Cascaded Poisson processes

Kuniaki Matsuo
Columbia Radiation Laboratory, Department of Electrical Engineering, Columbia University, New York, New York 10027

Bahaa E. A. Saleh
Columbia Radiation Laboratory, Department of Electrical Engineering, Columbia University, New York, New York 10027 and Department of Electrical and Computer Engineering, University of Wisconsin, Madison, Wisconsin 53706

Malvin Carl Teich
Columbia Radiation Laboratory, Department of Electrical Engineering, Columbia University, New York, New York 10027

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We investigate the counting statistics for stationary and nonstationary cascaded Poisson processes. A simple equation is obtained for the variance-to-mean ratio in the limit of long counting times. Explicit expressions for the forward-recurrence and inter-event-time probability density functions are also obtained. The results are expected to be of use in a number of areas of physics.

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I. INTRODUCTION

Multiplication, reduction, and branching processes have been examined in a broad variety of contexts.[1–6] Applications range from astrophysics to biological information transmission. In the great majority of mathematical treatments, the multiplication or branching is treated as an instantaneous effect [see Fig. 1(a)]. However, in many physical systems, a random time delay (or spatial dispersion) is inherent in the multiplication process. In this paper, we carry out an analysis of a cascade of Poisson multiplications that includes such time effects [see Fig. 1(b)]. Our results reduce to previously known descriptions, in the limit of instantaneous multiplication.

In a recent series of papers, we examined the two-stage multiplicative-Poisson process with random time delay. The particular model that we analyzed is the shot-noise-driven doubly stochastic Poisson point process (SNDP), in which each event of a Poisson point process generates an inhomogeneous rate function which, in turn, generates a second Poisson process. The SNDP is a doubly stochastic Poisson point process (DSPP)[7,8]; it is also a special case of the Neyman–Scott cluster process.[9,10]

A number of results were established in our study. We showed that the theoretical count variance is proportional to the count mean for an arbitrary inhomogeneous rate function [we call this the impulse-response function $h(t)$]. For long counting times, the theoretical counting distribution was shown to be the Neyman Type-A,[11,12] and this distribution was experimentally measured for radioluminescence from glass.[13] The forward-recurrence-time and inter-event-time probability densities were obtained, both in the absence and in the presence of self-excitation (dead time or refractoriness).[14] The results were used to describe the detection of optical fluorescence or scintillation generated by ionizing radiation. They were also used to fit the maintained-discharge interspike-interval histograms recorded from a cat's on-center retinal ganglion cell in darkness.[15]

General expressions for the count mean and variance were also obtained in the presence of small dead time, and the results were experimentally verified for radioluminescence from several transparent materials.[16] We showed that self-excitation could be used to constructively enhance or diminish the effects of point processes that display clustering, according to whether they are signal or noise. Finally, general expressions for the single- and multifold counting and time statistics, as well as for the power spectrum, were obtained for many cases of interest.[17] We presented a broad review of the application of such multiplied-Poisson noise to many areas in physics, optics, and electrical engineering (e.g., cathodoluminescence, x-ray radiography).[17] The statistics for a nonstationary SNDP were also obtained, and the counting distribution was found to reduce to the Neyman Type-A for input signals of short duration.[18] In this paper, we extend many of these results to the multistage case.

The results of our cascade analysis are likely to find use in problems where a series of multiplicative effects occur. Examples are the behavior of photon and charged-particle detectors, the production of certain types of cosmic rays, and the transfer of neural information. In Sec. II, we briefly review the results for the case of instantaneous multiplication. In Sec. III, we obtain the cascade counting and time statistics, as well as the autocovariance function, in the more general case, when time effects are incorporated into the model. The behavior of the resulting counting statistics is discussed in Sec. IV, and the Conclusion is presented in Sec. V.

II. THE INSTANTANEOUS MULTIPLICATION PROCESS

We briefly discuss the instantaneous multiplication process. Let $p(n)$ represent the probability that an event at the mth generation creates n events at the $(m+1)$st generation. The quantity $G_m(z)$, which is the probability generating function for the total number of events produced at the mth

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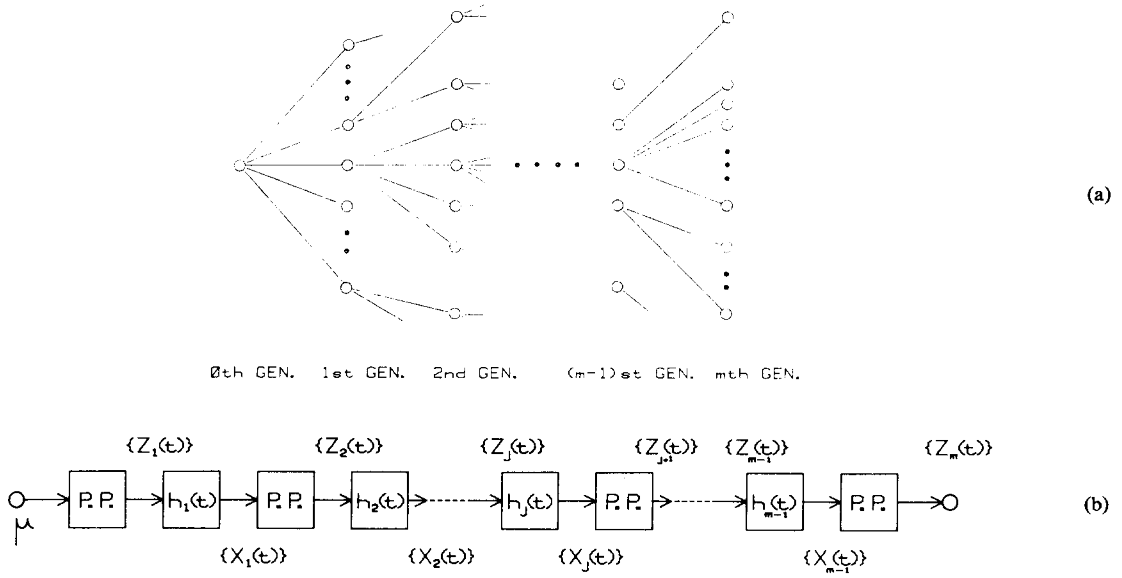


FIG. 1. Schematic representation of an m-stage cascaded system with Poisson multiplication at each stage. (a) Instantaneous multiplication; (b) time effects included. P. P. represents a Poisson process generator, whereas $h_j(t)$ represents a linear-filter impulse response function.

generation, is given by

$$G_m(z) = G_1(G_{m-1}(z)),$$

so that

$$G_m(z) = \underbrace{G_1(G_1(G_1(G_1 \cdots G_1(z))))}_{m \text{ times}}. \tag{1}$$

Here

$$G_0(z) = z$$

and

$$G_1(z) = \sum_{n=0}^{\infty} z^n p(n).$$

Assuming that $p(n)$ is Poisson distributed with mean a, and substituting $z = \exp(-s)$ in (1), we obtain the moment generating function $Q_m(s)$ at the mth generation for the cascaded Poisson instantaneous multiplication process: that is,

$$Q_m(s) = Q_1(Q_{m-1}(s)),$$

or

$$Q_m(s) = \underbrace{Q_1(Q_1(Q_1(Q_1 \cdots Q_1(s))))}_{m \text{ times}}. \tag{2}$$

Here

$$Q_0(s) = \exp(-s)$$

and

$$Q_1(s) = \exp(a(\exp(-s) - 1)).$$

III. POISSON MULTIPLICATION WITH TIME DELAY

As indicated in the Introduction, time delay can be an important effect in multiplication processes. In Subsec. A, we derive the counting-distribution moment generating function for an m-stage cascade of Poisson processes, for arbitrary T/τ_p. The quantity τ_p is the characteristic decay time of the inhomogeneous rate. This is followed by a calculation of the counting statistics for the single and multifold cases in Subsec. B and C, respectively. In Subsec. D, we derive the autocovariance function. The time statistics are obtained in Subsec. E. The counting statistics for the nonstationary case are considered in Subsec. F.

