to fit a particular set of data; one example is provided in the following section, in which we fit an experimental interspike time-interval histogram for the neural discharge in a cat's retinal ganglion cell.

5. APPLICATION OF INTEREVENT-TIME STATISTICS TO THE MAINTAINED DISCHARGE IN THE CAT RETINAL GANGLION CELL

In 1971, Barlow, Levick, and Yoon[14] carried out a fascinating series of experiments in which they obtained the pulse-number distributions for several dark-adapted on-center cat retinal ganglion cells. For one on-center brisk-sustained unit (BLF-1), a histogram for the time between successive spikes was also recorded in darkness. They concluded from their study that single absorbed quanta can cause multiple impulses at the ganglion cell, and that dark-light events behave like quantal absorptions in this respect. Indeed, they point out that the statistical properties of the maintained discharge are similar in darkness and for light-evoked responses, and they indicate that the discharge behaves as though it results from random unitary events in the receptors, each causing several impulses. They also observe that the deficit of short time intervals near $\tau = 0$ may be the result of relative refractoriness.

Based on this work, we have constructed a model for the generation of the point-process neural discharge in an on-center retinal ganglion cell (see Fig. 3). Poisson photons, or dark light (rate μ), excite a complex network of rods, bipolars, and other cells, which we brashly represent in terms of a linear-filter impulse-response function $h(t)$.[22] This produces a shot-noise process denoted by $\lambda(t)$ that, after a modification to be discussed below, provides the time-varying driving rate for the ganglion cell. Our model presupposes that the ganglion cell would, in the absence of all fluctuations of $\lambda_1(t)$,

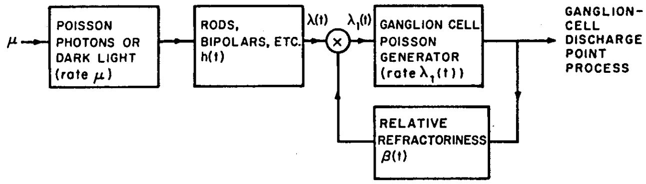

Fig. 3. Model for the generation of the point-process neural discharge in the on-center retinal ganglion cell. Observe the relation to Fig. 1, which is the mathematical model studied here.

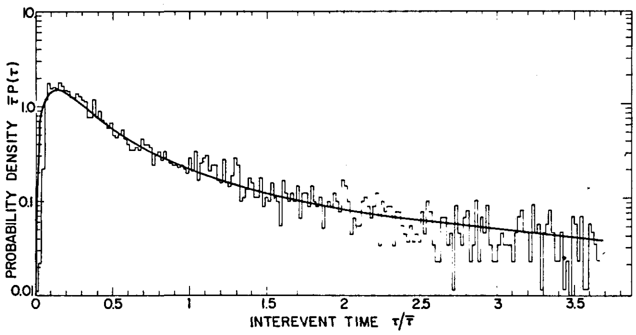

Fig. 4. Maintained discharge interspike-interval histogram recorded by Barlow, Levick, and Yoon[14] for dark-adapted on-center cat retinal ganglion cell in darkness (unit BLF-1). The mean time interval is 54.25 msec. Scales are normalized such that $\overline{\tau} = 1$. Theoretical fit (solid curve) is based on an exponential shot-noise filter impulse-response function with normalized time constant $\tau_p/\overline{\tau} = 0.5$ ($\tau_p \sim 27$ msec), an exponential recovery function with normalized time constant $\tau_d/\overline{\tau} = 0.1$ ($\tau_d \sim 5.4$ msec), and a multiplication parameter $\alpha = 2$.

produce a pure Poisson discharge. The modification alluded to above is relative refractoriness that, when a neural spike is generated, depresses the excitability of the cell. The recovery of the neuron is, again, represented in terms of a linear-filter impulse-response function $\beta(t)$ that multiplies the rate $\lambda(t)$ produced by the network before the ganglion cell. The output of the system represents the neural-discharge point process.

If we compare Figs. 1 and 3, we see immediately that there is an identity between them, and this is, of course, no accident. We posit that the ganglion–cell discharge point process be modeled as a self-exciting one-memory SNDP. We now proceed to relate theory and experiment as well as we can with the limited data available. The interevent-time histogram is used as the point of contact.

The maintained-discharge interspike-interval histogram for unit BLF-1 is presented in Fig. 4 with a bin width of 1 msec. Figure 4 differs slightly from Fig. 5(A) in the paper by Barlow, Levick, and Yoon,[14] in which pairs of adjacent channels were combined before plotting, so that the bin width there is 2 msec.[23] The two presentations also differ (insubstantially) in that the histogram in Fig. 4 has been normalized to the experimental mean time interval $\overline{\tau} = 54.25$ msec. This value will differ only slightly from the mean interevent time in the absence of self-excitation.

