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Photoelectron-Counting Distributions for Irradiance-Modulated Radiation*

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Photoelectron-counting distributions are obtained for modulated radiation sources, with arbitrary modulation waveforms and depths. Cases treated in detail include chaotic, amplitude-stabilized, and Risken sources, with square-wave, triangular, and sinusoidal modulation. Modulation is shown to broaden the counting distributions, an effect interpretable as accentuated photon bunching. The broadening is not marked for the chaotic source, but the shape of the distribution changes drastically for the amplitude-stabilized source. Varying the radiation statistics and the modulation waveforms can produce a variety of counting distributions, from double peaked to extremely flat. Modulation may serve to accentuate the distinctions between counting distributions associated with sources of different radiation statistics.
INDEX HEADINGS: Modulation of beams; Detection.

The relation between photoelectron-counting statistics and fluctuations of irradiance has been developed to a high degree of sophistication.[1] There has been little work, however, on the counting distributions to be expected from radiation that is modulated, as well as stochastically fluctuating. An investigation of radiation of such general time variation is fundamental to optical communications, particularly at low photon levels or short time scales, when the transmitted information is impressed as an intensity modulation and detectable only as an altered photoelectron-counting distribution. In this paper, we obtain the counting statistics associated with radiation of a variety of statistical types when modulated with a general waveform. The modulation may be either a deterministic function or random noise with a given probability density function. If the noise modulation arises from the interaction with a medium, then a good deal of information may be obtained about the medium from the counting statistics, as shown, for example, by Arecchi *et al.*[2] Although deterministic modulation is emphasized in this paper, the general results presented in Sec. I are valid when the modulation is inherently stochastic. Thus, such disturbances as atmospheric intensity modulation of a laser or a chaotic source can be treated by the general method.

We treat in detail the cases of square-wave, sinusoidal, and triangular modulation, for radiation sources of the chaotic, amplitude-stabilized, and nonlinear oscillator (Risken) types. Thus, analytical expressions are included for counting statistics in nine specific cases, covering the radiation statistics and modulation waveforms that are likely to be of interest, each considered for arbitrary modulation depth.

Previous work has been concerned primarily with the modulated amplitude-stabilized source,[3,4] which is usually a single-mode laser. Small-depth square-wave and sinusoidal modulation of such a source has been considered by Fray *et al.*,[5] and Pearl and Troup[6] have investigated the 100% sinusoidally modulated case. Results for the triangularly modulated case for the same underlying Poisson process, with arbitrary modulation depth, were shown by Teich and Diament[7] to possess the unusual property of extreme flatness over a tunable region of counts. This property was further demonstrated to hold for any Poisson process with a linearly swept mean, and has been experimentally confirmed for γ-ray counting.[8]

I. THEORY

We consider quasimonochromatic, linearly polarized radiation illuminating a photoelectric detector. The irradiance fluctuates about some mean value with a statistical distribution, and with a coherence time, characteristic of the radiation source and the intervening medium. The probability of counting n photoelectrons in an observation time interval from t to $t+T_0$ is given by the well-established Mandel formula[9]

$$p(n,T_0,t)=\langle M^n e^{-M}/n!\rangle, \tag{1}$$

where

$$M=\int_t^{t+T_0} \alpha I(t')dt'. \tag{2}$$

This expresses an ensemble average, over the irradiance-fluctuation statistics, of the Poisson process of photoemission, and is independent of t for the stationary, ergodic sources considered here. The factor α is the quantum efficiency of the detector, including any dependence on the photosensitive area, which is small compared to the coherence area.

We now investigate the effects on the photoelectron-counting statistics of an imposed modulation of the effective irradiance level of the radiation. The stochastic fluctuations, on a time scale of the order of the coherence time τ of the radiation, are superimposed on some variation of their mean, which is imparted on the time scale of the modulation repetition period T_1. The two processes are independent, and it is assumed that modulation of the irradiance does not affect the statistical nature of the source. Under these conditions, variations imposed on the instantaneous irradiance are equivalent to modulation of its mean. The Mandel formula remains operative for the determination of the counting distribution, when reinterpreted to include an ensemble average not only over the stochastic fluctuations of the irradiance, but over its secular variation as well. The randomness of the latter stems from that of the time of observation, t, which is taken to be uniformly distributed and uncorrelated with the modulation waveform.[6] The results given in Sec. II pertain specifically to observation intervals T_0, short compared to both the coherence time τ and the modulation period T_1. Note that τ and T_1 need not be related. Verification of the expected distributions would, however, require a series of many observations, lasting longer than both τ and T_1.