A. Moment generating function for the counting process at the mth stage

We consider the system illustrated in Fig. 1(b). The quantity μ is the initial deterministic driving rate, $\{Z_j(t)\}$ is a process of impulses corresponding to the point process at the jth stage, and $\{X_j(t)\}$ is the linearly filtered point process at the jth stage which, in turn, provides the driving rate process for the $(j+1)$st stage. The boxes labeled P.P. and $h_j(t)$ represent Poisson point process generators and linear filters, respectively. The moment generating functional for the filtered point process at the jth stage is defined by

$$L_{X_j}(s) \triangleq \left\langle \exp\left(-\int_{-\infty}^{\infty} s(t) X_j(t)\, dt \right) \right\rangle,$$
$$j = 1,2,\ldots,m-1. \tag{3}$$

It can be shown (see Appendix A) that (3) can be written as

$$L_{X_j}(s) = \left\langle \exp\left\{ \int_{-\infty}^{\infty} X_{j-1}(t) \times \left[\exp\left(-\int_{-\infty}^{\infty} h_j(\tau - t) s(\tau)\, d\tau \right) - 1 \right] dt \right\} \right\rangle. \tag{4}$$

If we replace $\exp(-\int_{-\infty}^{\infty} h_j(\tau - t) s(\tau)\, d\tau) - 1$ by $-s(t)$, the right-hand side of (4) is, by definition, the moment generating functional of the process $\{X_{j-1}(t)\}$; that is,

$$\left\langle \exp\left\{ \int_{-\infty}^{\infty} X_{j-1}(t) \left[\exp\left(-\int_{-\infty}^{\infty} h_j(\tau - t) s(\tau)\, d\tau \right) - 1 \right] dt \right\} \right\rangle$$
$$\Rightarrow \left\langle \exp\left(-\int_{-\infty}^{\infty} s(t) X_{j-1}(t)\, dt \right) \right\rangle = L_{X_{j-1}}(s). \tag{5}$$

Therefore, we have a recursive formula for the moment gen-

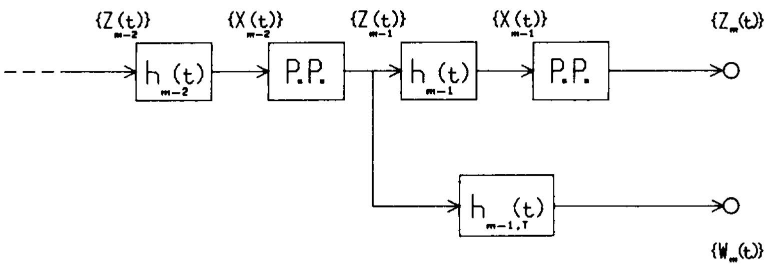


FIG. 2. Block diagram for generation of the integrated rate $\{W_m(t)\}$.

erating functional of the process $\{X_j(t)\}$,

$$L_{X_j}(s) = L_{X_{j-1}}\left\{1 - \exp\left(-\int_{-\infty}^{\infty} h_j(\tau - t)s(\tau)\,d\tau\right)\right\}, \quad (6)$$

where the moment generating functional of the first stage is

$$L_{X_1}(s) = \exp\left\{\mu\int_{-\infty}^{\infty}\left[\exp\left(-\int_{-\infty}^{\infty} h_1(\tau - t)s(\tau)\,d\tau\right) - 1\right]dt\right\}. \quad (7)$$

For convenience, we define the following operator:

$$q_j(\cdot) = \exp\left\{\int_{-\infty}^{\infty} h_j(t - t_j)[(\cdot) - 1]\,dt\right\},$$
$$j = 1,2,\ldots,m-1.$$

By using the above equations recursively, the moment generating functional for the process $\{X_{m-2}(t)\}$ becomes

$$L_{X_{m-2}}(s) = \exp\left\{\mu\int_{-\infty}^{\infty}\left[q_1\left(q_2\left(q_3\cdots\left(q_{m-3}\left(\exp\left(-\int_0^{\infty} h_{m-2}(t_{m-1} - t_{m-2})s(t_{m-1})\,dt_{m-1}\right)\right)\right)\cdots\right)\right) - 1\right]dt_1\right\}. \quad (8)$$

The integrated driving rate process at the mth stage is shown schematically in Fig. 2, and the moment generating functional for the process $\{W_m(t)\}$ is defined by

$$L_{W_m}(s) \triangleq \left\langle \exp\left(-\int_{-\infty}^{\infty} s(t)W_m(t)\,dt\right)\right\rangle. \quad (9)$$

It can be shown that the above equation can be written as

$$L_{W_m}(s) = \left\langle \exp\left\{\int_{-\infty}^{\infty} X_{m-2}(t) \times \left[\exp\left(-\int_{-\infty}^{\infty} h_{m-1,T}(\tau - t)s(\tau)\,d\tau\right) - 1\right]dt\right\}\right\rangle, \quad (10)$$

where the linear filter $h_{m-1,T}(t)$ is a convolution of $h_{m-1}(t)$ with an integrator (assumed to be noncausal for convenience) on the time interval $(0,T)$, i.e.,

$$h_{m-1,T}(t) = \int_0^T h_{m-1}(t + t')\,dt'. \quad (11)$$

To find the moment generating function of $\{W_m(t)\}$, we let $s(t) = s\delta(t)$ and we obtain

$$Q_{W_m}(s) = \left\langle \exp\left\{\int_{-\infty}^{\infty} X_{m-2}(t)[\exp(-sh_{m-1,T}(-t)) - 1]\,dt\right\}\right\rangle$$
$$= L_{X_{m-2}}\{1 - \exp(-sh_{m-1,T}(-t))\}. \quad (12)$$

Combining (8) and (12) yields

$$Q_{W_m}(s) = \exp\left\{\mu\int_{-\infty}^{\infty}[q_1(q_2(q_3\cdots q_{m-2}(\exp(-sh_{m-1,T}(-t_{m-1})))\cdots)) - 1]\,dt_1\right\}. \quad (13)$$

This result will be used subsequently to find the counting and time statistics.

The moment generating function for the counting process at the mth stage is related to that of $\{W_m(t)\}$ by[19]

$$Q_{N_m}(s) = Q_{W_m}\{1 - \exp(-s)\}. \quad (14)$$

Inserting (13) into (14) then yields the final result

$$Q_{N_m}(s) = \exp\left\{\mu\int_{-\infty}^{\infty}[q_1(q_2(q_3\cdots q_{m-2}(\exp(1 - \exp(-s)) \times h_{m-1,T}(-t_{m-1}))\cdots)) - 1\,dt_1\right\}. \quad (15)$$

B. Singlefold counting statistics at the *m*th stage

The probability distribution for the occurrence of n events in a fixed time interval $(0,T)$, at the mth stage, can be computed by using the formula[19]

$$p_m(n) = \frac{(-1)^n\partial^n}{n!\,\partial s^n} Q_{W_m}(s)\Big|_{s=1}. \quad (16)$$

With the help of the results derived in Appendix B, we have

$$p_m(0) = \exp\left\{\mu\int_{-\infty}^{\infty}[D_1^{(0)}(t) - 1]\,dt\right\}, \quad m \geqslant 2 \quad (17a)$$

and

$$(n+1)p_m(n+1) = \mu\sum_{k=0}^{n}\frac{(-1)^{k+1}}{k!}p_m(n-k)I^{(k+1)}, \quad m \geqslant 2, \quad (17b)$$

where

$$I^{(k+1)} = \int_{-\infty}^{\infty} D_1^{(k+1)}(t)\,dt, \quad k \geqslant 0,$$

$$D_j^{(k+1)}(t) = \sum_{r=0}^{k}\binom{k}{r}D_j^{(k-r)}(t) \times \int_{-\infty}^{\infty} h_j(\tau - t)D_{j+1}^{(r+1)}(\tau)\,d\tau, \quad k \geqslant 0,$$
$$j = 1,2,3,\ldots,m-2,$$

$$D_j^{(0)}(t) = \exp\left\{\int_{-\infty}^{\infty} h_j(\tau - t)\left[D_{j+1}^{(0)}(\tau) - 1\right] d\tau\right\},$$
$$j = 1,2,\ldots,m-2,$$
$$D_{m-1}^{(k)}(t) = \{-h_{m-1,T}(-t)\}^k \exp\{-h_{m-1,T}(-t)\},$$
$$m \geqslant 2,\ k \geqslant 0,$$
$$D_j^{(k+1)}(t) = \frac{\partial^{k+1}}{\partial s^{k+1}} \exp(\theta_j(t,s))\bigg|_{s=1}, \qquad k \geqslant 0,$$
$$j = 1,2,\ldots,m-2,$$
$$\theta_{m-1}(t,s) = -s h_{m-1,T}(-t), \qquad m \geqslant 2.$$