The theoretical interevent-time density function for exponential shot-noise pulses and gradual exponential recovery expressed in Eqs. (28)–(35) is graphed as the solid curve in Fig. 4. With the benefit of the parametric study conducted in Section 4, we were able to obtain a good fit with relative ease. This has enabled us to extract values for the three normalized parameters: $\tau_d/\overline{\tau} = 0.1$, $\tau_p/\overline{\tau} = 0.5$, and $\alpha = 2$. Removing the normalization, we determine that, for unit BLF-1 in darkness, the recovery time τ_d is ~5.4 msec, the shot-noise decay time τ_p is ~27 msec, and the multiplication parameter α is ~2. All these values are reasonable. (It is, perhaps, worth mentioning that the maintained discharge recorded from the lobula complex of the blowfly exhibits very similar behavior.[24])

There is a point of connection with the pulse-number distributions analyzed by Barlow, Levick, and Yoon[14] We have previously shown that, when the counting time is much greater than the decay time of the shot-noise filter, the pulse-number distribution for an SNDP without self-excitation (refractoriness) reduces to the Neyman Type-A distribution[6,21] for arbitrary $h(t)$. The form of the mean and variance for the Neyman Type-A is precisely that used by Barlow, Levick, and Yoon[14] in one of their analyses of the pulse-number data.

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REFERENCES

1. D. L. Snyder, *Random Point Processes* (Wiley-Interscience, New York, 1975).
2. D. R. Cox, "Some statistical methods connected with series of events," J. R. Statist. Soc. B **17**, 129–164 (1955).
3. A relatively recent book in the area is B. E. A. Saleh, *Photoelectron Statistics* (Springer-Verlag, New York, 1978).
4. M. S. Bartlett, "The spectral analysis of two-dimensional point processes," Biometrika **51**, 299–311 (1964).
5. A. J. Lawrance, "Some models for stationary series of univariate events," in *Stochastic Point Processes: Statistical Analysis, Theory, and Applications*, P. A. W. Lewis, ed. (Wiley-Interscience, New York, 1972).
6. B. E. A. Saleh and M. C. Teich, "Multiplied Poisson noise in pulse, particle, and photon detection," submitted to Proc. IEEE.
7. Synder has shown that a doubly stochastic Poisson point process may be expressed in terms of an equivalent self-exciting point process (see Ref. 1, pp. 292–293).
8. W. L. Smith, "Renewal theory and its ramifications," J. R. Statist. Soc. B **20**, 243–302 (1958).
9. L. Takács, "On a probability problem in the theory of counters," Ann. Math. Stat. **29**, 1257–1263 (1958).
10. E. Parzen, *Stochastic Processes* (Holden-Day, San Francisco, 1962).
11. J. W. Müller, ed., "Bibliography on dead time effects," Rep. no. BIPM-75/6 (Bureau International des Poids at Mesures, Sèvres, France, 1975).
12. M. C. Teich, L. Matin, and B. I. Cantor, "Refractoriness in the maintained discharge of the cat's retinal ganglion cell," J. Opt. Soc. Am. **68**, 386–402 (1978).
13. M. C. Teich and P. Diament, "Relative refractoriness in visual information processing," Biol. Cybernet. **38**, 187–191 (1980).
14. H. B. Barlow, W. R. Levick, and M. Yoon, "Responses to single quanta of light in retinal ganglion cells of the cat," Vision Res. **11**, Suppl. 3, 87–101 (1971).
15. D. R. Cox and P. A. W. Lewis, *The Statistical Analysis of Series of Events* (Methuen, London, 1968).
16. A. Papoulis, *Probability, Random Variables, and Stochastic Processes* (McGraw-Hill, New York, 1965).
17. J. B. Birks, *The Theory and Practice of Scintillation Counting*

(Pergamon, New York, 1964).

18. The rate function is sometimes more precisely described as a sum of two time-decaying exponential functions, a strong fast component plus a much weaker slow component. See G. Walter and A. Coche, "Mesure de la composante lente de l'émission de lumière dans les scintillateurs organiques: application à la discrimination des particules selon leur pouvoir ionisant," Nucl. Instrum. Methods **23**, 147–151 (1963).
19. W. Viehmann and A. G. Eubanks, "Noise limitations of multiplier phototubes in the radiation environment of space," NASA Tech. Note D-8147 (Goddard Space Flight Center, Greenbelt, Maryland, 1976).
20. W. Viehmann, A. G. Eubanks, G. F. Piper, and J. H. Bredekamp, "Photomultiplier window materials under electron irradiation: fluorescence and phosphorescence," Appl. Opt. **14**, 2104–2115 (1975).
21. M. C. Teich, "Role of the doubly stochastic Neyman Type-A and Thomas counting distributions in photon detection," Appl. Opt. (to be published).
22. W. R. Levick and J. L. Zacks, "Responses of cat retinal ganglion cells to brief flashes of light," J. Physiol. London **206**, 677–700 (1970).
23. W. R. Levick, John Curtin School of Medical Research, Australian National University, Canberra City, Australia, personal communication.
24. H. A. K. Mastebroek, W. H. Zaagman, and J. W. Kuiper, "Intensity and structure of visually evoked neural activity: rivals in modeling a visual system," Vision Res. **17**, 29–35 (1977).