The radiation exhibits statistical fluctuations with some irradiance probability distribution $P(I)$, whose mean I_m is caused to undergo excursions between two levels, I_a and I_b, with any modulation waveform. The modulation depth, $m\equiv(I_b-I_a)/(I_b+I_a)$, is arbitrary. Since the statistical fluctuations and the modulation waveform are independent, the ensemble average required by Mandel's formula can be effected by successive averaging over the statistics of the two processes. The final probability of observing n photoelectrons during an interval T_0 is, therefore,

$$p(n,m,T_0)=\int_{I_a}^{I_b} p(n,T)P_m(I_m)dI_m. \tag{3}$$

Here, $p(n,T)$ is the corresponding counting distribution in the absence of modulation and depends on the mean irradiance I_m, the integration variable in Eq. (3). This mean level I_m becomes a random variable by virtue of the randomness of the uniformly distributed time of observation t. Hence,

$$P_m(I_m)dI_m=dt/T_1, \tag{4}$$

so that $P_m(I_m)=(dt/dI_m)/T_1$ is determined entirely by the modulation waveform $I_m(t)$. For the noise-modulation case, $P_m(I_m)$ is specified by the statistics of the noise modulation, and the randomization of observation times is generally not necessary.

For any particular combination of radiation statistics $P(I)$ and modulation waveform $I_m(t)$, the actual calculation of the counting distribution $p(n,m,T_0)$ is more easily performed in terms of the generating function. This is a function of a continuous parameter, s, rather than of the discrete one, n, and is defined as

$$\langle s^n\rangle_n=\sum_{n=0}^{\infty} p(n,m,T_0)s^n. \tag{5}$$

The counting distribution can be recovered from this whenever the function can be expanded in powers of s. Other observable statistical parameters are also readily extracted from the generating function. The factorial moments of the counting distribution, $\langle n!/(n-m)!\rangle_n$, are exhibited in a Taylor-series expansion of $\langle s^n\rangle_n$ about $s=1$ because

$$\langle s^n\rangle_n=\sum_{m=0}^{\infty} \left\langle \binom{n}{m} \right\rangle_n (s-1)^m. \tag{6}$$

Most important, the generating function of the counting distribution, $\langle s^n\rangle_n$, is directly related to the characteristic function of the radiation statistics, i.e., the Laplace transform of $P(I)$, as follows immediately from Mandel's formula, Eq. (1). In the unmodulated case,

$$\langle s^n\rangle_n=\sum_{n=0}^{\infty} p(n,T,t)s^n=\langle \exp-[M(1-s)]\rangle_I$$

$$=\int_0^{\infty} \exp-[M(1-s)]P(I)dI, \tag{7}$$

which becomes a Laplace transform when M is proportional to I, e.g., in the case of short observation intervals, for which $M=\alpha TI$. In terms of the dimensionless transform variable

$$\sigma=\alpha TI_1(1-s), \tag{8}$$

normalized to some convenient reference irradiance I_1, the generating function is just the Laplace transform, $\langle s^n\rangle_n=F(\sigma)=\langle\exp-(\sigma I/I_1)\rangle_I$.

To account for the effects of modulation there remains only to average $F(\sigma)$ over the modulation waveform to obtain the new generating function $F_m(s)$. Because I_1 is equal or proportional to I_m, excursions of σ correspond to those of I_m, so that Eq. (3) simplifies to

$$F_m(s)=\int_{\sigma_a}^{\sigma_b}F(\sigma)P_m(\sigma)d\sigma, \tag{9}$$

which then yields $p(n,m,T_0)$ by expansion in powers of s, via Eqs. (5) and (8).

Depending on the complexity of the characteristic function $F(\sigma)$, it may be more expedient to perform the integrations in the reverse order, i.e., to average the Laplace transform of the modulation distribution $P_m(\sigma)$ over the radiation statistics $P(I)$. The general result is expressible as the double integral

$$F_m(s)=\int_0^\infty\int_{\sigma_a}^{\sigma_b}P_m(\sigma)e^{-\sigma x}P(x)d\sigma dx, \tag{10}$$

with $x=I/I_1$.