The count mean and variance at the mth stage can be derived by using the relations for the cumulant generating function[19]

$$\langle N_m(T)\rangle = -\frac{\partial}{\partial s} \ln Q_{N_m}(s)\bigg|_{s=0} \tag{18a}$$

and

$$\mathrm{Var}(N_m(T)) = \frac{\partial^2}{\partial s^2} \ln Q_{N_m}(s)\bigg|_{s=0}. \tag{18b}$$

However, a moment's thought will demonstrate that the system in Fig. 3 is equivalent to the one in Fig. 1(b), as far as the first and the second moments are concerned. Here $\{V_0(t)\}$, $\{V_1(t)\},\ldots,\{V_{m-1}(t)\}$ are zero-mean, unit-variance white-processes, and the cross-correlation function of $\{V_i(t)\}$ and $\{V_j(t)\}$ is

$$R_{V_iV_j}(\tau) = \langle V_i(t+\tau)V_j(t)\rangle = \begin{cases}\delta(\tau) & \text{for } i = j,\\ 0 & \text{for } i \neq j.\end{cases} \tag{19}$$

The identity is provided by the theory of random processes in linear systems.[20,21] It can be shown that the mean and the autocorrelation function at the input to the mth stage are, respectively,

$$\langle X_{m-1}(t)\rangle = \mu \prod_{j=1}^{m-1} \alpha_j, \quad m \geqslant 2, \tag{20a}$$

and

$$R_{X_{m-1}}(\tau) = \mu^2 \prod_{j=1}^{m-1} \alpha_j^2 + \mu \sum_{i=1}^{m-1} \left\{\prod_{j=0}^{i-1} \alpha_j \mathop{*}_{k=i}^{m-1} g_k(\tau)\right\}, \quad m \geqslant 2, \tag{20b}$$

where

$$\alpha_0 = 1,$$
$$\alpha_k \triangleq \int_{-\infty}^{\infty} h_k(t)\, dt,$$
$$g_k(\tau) \triangleq h_k(\tau) * h_k(-\tau),$$
$$\mathop{*}_{k=i}^{m-1} g_k(\tau) \triangleq g_i(\tau) * g_{i+1}(\tau) * \cdots * g_{m-1}(\tau).$$

The symbol $*$ indicates convolution. The counting statistics are easily derived by using the above equations.[19] The mean number of counts at the mth stage, in the counting interval $(0,T)$, is[22]

$$\langle N_m(T)\rangle = \langle W_m(T)\rangle = \left\langle \int_0^T X_{m-1}(t)\, dt \right\rangle = \mu T \prod_{j=1}^{m-1} \alpha_j. \tag{21}$$

The variance of the number of counts at the mth stage can be expressed as[19]

$$\mathrm{Var}(N_m(T)) = \mu T \prod_{j=1}^{m-1} \alpha_j + \mu \sum_{i=1}^{m-1} \left\{\prod_{j=0}^{i-1} \alpha_j \int_{-T}^{T} (T - |\tau|)\left[\mathop{*}_{k=i}^{m-1} g_k(\tau)\right] d\tau\right\}. \tag{22}$$

In the limit of long counting times, which is a special case of substantial interest, the results can be found by substituting $\alpha_k \delta(t)$ for $h_k(t)$ in the above equations, which yields[22]

$$\mathrm{Var}(N_m(T)) = \mu T \prod_{j=1}^{m-1} \alpha_j + \mu T \sum_{i=1}^{m-1} \left\{\prod_{j=0}^{i-1} \alpha_j \cdot \prod_{k=i}^{m-1} \alpha_k^2\right\}. \tag{23}$$

The variance-to-mean ratio (Fano factor) is then expressed quite simply as

$$F_m = \frac{\mathrm{Var}(N_m(T))}{\langle N_m(T)\rangle} = 1 + \sum_{i=1}^{m-1} \left\{\prod_{k=i}^{m-1} \alpha_k\right\}, \quad m \geqslant 2. \tag{24a}$$

When all $\alpha_j = \alpha$, Eq. (24a) reduces to

$$F_m = 1 + \alpha[(1 - \alpha^{m-1})/(1 - \alpha)], \quad m \geqslant 1. \tag{24b}$$

For $m = 1$ and $m = 2$, we recover the usual expressions for the Poisson and Neyman Type-A distributions, respectively.

C. Multifold counting statistics at the mth stage

The joint probability for the number of counts N_j in L time intervals $[\tau_j, \tau_j + T_j]$, $j = 1,2,3,\ldots,L$, for the m-stage cascaded Poisson system, can be written as[19]

$$p_m(\mathbf{n}) = \prod_{j=1}^{L} \frac{(-1)^{n_j}\partial^{n_j}}{n_j!\, \partial s_j^{n_j}} Q_{\mathbf{W}_m}(\mathbf{s})\bigg|_{\mathbf{s}=\mathbf{1}}, \tag{25}$$

where

$$\mathbf{n} = (n_1, n_2, \ldots, n_L),$$
$$\mathbf{s} = (s_1, s_2, \ldots, s_L),$$
$$\mathbf{1} = (1,1,\ldots,1),$$
$$\mathbf{W}_m = (W_{m_1}, W_{m_2}, \ldots, W_{m_L}),$$

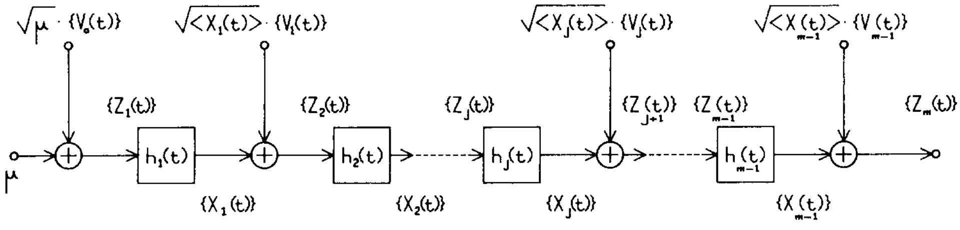


FIG. 3. Equivalent model to that presented in Fig. 1(b) as far as the count mean and variance are concerned. The $\{V_j(t)\}$ represent zero-mean unit-variance white processes.

provided that the integrated rate processes $\{W_{m_j}(t)\}$ at the mth stage are[17]

$$W_{m_j} = \int_{\tau_j}^{\tau_j + T_j} X_{m-1}(t)\,dt. \tag{26}$$

Here $Q_{\mathbf{W}_m}(\mathbf{s})$ is the L-dimensional multifold moment generating function of the integrated rate process at the mth stage, and can be expressed as

$$Q_{\mathbf{W}_m}(\mathbf{s}) = \exp\Big\{\mu\int_{-\infty}^{\infty}\Big[q_1\Big(q_2\Big(q_3\cdots q_{m-2}\Big(\exp\Big(-\sum_{j=1}^{L} s_j h_{m-1,T_j}(t_{m-1}+\tau_j)\Big)\Big)\cdots\Big)\Big) - 1\Big]dt_1\Big\}. \tag{27}$$

However, it is quite difficult to obtain the joint probability distribution function using (25) and (27), and we therefore carry this result no further.