Our expressions are actually valid also for a number of combinations of relative time scales other than $T_0\ll\tau,T_1$, as may be seen directly from Eq. (2). For the case $\tau\ll T_0\ll T_1$, the distributions for any modulation function are identical to those of the amplitude-stabilized case. Fluctuations resulting from the radiation statistics alone are averaged out in this case, as discussed by Troup and Lyons.[10] In the opposite extreme, when $T_1\ll T_0\ll\tau$, the modulation may be neglected, as only the over-all average irradiance is seen during any measurement time T_0. The results are then those for the radiation statistics of the appropriate source, with $m\equiv 0$. Lastly, in the limit of very long observation time, i.e., $T_0\gg\tau,T_1$, fluctuations arising from both the statistics of the unmodulated radiation and from the modulation itself are averaged out, and the appropriate result is simply the Poisson distribution. The counting distribution for a modulated chaotic source with arbitrary τ/T_0 (but still subject to $T_0\ll T_1$ or $T_0\gg T_1$) could be obtained in approximate form by using Mandel's empirical distribution for this case.[11]

Finally, we note that the counting distributions cited here do not depend on either the frequency or the periodicity of the modulation waveform, so long as the relative time scales are appropriately related, as indicated above.

II. ANALYTICAL RESULTS

To obtain specific counting statistics for the several radiation sources modulated by various waveforms to arbitrary depths, the radiation statistics $P(I)$ are specified, together with the characteristic function $F(\sigma)$. To each modulation waveform $I_m(t)$, there corresponds a distribution $P_m(I_m)$; this is also expressible in terms of σ, which is proportional to I_m. For each combination of source statistics and modulation, there remains to integrate $F(\sigma)P_m(\sigma)$ to obtain the generating function $F_m(s)$, which then yields the counting distribution $p(n,m,T)$ by expansion in a power series in s. The results are most usefully expressed in terms of the modulation depth $m=(\sigma_b-\sigma_a)/(\sigma_b+\sigma_a)$ about the over-all mean irradiance I_M or the over-all mean count $N=\alpha TI_M$.

The three types of radiation sources treated are the following.

(a) Narrow-band gaussian noise. Chaotic sources (e.g., thermal radiator, laser below threshold) with a gaussian amplitude distribution and hence an exponential intensity distribution

$$P(I)=\exp(-I/I_1)/I_1. \tag{11}$$

The characteristic function is

$$F(\sigma)=1/(1+\sigma) \tag{12}$$

and I_1 is simply the mean irradiance I_m.

(b) Amplitude-stabilized source. With only phase fluctuations, the irradiance is constant at $I_1=I_m$; i.e.,

$$P(I)=\delta(I-I_1),\quad F(\sigma)=e^{-\sigma}. \tag{13}$$

This is the case for a well-stabilized laser operating far above threshold.

(c) Nonlinear oscillator. A model of a laser applicable throughout the range from well below, to near, to well above threshold yields the Risken irradiance distribution[1]

$$P(I)=(2/\pi^{1/2})(\{\exp-[(I/I_1)-w]^2\}/I_1\mathrm{erfc}(-w)). \tag{14}$$

The parameter w describes the state of excitation of the laser; $w<0$ below, $w=0$ at, $w>0$ above threshold. The mean irradiance varies with excitation as

$$I_m=I_1Z_1(-w)/Z_0(-w). \tag{15}$$

For convenience we define

$$Z_n(x)=\exp(x^2)\mathrm{i}^n\mathrm{erfc}x, \tag{16}$$

where the tabulated function $\mathrm{i}^n\mathrm{erfc}x$ is the nth repeated integral of the error function.[12] Well above threshold ($w>2$), the irradiance distribution is gaussian, with mean wI_1. At threshold ($w=0$), the distribution has the shape of half the gaussian curve, with $I_m=I_1/\pi^{1/2}$. Far below threshold ($w<-3$), the distribution approaches the exponential one of gaussian noise, with $I_m=I_1/(-2w)$. The characteristic function is

$$F(\sigma)=Z_1[(\sigma/2)-w]/Z_1(-w). \tag{17}$$

The three modulation waveforms considered include the following.

(a) Square wave. Abrupt transitions between two levels, sustained for equal periods. The result is

$$F_m(s)=\tfrac{1}{2}F(\sigma_b)+\tfrac{1}{2}F(\sigma_a). \tag{18}$$

The more general case of an asymmetrical square wave is a trivial extension of this case.

(b) Triangular. Linear sweep, back and forth between two levels. Hence, $P_m(\sigma)=1/(\sigma_b-\sigma_a)$ within the range of integration and

$$F_m(s)=\frac{1}{\sigma_b-\sigma_a}\int_{\sigma_a}^{\sigma_b}F(\sigma)d\sigma. \tag{19}$$

The rate of sweep, or even its periodicity, is of no consequence, so long as T_0 remains small compared to any sweep period.

(c) Sinusoidal. For arbitrary amplitude and modulation depth,

$$P_m(\sigma)=(1/\pi)/[(\sigma_b-\sigma)(\sigma-\sigma_a)]^{1/2}. \tag{20}$$

The frequency of the sinusoidal modulation does not enter, and need not even be constant, provided that T_0 remains small compared to the modulation period.

A. Square-Wave Modulation

The calculations for square-wave modulation involve merely an average of the unmodulated counting distribution, evaluated at the two mean-count levels, $N(1\pm m)$. The resultant counting distributions are

$$p(n,m,N)=\frac{[N(1+m)]^n}{2[1+N(1+m)]^{n+1}}+\frac{[N(1-m)]^n}{2[1+N(1-m)]^{n+1}}, \tag{21}$$

$$p(n,m,N)=(N^n/n!)e^{-N}\times\tfrac{1}{2}[(1+m)^ne^{-mN}+(1-m)^ne^{mN}] \tag{22}$$

and

$$p(n,m,N)=[N_0{}^n/Z_0(-w)]\tfrac{1}{2}[(1+m)^nZ_n(b)+(1+m)^nZ_n(a)], \tag{23}$$

where

$$a=\tfrac{1}{2}N_0(1-m)-w,\quad b=\tfrac{1}{2}N_0(1+m)-w, \tag{24}$$

and

$$N_0=NZ_0(-w)/Z_1(-w), \tag{25}$$

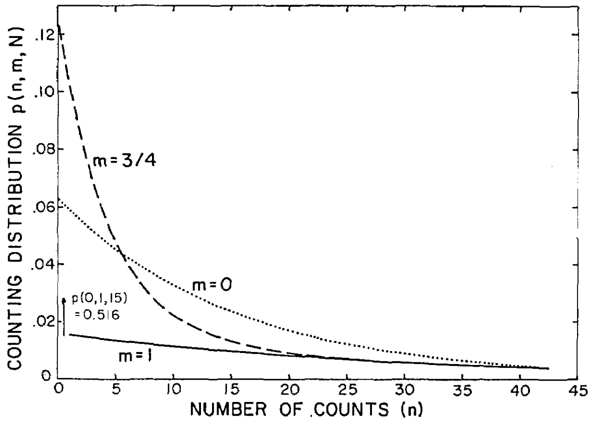

FIG. 1. Counting distribution $p(n,m,N)$ for a square-wave-modulated chaotic source with the modulation depth m as a parameter. The mean count is $N=15$ for all distributions shown.

for gaussian, stable, and Risken radiation, respectively. Figures 1–3 present the counting distributions for these three cases, with the modulation depth as a parameter. The over-all mean count N for the three cases plotted is 15, 15, and 5, respectively.

B. Triangular Modulation

The general result for the photocounting distribution for triangularly modulated radiation is obtainable, by expanding Eq. (19) in a power series in s, as

$$p(n,m,N)=\frac{1}{2mN}\int_{N(1-m)}^{N(1+m)}F(\sigma)d\sigma-\sum_{k=0}^{n-1}\frac{(1+m)p[k,N(1+m)]-(1-m)p[k,N(1-m)]}{2m(k+1)} \tag{26}$$

in terms of the unmodulated distribution $p(n,N)$.

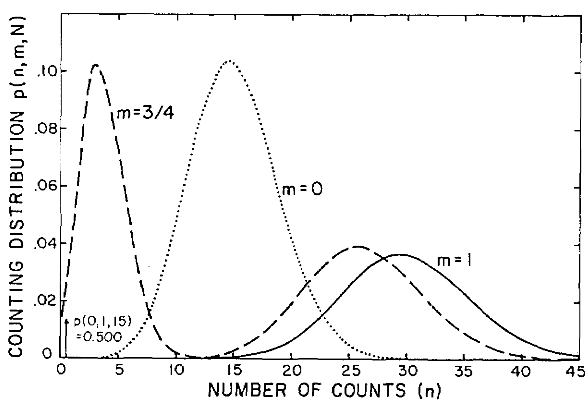

FIG. 2 Counting distribution $p(n,m,N)$ for a square-wave-modulated Poisson process with the modulation depth m as a parameter. The mean count is $N=15$ for all distributions shown.