The L-dimensional multifold moment generating function for the counting process at the mth stage can be determined by using the formula[19]

$$Q_{\mathbf{N}_m}(\mathbf{s}) = Q_{\mathbf{W}_m}\{\mathbf{1} - \exp(-\mathbf{s})\}, \tag{28}$$

where

$$\mathbf{N}_m = (N_{m,1}, N_{m,2}, \ldots, N_{m,L}),$$

$$\mathbf{1} - \exp(-\mathbf{s}) = (1 - \exp(-s_1), 1 - \exp(-s_2), \ldots, 1 - \exp(-s_L)).$$

Finally, we obtain the general expression

$$Q_{\mathbf{N}_m}(s) = \exp\Big\{\mu\int_{-\infty}^{\infty}\Big[q_1\Big(q_2\Big(q_3\cdots q_{m-2}\Big(\exp\Big(-\sum_{j=1}^{L}(1-\exp(-s_j))h_{m-1,T_j}(t_{m-1}+\tau_j)\Big)\Big)\cdots\Big)\Big) - 1\Big]dt\Big\}. \tag{29}$$

D. Autocovariance function at the mth stage

In this subsection we derive the autocovariance function for the number of counts N_m, registered in a time interval of duration T, for the m-stage cascaded Poisson process. The time separation between the intervals is $\tau = t_2 - t_1$. Using the definition of the autocovariance function and (29) we have

$$C_{N_m}(t_1,t_2) = \frac{\partial^2}{\partial s_1 \partial s_2}\ln Q_{N_m(t_1)N_m(t_2)}(s_1,s_2)\Big|_{s_1=s_2=0} = \mu\int_{-\infty}^{\infty}\{U_{m-1}(t,t_1)U_{m-1}(t,t_2) + V_{m-1}(t,t_1,t_2)\}\,dt, \tag{30a}$$

where

$$U_k(t,t_j) = \int_{-\infty}^{\infty} h(\tau - t)U_{k-1}(\tau,t_j)\,d\tau, \quad k = 1,2,\ldots,m-1, \quad j = 1,2, \tag{30b}$$

$$U_0(t,t_1) = -[u(t-t_j) - u(t-t_j-T)], \quad j = 1,2, \tag{30c}$$

$$V_k(t,t_1,t_2) = \int_{-\infty}^{\infty} h(\tau - t)[U_{k-1}(\tau,t_1)U_{k-1}(\tau,t_2) + V_{k-1}(\tau,t_1,t_2)]\,d\tau, \quad k = 1,2,\ldots,m-1, \tag{30d}$$

$$V_0(t,t_1,t_2) = 0. \tag{30e}$$

It can be shown (see Appendix C) that, when all stages are identical, (30a) can be rewritten as

$$C_{N_m}(\tau) = \mu T\int_{-\infty}^{\infty}\Big\{\alpha\Big[\frac{|H(\omega)|^{2(m-1)} - \alpha^{m-1}}{|H(\omega)|^2 - \alpha}\Big] + |H(\omega)|^{2(m-1)}\Big\}\Phi_T(\omega)e^{j\omega\tau}\frac{d\omega}{2\pi}, \tag{31}$$

where

$$\tau = t_2 - t_1, \quad \alpha = \int_{-\infty}^{\infty} h(t)\,dt,$$

$$H(\omega) = \text{F.T. of } h(t), \quad \Phi_T(\omega) = T\Big[\frac{\sin(\omega T/2)}{(\omega T/2)}\Big]^2.$$

From (31), we can obtain the variance of the counting process by simply setting $\tau = 0$, so that

$$\mathrm{Var}(N_m(T)) = \mu T\alpha^{m-1} + \mu\sum_{i=1}^{m-1}\Big\{\alpha^{i-1}\int_{-T}^{T}(T - |\tau|)\Big[\overset{m-i}{*}\, g(\tau)\Big]\,d\tau\Big\}. \tag{32}$$

Equation (32) can also be obtained from (22), and the definition of $g_k(\tau)$ given in (20b), by substituting $h_k(t) = h(t)$ for all k. The power spectral density for the process is obtained by taking the Fourier transform of (31).

E. Time statistics at the mth stage

The forward-recurrence-time probability density $P_m^{(1)}(t)$ and the inter-event-time probability density $P_m^{(2)}(t)$, for the m-stage cascaded Poisson system can be derived from the explicit expression for $Q_{W_m}(s)$.[19] The calculations are straightforward and lead to

$$P_m^{(1)}(T) = -\frac{\partial}{\partial T}Q_{W_m}(1) = \mu Q_{W_m}(1)\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{(m-1)\text{-fold integrals}} h_{m-1}(t_{m-1}+T) \times h_{m-2}(t_{m-1}-t_{m-2}) \cdots h_2(t_3-t_2)h_1(t_2-t_1)\exp\Big\{\sum_{j=1}^{m-1}\theta_j(t_j)\Big\} \times dt_1\,dt_2\cdots dt_{m-1}, \tag{33}$$

and

$$P_m^{(2)}(T) = \frac{-1}{\langle X_{m-1}(t)\rangle}\frac{\partial}{\partial T}P_m^{(1)}(T)$$

$$= \Big(Q_{W_m}(1)\Big/\prod_{j=1}^{m-1}\alpha_j\Big)\Big[\mu\Big\{\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{(m-1)\text{-fold integrals}} h_{m-1}(t_{m-1}+T) \times h_{m-2}(t_{m-1}-t_{m-2}) \cdots h_2(t_3-t_2)h_1(t_2-t_1)\exp\Big\{\sum_{j=1}^{m-1}\theta_j(t_j)\Big\}dt_1\,dt_2\cdots dt_{m-1}\Big\}^2$$

$$- \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{(m-1)\text{-fold integrals}}\Big\{h_{m-1}(t_{m-1}+T) \times \frac{\partial}{\partial T}\sum_{j=1}^{m-1}\theta_j(t_j) + \frac{\partial}{\partial T}h_{m-1}(t_{m-1}+T)\Big\} \times h_{m-2}(t_{m-1}-t_{m-2})\cdots h_2(t_3-t_2)h_1(t_2-t_1) \times\exp\Big\{\sum_{j=1}^{m-1}\theta_j(t_j)\Big\}dt_1\,dt_2\cdots dt_{m-1}\Big], \tag{34}$$

where

$$Q_{W_m}(1) = [\text{Eq. } (13)]|_{s=1},$$

$$\theta_j(t) = \int_{-\infty}^{\infty} h_j(\tau - t)[\exp(\theta_{j+1}(\tau)) - 1]\, d\tau, \qquad j = 1,2,\ldots,m-1,$$

$$\theta_{m-2}(t) = -h_{m-1,T}(-t).$$

F. Counting statistics for the nonstationary case

In this section, we obtain the moment generating function for the counting statistics, together with its mean and variance, for a nonstationary cascaded Poisson process (i.e., μ is a function of time).

A schematic diagram illustrating the generation of the process can be obtained by replacing μ by $\mu(t)$ in Fig. 1(b). The moment generating function for the m-stage integrated rate process $\{W_m(t,T)\}$ can be found by using a similar approach to that used in Subsec. A, giving rise to

$$Q_{W_m(t,T)}(s) = \exp\left\{\int_{-\infty}^{\infty} \mu(t-t_1)[q_1(q_2(q_3 \cdots q_{m-2}(\exp(-sh_{m-1,T}(-t_{m-1})))\cdots)) - 1]\, dt_1\right\}, \tag{35}$$

where

$$q_j(\cdot) = \exp\left\{\int_{-\infty}^{\infty} h_j(t-t_j)[(\cdot) - 1]\, dt\right\}.$$

Given the statistics of the integrated rate process $\{W_m(t,T)\}$, we readily obtain the statistics of the mth stage counting process. The mean and variance are, respectively,

$$\langle N_m(t,T)\rangle = \mu(t) * h_{m-1,T}(t) \mathop{*}_{j=1}^{m-2} h_j(t), \tag{36}$$

$$\operatorname{Var}(N_m(t,T)) = \mu(t) * h_{m-1,T}(t) \mathop{*}_{j=1}^{m-2} h_j(t) + \sum_{k=1}^{m-1} \mu(t) \mathop{*}_{j=1}^{k-1} h_j(t) * \left[h_{m-1,T}(t) \mathop{*}_{r=k}^{m-2} h_r(t)\right]^2. \tag{37}$$

Here

$$\mathop{*}_{r=i}^{j} h_r(t) = \delta(t) \quad \text{for } j < i,$$

and the moment generating function is

$$Q_{N_m(t,T)}(s) = \exp\left\{\int_{-\infty}^{\infty} \mu(t-t_1)[q_1(q_2(q_3 \cdots q_{m-2}(\exp(1-\exp(-s)) \times h_{m-1,T}(-t_{m-1}))\cdots)) - 1]\, dt_1\right\}. \tag{38}$$

Note that (35), (36), (37), and (38) are identical to (13), (21), (22), and (15), where $\mu(t)$ is not a function of time.