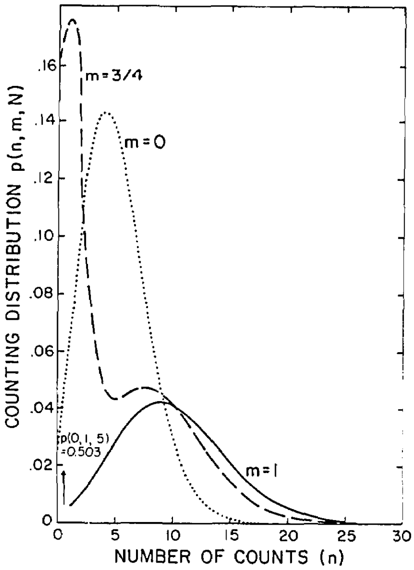

Fig. 3. Counting distribution $p(n,m,N)$ for a square-wave-modulated Risken source with the modulation depth m as a parameter. The mean count is $N=5$ and the excitation parameter is $w=2$ for all distributions shown.

Specifically, for the cases of chaotic and stable radiation,

$$p(n,m,N)=\frac{1}{2mN}\left\{\ln\left(\frac{b}{a}\right)-\sum_{k=1}^{n}\frac{1}{k}\left[\left(\frac{1+m}{b}\right)^k-\left(\frac{1-m}{a}\right)^k\right]\right\}, \quad (27)$$

where $a=1+N^{-1}-m$, $b=1+N^{-1}+m$, and

$$p(n,m,N)=\frac{\exp[-N(1-m)]}{2mN}\sum_{k=0}^{n}\frac{[N(1-m)]^k}{k!}-\frac{\exp[-N(1+m)]}{2mN}\sum_{k=0}^{n}\frac{[N(1+m)]^k}{k!}. \quad (28)$$

For the Risken irradiance distribution, the form of Eq. (26), with $F(\sigma)$ given in Eq. (17) and with [13]

$$p(k,N)=N^kZ_k[(N/2)-w]/Z_0(-w), \quad (29)$$

is convenient for some values of the parameters. The alternative obtained by bypassing $F(\sigma)$ as in Eq. (10) is more expedient in other ranges,

$$p(n,m,N)=\sum_{k=n}^{\infty}(-1)^{n+k}N^k\left[\frac{Z_k(-w)}{Z_0(-w)}\right]\times\binom{k}{n}\sum_{l\text{ even}}\binom{k}{l}\frac{m^l}{l+1}, \quad (30)$$

where the summation on l is over the range $0\leq l\leq k$, but only with l even. These distributions are shown in Figs. 4, 5, and 6, respectively, with m as a parameter and N chosen as before, for convenient comparison with Figs. 1, 2, and 3.

C. Sinusoidal Modulation

For sinusoidal modulation, the results for the three types of radiation are obtained by use of Eq. (20) in Eq. (9).

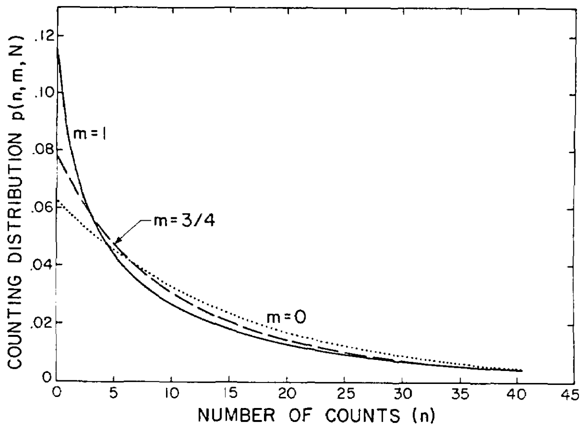

Fig. 4. Counting distribution $p(n,m,N)$ for a triangularly modulated chaotic source with the modulation depth m as a parameter. The mean count is $N=15$ for all distributions shown.

For the chaotic source,

$$F_m(s)=\{[1+N(1-m)(1-s)][1+N(1+m)(1-s)]\}^{-\frac{1}{2}} \quad (31)$$

and power-series expansion yields

$$p(n,m,N)=(N^n\mu^n/R^{n+1})P_n[(N\mu+\mu^{-1})/R], \quad (32)$$

where

$$\mu^2=1-m^2,\quad R^2=1+2N+N^2\mu^2, \quad (33)$$

and $P_n(x)$ is the Legendre polynomial. The indeterminate limiting case of $m=1$ reduces to

$$p(n,1,N)=[(2n)!/2^n(n!)^2]N^n/(1+2N)^{n+\frac{1}{2}}. \quad (34)$$

For the stable source,

$$F_m(s)=\exp[-N(1-s)]\mathrm{I}_0[mN(1-s)], \quad (35)$$

where $\mathrm{I}_0(x)$ is the modified Bessel function. Series expansion yields the finite sum of Bessel functions,

$$p(n,m,N)=\frac{N^ne^{-N}}{n!}\sum_{l=0}^{n}\binom{n}{l}\left(\frac{-m}{2}\right)^l\sum_{k=0}^{l}\binom{l}{k}\mathrm{I}_{|l-2k|}(mN). \quad (36)$$

An alternate form, more convenient for great modulation depths, is

$$p(n,m,N)=\frac{N^n}{\pi^{\frac{1}{2}}}\exp[-N(1+m)]\sum_{k=0}^{n}\frac{(2m)^k(1-m)^{n-k}}{k!k!(n-k)!}\times\Gamma(k+\tfrac{1}{2})M(\tfrac{1}{2},k+1,2mN),\quad(37)$$

where $\Gamma(x)$ and $M(a,b,x)$ are the gamma function and the confluent hypergeometric function, respectively. In particular, for 100% sinusoidal modulation, one term in Eq. (37) suffices

$$p(n,1,N)=[(2N)^n/(n!)^2]\times\exp(-2N)[\Gamma(n+\tfrac{1}{2})/\pi^{\frac{1}{2}}]M(\tfrac{1}{2},n+1,2N).\quad(38)$$

This form differs slightly from that reported by Pearl and Troup[6]; the discrepancy is apparently due to printing errors in the latter. Figure 7 presents plots of these distributions, for $N=15$ and $m=0$, $\frac{3}{4}$, and 1.

For the source with the Risken distribution, it is more expedient to reverse the order of integrations in Eq. (10). This amounts then to an average of Eq. (35), the generating function for the stable source, over the actual source fluctuations $P(x)=\exp[-(x-w)^2]$, suitably

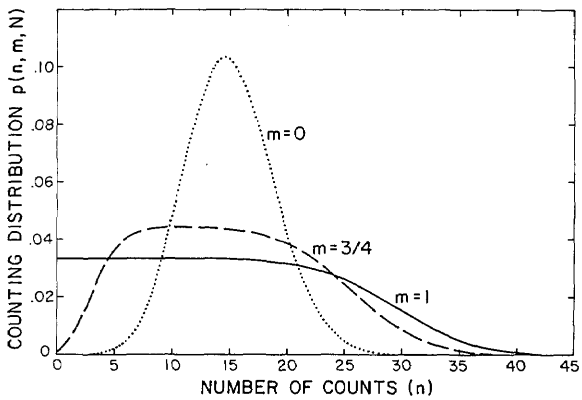

FIG. 5. Counting distribution $p(n,m,N)$ for a triangularly modulated Poisson process with the modulation depth m as a parameter. The mean count is $N=15$ for all distributions shown.

normalized as in Eq. (14). After expansion of $F_m(s)$ in powers of s, the final result is

$$p(n,m,N)=\sum_{k=n}^{\infty}(-1)^{n+k}N_0^k\left[\frac{Z_k(-w)}{Z_0(-w)}\right]\binom{k}{n}Q_k(m),\quad(39)$$

where

$$Q_k(m)=\sum_l\frac{k!}{(k-2l)!}\frac{(\frac{1}{2}m)^{2l}}{l!l!},\quad(40)$$

with the summation range $0\leq l\leq\frac{1}{2}k$. The parameter N_0 is related to the over-all mean count N as in Eq. (25).

III. DISCUSSION

For square-wave modulation, the counting distribution is simply an average of the unmodulated ones corresponding to the two discrete levels of the radiation. In Fig. 1, we show the result for the chaotic source, Eq. (21). As the modulation depth m increases, the counting probability becomes greater at low (as well as at high) count numbers n, at the expense of counts near the mean. Square-wave modulation of the gaussian source in effect sharpens the peak that occurs near $n=0$.

These results can be interpreted in terms of photon bunching. The counting distributions for the square-wave-modulated gaussian resemble those obtained with a two-photon detector[13] for the gaussian source, when displayed on the same scale. These have a distinctly more positive second derivative than the Bose–Einstein distribution generated by the ordinary one-quantum detector for an unmodulated gaussian source. In the modulated case this results from the accentuation of photon bunching induced by the alternately high and low irradiance levels, whereas in the two-quantum case it stems from the bunching observed preferentially because of the requirement of two photons to effect a photoelectric transition.