The L-dimensional multifold moment generating functions of the integrated rate process and the counting process at the mth stage are easily obtained, and they are, respectively,

$$Q_{\mathbf{W}_m(t,T)}(\mathbf{s}) = \exp\left\{\int_{-\infty}^{\infty} \mu(t-t_1)\left[q_1\left(q_2\left(q_3 \cdots q_{m-2}\left(\exp\left(-\sum_{j=1}^{L} s_j h_{m-1,T_j}(t_{m-1}+\tau_j)\right)\right)\cdots\right)\right) - 1\right] dt_1\right\}, \tag{39}$$

and

$$Q_{\mathbf{N}_m(t,T)}(\mathbf{s}) = \exp\left\{\int_{-\infty}^{\infty} \mu(t-t_1)\left[q_1\left(q_2\left(q_3 \cdots q_{m-2}\left(\exp\left(-\sum_{j=1}^{L}(1-\exp(-s_j)) \times h_{m-1,T_j}(t_{m-1}+\tau_j)\right)\right)\cdots\right)\right) - 1\right] dt_1\right\}. \tag{40}$$

These equations correspond to (27) and (29), respectively.

We now consider an important limiting case in which the rate $\mu(t)$ has a time course τ_s that is very short in duration, compared with the counting time T, added to the total linear filter correlation time $(m-1)\tau_c$ $(\tau_s \ll T + (m-1)\tau_c)$. In that case, the quantity $\mu(t)$ can be mathematically represented by the limiting distribution

$$\mu(t) = E\delta(t), \tag{41}$$

where E is the strength of the excitation (number of points) and $\delta(t)$ is the Dirac delta function. Substituting (41) in (36)

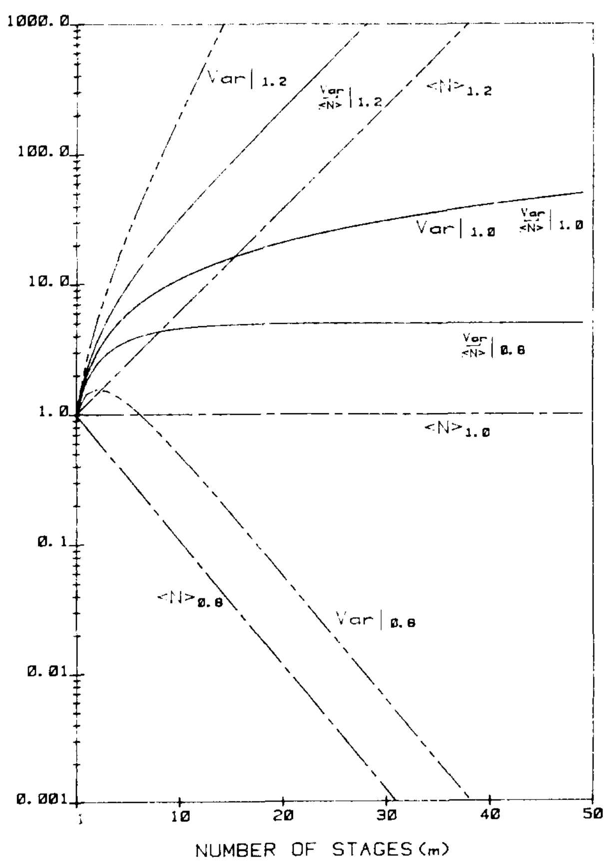
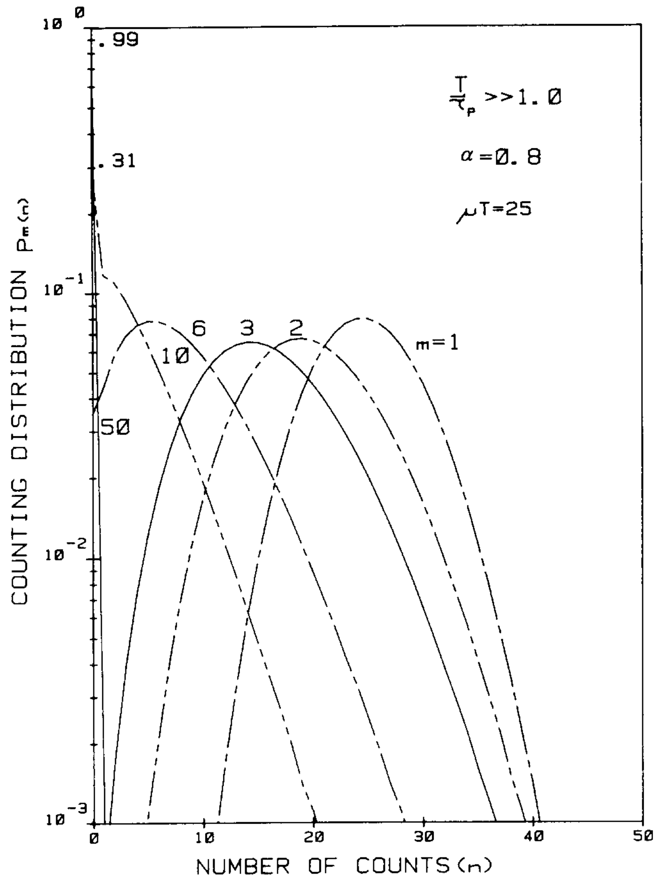
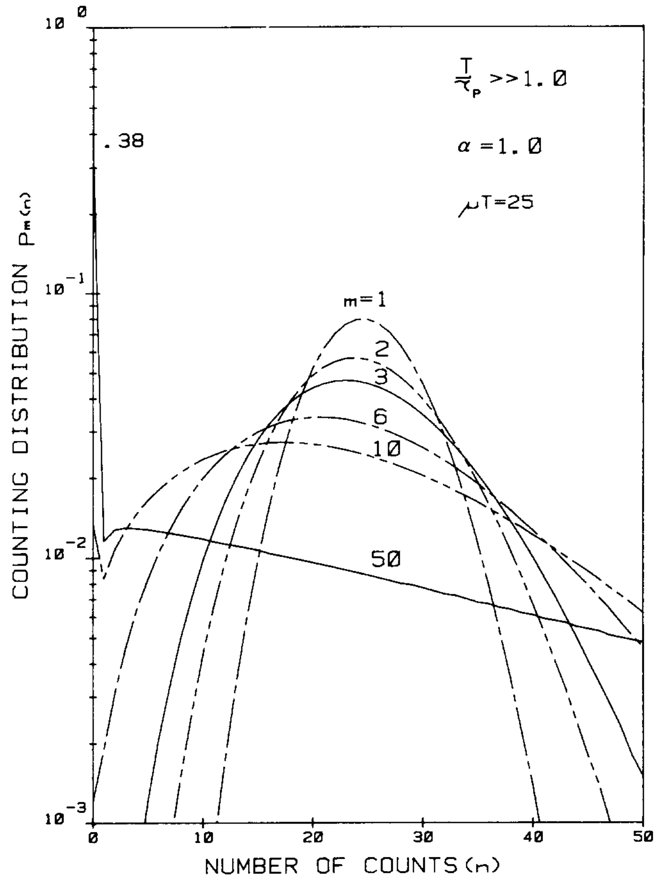


FIG. 4. Count mean $\langle N_m(T)\rangle$, count variance $\operatorname{Var}(N_m(T))$, and variance-to-mean ratio $\operatorname{Var}(N_m(T))/\langle N_m(T)\rangle$, vs number of stages m, with α as a parameter.