A characteristic of 100%, symmetrical, square-wave modulation is the increase of counting probability by 0.5 for $n=0$, regardless of the radiation statistics, as may be seen in the figures. We note in Fig. 1 that the curves for various values of m have similar shapes. This occurs because of the rather strong pre-existent fluctuations of the gaussian source; the change of mean irradiance introduced by the modulation, even when it is as drastic as the discontinuous square wave, does not have a great effect on the already broad counting distribution. The compounding of two broad distributions will not differ greatly in form from the original distributions. By

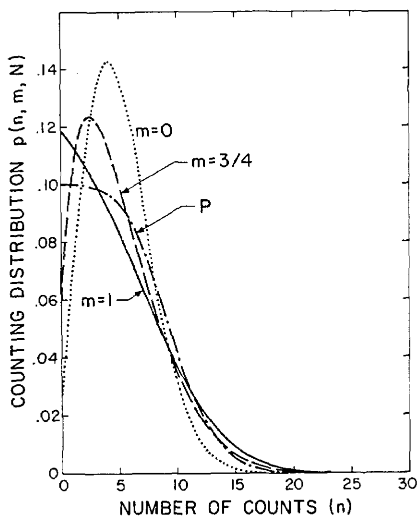

FIG. 6. Counting distribution $p(n,m,N)$ for a triangularly modulated Risken source with the modulation depth m as a parameter. The mean count is $N=5$ and the excitation parameter is $w=2$ for all distributions shown. For comparison, curve P is the distribution for the 100%-triangularly modulated amplitude-stabilized source, with the same mean count.

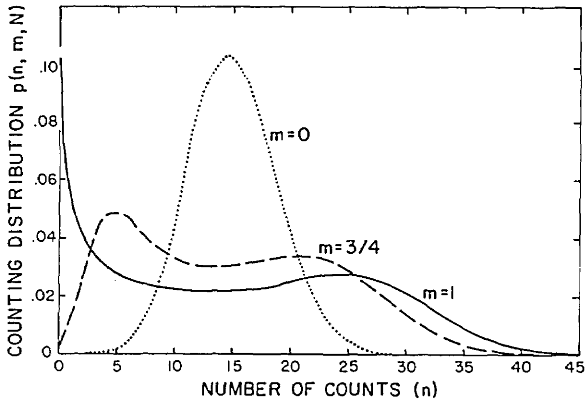

FIG. 7. Counting distribution $p(n,m,N)$ for a sinusoidally modulated Poisson process with the modulation depth m as a parameter. The mean count is $N=15$ for all distributions shown.

contrast, peaked distributions will be markedly altered by this process.

This is amply evident in Fig. 2, which illustrates square-wave modulation, with the same parameters, imposed on an amplitude-stabilized source, Eq. (22). This result applies also to any other square-wave-modulated source when $\tau \ll T_0 \ll T_1$. The peaked nature of the Poisson distribution leads to a double-peaked curve for values of m above a certain threshold. Counts near the mean become relatively unlikely. With the mean count preserved, square-wave modulation virtually eliminates overlap of the high-probability count number regions. This case has been studied experimentally for low modulation depths by Fray *et al.*,[5] who found agreement with theory.

The distribution Eq. (23) for the square-wave-modulated Risken source (with $w=2$ and $N=5$) is shown in Fig. 3. The value of $w=2$ corresponds to a laser somewhat above threshold, and the results are similar to those obtainable with a Poisson source of the same mean. Nevertheless, minor differences between the unmodulated Risken and stable-source counting statistics are accentuated by even partial modulation. For example, the subsidiary peak in the $m=\frac{3}{4}$ curve in Fig. 3 is considerably less pronounced than for the corresponding stable source.

Whereas square-wave modulation concentrates the mean irradiance at two extreme values, allowing no intermediate levels, the triangular waveform provides equal weight to all intermediate irradiance values between the two extremes. No sharp jump of probability occurs at $n=0$ for unity modulation depth, as observed with square-wave modulation, because the average irradiance is zero only instantaneously at the bottom of the triangular waveform. The result for triangular modulation of the gaussian source [Eq. (27)] is largely independent of modulation depth, as shown in Fig. 4.