(a)



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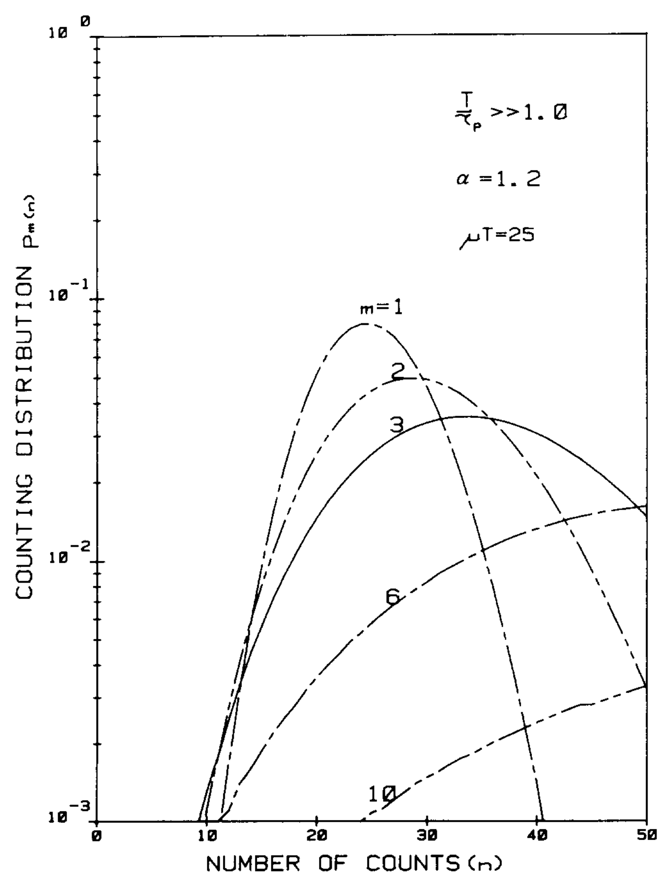


(c)

FIG. 5. Counting distribution $p_m(n)$ vs count number n for $T/\tau_p \gg 1.0$, $\mu T = 25$, and $m = 1,2,3,6,10$, and 50. (a) $\alpha = 0.8$; (b) $\alpha = 1.0$; (c) $\alpha = 1.2$.

and (37) yields the mean and variance

$$\langle N_m(t,T)\rangle = Eh_{m-1,T}(t) \mathop{*}_{j=1}^{m-2} h_j(t), \tag{42}$$

$$\mathrm{Var}(N_m(t,T)) = Eh_{m-1,T}(t) \mathop{*}_{j=1}^{m-2} h_j(t) + E\sum_{k=1}^{m-1} \mathop{*}_{j=1}^{k-1} h_j(t) * \left[h_{m-1,T}(t) \mathop{*}_{r=k}^{m-2} h_r(t)\right]^2. \tag{43}$$

The variance-to-mean ratio (Fano factor) is

$$F_m = 1 + \left\{\sum_{k=1}^{m-1} \mathop{*}_{j=1}^{k-1} h_j(t) * \left[h_{m-1,T}(t) \mathop{*}_{r=k}^{m-2} h_r(t)\right]^2 \Big/ h_{m-1,T}(t) \mathop{*}_{j=1}^{m-2} h_j(t)\right\}. \tag{44}$$

IV. BEHAVIOR OF THE COUNTING STATISTICS

In this section, we discuss the behavior of the counting distributions given by (17). For simplicity, we assume that the impulse response functions for all stages are identical exponential functions with areas α and time constants $\tau_p/2$, so that

$$h(t) = (2\alpha/\tau_p)\exp(-2t/\tau_p)u(t). \tag{45}$$

Here $u(t)$ is the unit step function.

In Fig. 4, we plot the count mean $\langle N_m(T)\rangle$, the count variance $\mathrm{Var}(N_m(t))$, and the ratio $F_m = \mathrm{Var}(N_m(T))/\langle N_m(T)\rangle$ versus the number of stages m, with α as a parameter, when $T/\tau_p \gg 1.0$. For $\alpha = 0.8$ (<1.0), $\langle N_m(T)\rangle$ and $\mathrm{Var}(N_m(T))$ have exponentially decaying behavior for large m; however, the ratio F_m approaches a constant as m becomes large, as is evident from (24). This is the same as for the SNDP, or in fact for any two-stage multiplied process in which the first stage is Poisson.[6,17] This is clearly a result of the decrease in mean and variance at each stage.

For $\alpha = 1.0$, $\langle N_m(t)\rangle$ is independent of m, but $\mathrm{Var}(N_m(T))$ and F_m are identical, monotonically increasing functions of m, thereby transparently reflecting the broadening of the distributions as the number of stages increases. For $\alpha = 1.2$, the three functions, $\langle N_m(T)\rangle$, $\mathrm{Var}(N_m(T))$, and F_m are dramatically increasing functions of m, as expected from (21), (23), and (24).

In Fig. 5, we exhibit the behavior of the counting distributions at the output of the mth stage ($m = 1,2,3,6,10$, and 50), with $T/\tau_p \gg 1.0$, for three different values of α, with μT constant. In Fig. 5(a) ($\alpha = 0.8$), the distributions move to the left as the mean decreases, and the variance also decreases as m increases. This is apparent from (21) and (23). In Fig. 5(b) ($\alpha = 1.0$), the mean remains fixed, but the character of the distributions changes dramatically as the number of stages increases. This reflects the accentuation of the clustering in the process by increasing m. If we consider the curves for $m = 10$ and 50 in Fig. 5(b), small dips around $n = 1$ can be observed. It can be shown that under certain conditions for μ, T, α, and m, $p_m(1) < p_m(0)$ and $p_m(1) < p_m(2)$. In Fig. 5(c) ($\alpha = 1.2$), the distributions move to the right, and the variances increase as m increases (the case for $m = 50$ is not shown). This can be understood from (21) and (23).

The counting distributions for a large number of stages ($m = 50$) is shown in Fig. 6, when $T/\tau_p \gg 1.0$. The mean of the output count is fixed at 10, and α is a parameter. An increasing multiplication parameter gives rise to an increasingly flat counting distribution for $n \neq 0$.

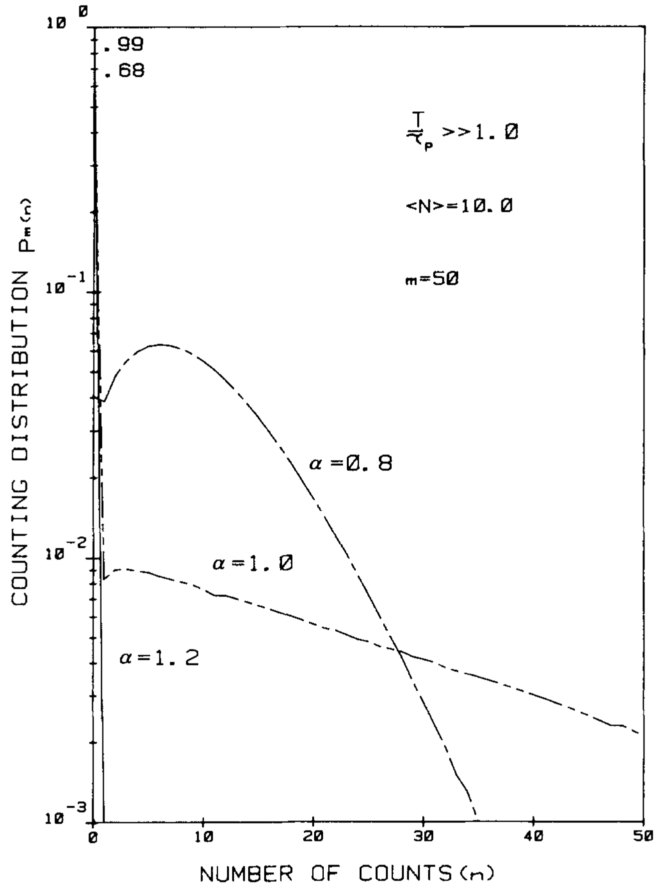


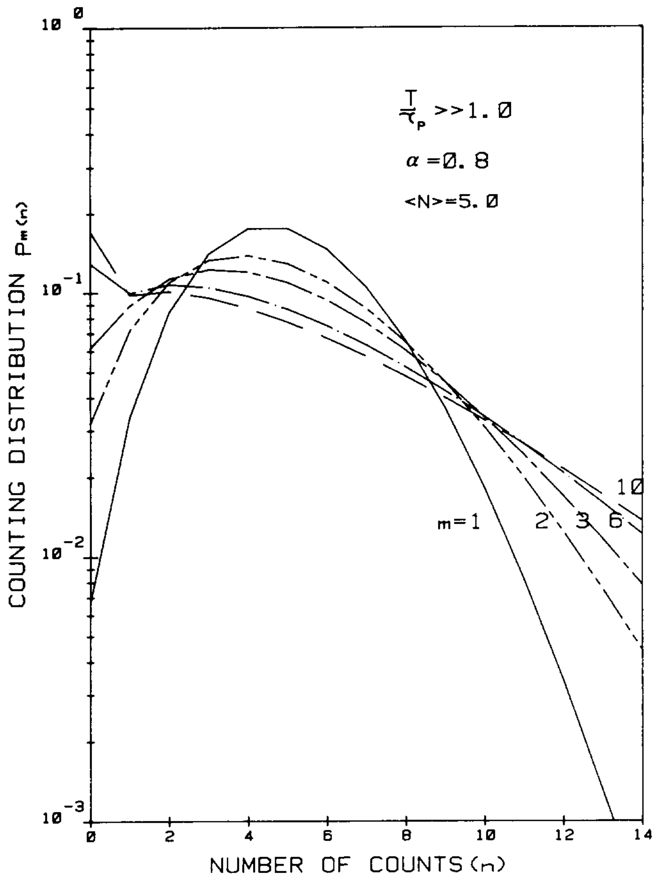
FIG. 6. Counting distribution $p_m(n)$ vs count number n for $m = 50$, $T/\tau_p \gg 1.0$, and $\langle N_m(T)\rangle = 10$.

In Fig. 7, we display the counting distributions for large T/τ_p with the output count mean $\langle N_m(T)\rangle$ fixed at 5, and with α as a parameter. Note that for fixed m, the distributions broaden as α increases. The distribution for large m and large α assumes a character resembling a delta function at $n = 0$, together with a flat component.

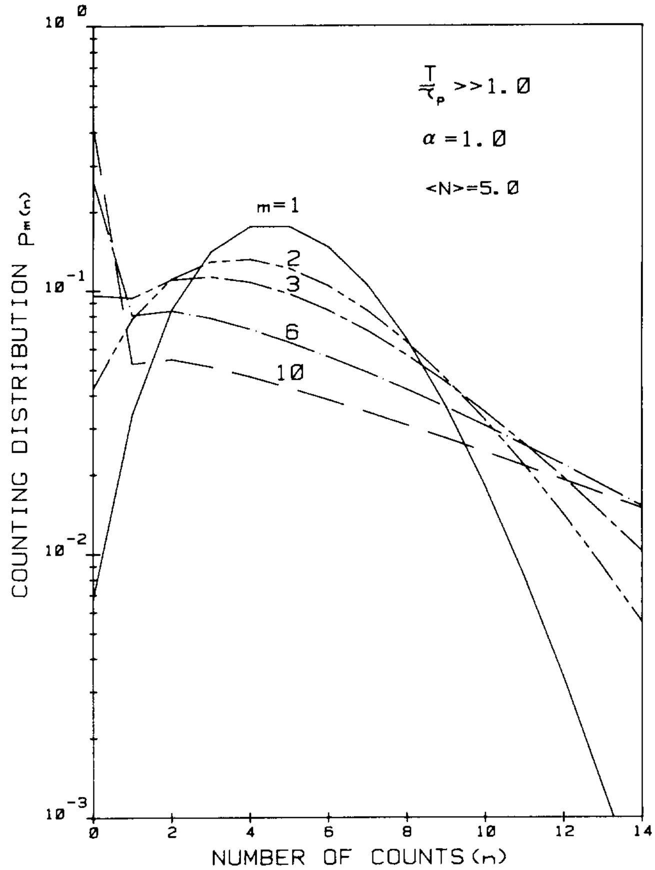
In Fig. 8, we display the dependence of the counting distributions on the ratio T/τ_p, the number of stages m, and the area of the impulse response function α. For all cases, the average number of counts $\langle N_m(T)\rangle$ is fixed at 5. In the limit where $\alpha T/\tau_p \ll 1.0$ and $T/\tau_p \ll 1.0$, the output of the first stage will be Poisson[13] so that, by induction, it is clear that the output of the cascade is also Poisson. Because of cumulative truncation and integration errors in the numerical calculations, it is quite difficult to obtain accurate counting statistics for arbitrary T/τ_p, for $m > 4$.

V. CONCLUSION

We have developed the statistics of a point process generated by a cascade of independent Poisson processes, and have found the moment generating function, as well as the counting and time statistics when dynamics are included.



(a)



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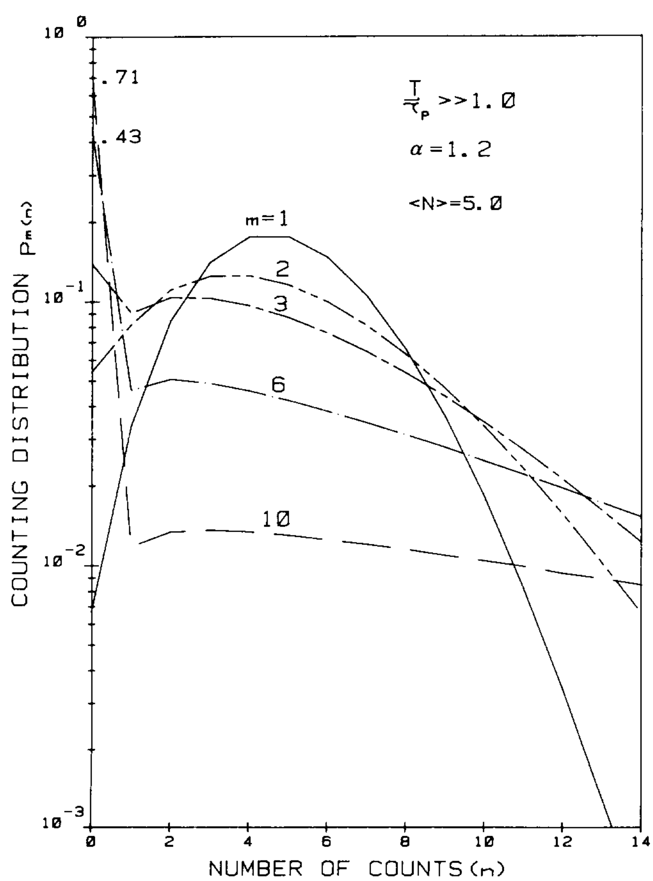
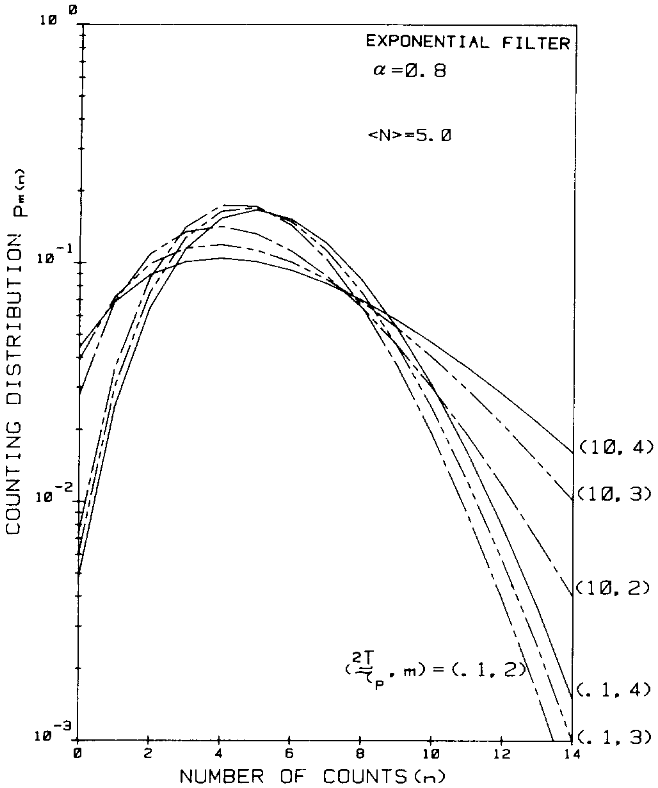
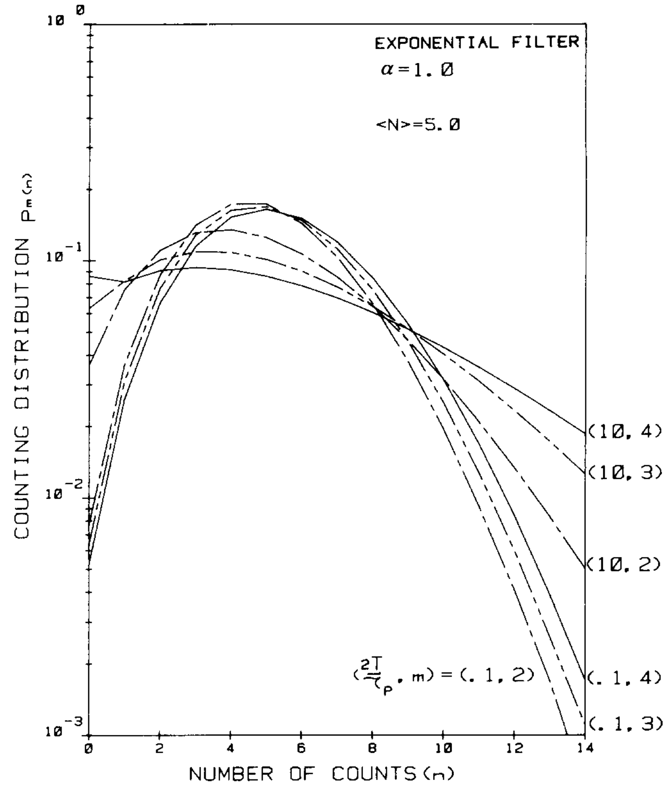