The counting distribution for the amplitude-stabilized source with triangular irradiance modulation possesses the interesting property of extreme flatness over some range of counts that increases with modulation depth. For the unity-modulation-depth curve shown in Fig. 5, for example, the flatness is better than 1 part in 10^4 on the interval $0\leq n\leq 15$. This result has recently been discussed in detail by Teich and Diament[7,8] and may be inferred directly from Eq. (28), specialized to $m=1$. It depends only on the existence of a Poisson process whose mean is triangularly modulated and is therefore not restricted to photoelectron counting. The flat nature of the distribution may be useful for improving the signal-to-noise ratio in an optical communications system.

The counting distribution for a triangularly modulated Risken source [Eq. (30)] with $w=2$ is shown in Fig. 6. It is intermediate between the triangularly modulated gaussian and stable sources. The general shape of the distribution does not vary greatly as the parameter w is increased, i.e., as the Risken distribution becomes more and more sharp. Comparison of the $m=1$ curve with the dash–dot curve in Fig. 6 for a triangularly modulated stable source (also $N=5$, $m=1$) shows a readily identifiable distinction, illustrating two important points. First, the usually slight distinction between the stable and the Risken source with $w=2$ becomes more pronounced when a distinctive characteristic, such as flatness near $n=0$, is induced in the counting distribution by deep modulation. Second, the delicate balance yielding an extremely flat distribution arises from the unique combination of triangular modulation of a Poisson source.

The counting distribution for the sinusoidally modulated gaussian source [Eq. (32)] looks very much like that for triangular modulation (Fig. 4), and we do not display it. Although there is a larger spread in $p(0)$ for various values of m than in the triangular case, this is not so large as that observed for square-wave modulation (Fig. 1).

For the case of sinusoidally modulated amplitude-stabilized radiation, the equivalent expressions [Eqs. (36) and (37)] are plotted in Fig. 7. For modulation depths greater than a certain threshold, as for the square-wave case, the distribution is double peaked. Comparison with Figs. 2 and 5 for the square-wave and the triangular cases, respectively, shows clearly that the counting distribution for the sinusoidal-stable combination has characteristics intermediate between the other two.

Fray *et al.*[5] have derived, and experimentally verified, an expression for this case, valid for small values of m, and Pearl and Troup[6] have done the same for 100% modulation depth.

The Risken distribution provides the connection between the gaussian and stable sources, and sinusoidal modulation generates effects intermediate between square-wave and triangular modulation. Consequently, the sinusoidally modulated Risken source [Eq. (39)] has an undistinguished counting distribution somewhat like an average of the previous cases, and is not shown.

IV. CONCLUSION

The counting distributions for radiation with arbitrary statistics, modulation, and modulation depth are expected to be of use in noiseless low-level direct communications problems where the information is transmitted by irradiance modulation and the source does not necessarily arise from a stable single-mode laser. The nine specific cases examined in detail range from gaussian to amplitude-stabilized radiation (including chaotic and laser sources) and from square-wave to triangular modulation.

The modulation effects can be interpreted as an accentuation of photoelectron bunching. Both the low- and the high-count probabilities are increased at the expense of counts near the mean. However, the extent of the change which the modulation causes in the counting statistics has been found to depend strongly on the irradiance distribution of the underlying radiation. Of the cases studied, the smallest effect has been found for gaussian radiation, which has large fluctuations and therefore a broad counting distribution even in the absence of modulation. The largest effect was observed for the initially sharp, single-peaked counting distribution for the amplitude-stabilized source. Deep modulation generates a double-peaked counting distribution having very little overlap with the original unmodulated Poisson distribution. Thus, the distinction between sources with different statistics may be enhanced by modulation. By varying both the radiation statistics and the modulation waveform, one may synthesize counting distributions of a wide variety of shapes.

Triangular modulation (or linear sweep) yields some particularly interesting results. For a stable source, the resultant counting distribution is extremely flat; for a gaussian source, it is relatively independent of the modulation depth m. The most dramatic change in a counting distribution arises from (discontinuous) square-wave modulation. Sinusoidal modulation causes effects intermediate between the square-wave and triangular cases, much as for the Risken distribution, which has statistics intermediate between the gaussian and stable sources. For modulation of a stable source with random noise having a gaussian distribution,[14] the counting statistics are those obtained from the corresponding unmodulated Risken source.

Of all the distributions considered, the unmodulated amplitude-stabilized source retains the narrowest width in the count-number domain, and remains the optimum source for the noiseless binary pulse-code-modulation communications scheme.

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