(c)

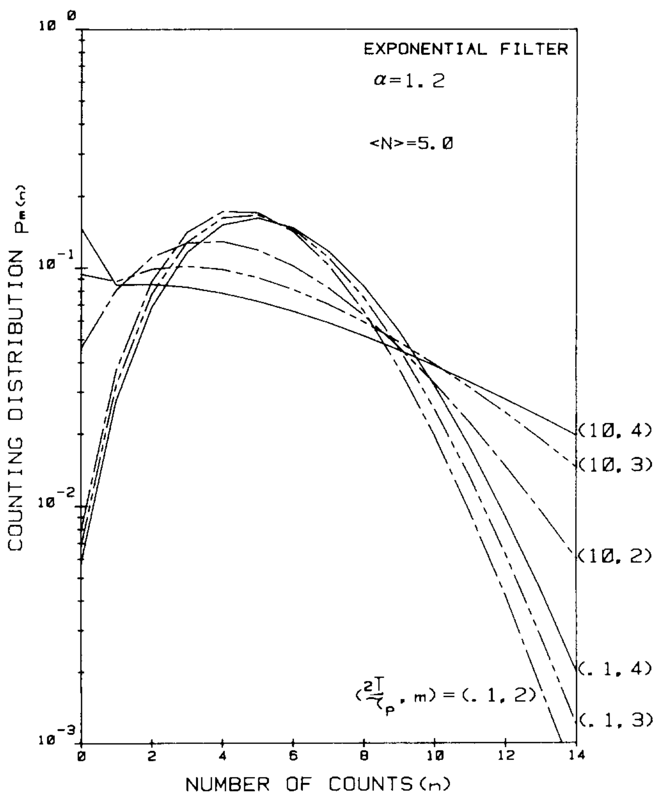
FIG. 7. Counting distribution $p_m(n)$ vs count number n for $T/\tau_p \gg 1.0$, $\langle N_m(T)\rangle = 5$, and $m = 1,2,3,6$, and 10. (a) $\alpha = 0.8$; (b) $\alpha = 1.0$; (c) $\alpha = 1.2$.



(a)



(b)



(c)

FIG. 8. Counting distribution $p_m(n)$ vs count number n with $2T/\tau_p$, m, and α as parameters. The impulse response function $h(t)$ is exponential with characteristic decay time $\tau_p/2$, and the mean count $\langle N_m(T)\rangle = 5.0$ for all cases. In the limit $T/\tau_p \to 0$, the counting distributions approach the Poisson, independent of m and α, whereas in the limit $T/\tau_p \to \infty$, the counting distributions approach those derived with instantaneous multiplication. (a) $\alpha = 0.8$; (b) $\alpha = 1.0$; (c) $\alpha = 1.2$.

Both the stationary and nonstationary cases have been considered. A simple expression for the variance-to-mean ratio at the mth stage has been obtained. We have carried out a parametric study of the counting distributions, by employing the DEC PDP 11/60 and IBM 4341 computers.

In some of the aforementioned applications of cascaded Poisson processes, a statistically independent additive Poisson point process may also be present, representing for example, broadband background light and/or thermionic emission in a photomultiplier tube. The counting statistics for the superposition process can be simply determined by the use of numerical discrete convolution. Our approach may be useful for describing the detection of light by the human visual system at threshold.[23,24] We have applied a similar analysis to branching Poisson processes, in which all initiating events are included in the final point process. The results of this study will be reported shortly.[25]

ACKNOWLEDGMENTS

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APPENDIX A: DERIVATION OF THE MOMENT GENERATING FUNCTIONAL FOR A FILTERED POISSON PROCESS AT THE jth STAGE

Let $\{X_j(t)\}$ be a filtered Poisson process in which points occur with intensity $\{X_{j-1}(t)\}$. The moment generating functional in the interval $(0,T)$ is given by[21]

$$L_{X_j}(s) \triangleq \left\langle \exp\left(-\int_0^T s(t) X_j(t)\, dt \right) \right\rangle, \tag{A1}$$

which is evaluated to be

$$L_{X_j}(s) = \left\langle \exp\left\{ \int_0^T X_{j-1}(t) \times \left[\exp\left(-\int_0^T h_j(\tau - t) s(\tau)\, d\tau \right) - 1 \right] dt \right\} \right\rangle. \tag{A2}$$

Proof: By using the conditional expectation and the property of the Poisson process, we have

$L_{X_j|X_{j-1}}(s)$

= the moment generating functional of $\{X_j(t)\}$ conditioned on the driving process $\{X_{j-1}(t)\}$

$$= \mathrm{Prob}(N_T = 0) + \sum_{k=1}^{\infty} \mathrm{Prob}(N_T = k) \times \left\langle \exp\left\{ -\sum_{n=1}^{k} \int_0^T h_j(t - \tau_n) s(t)\, dt \right\} \middle| N_T = k \right\rangle, \tag{A3}$$

where $\mathrm{Prob}(N_T = k)$ is the probability of having k events in $0 < t < T$. The summation within the expectation is unchanged by a random reordering of the occurrence times, $\tau_1, \tau_2, \ldots, \tau_k$. With this reordering, the occurrence times, given $N_T = k$, are independent and identically distributed, and the common density is

$$P_{\tau_n}(\tau) = X_{j-1}(\tau) \Big/ \int_0^T X_{j-1}(t)\, dt, \quad n = 1,2,\ldots,k.$$

Thus we obtain

$$\left\langle \exp\left\{ -\sum_{n=1}^{k} \int_0^T h_j(t - \tau_n) s(t)\, dt \right\} \middle| N_T = k \right\rangle$$

$$= \left\{ \int_0^T X_j(\tau) \exp\left(-\int_0^T h_j(t - \tau) s(t)\, dt \right) d\tau \Big/ \int_0^T X_{j-1}(t)\, dt \right\}^k.$$

Substituting this expression into (A3), and using a straightforward calculation with the Poisson distribution provides

$$L_{X_j|X_{j-1}}(s) = \exp\left\{ \int_0^T X_{j-1}(t) \times \left[\exp\left(-\int_0^T h_j(\tau - t) s(\tau)\, d\tau \right) - 1 \right] dt \right\}. \tag{A4}$$

To remove the conditioning of the process $\{X_j(t)\}$, we average (A4) over $\{X_{j-1}(t)\}$, to obtain (A2). Finally, setting $t = -\infty$ (assuming the process starts at $-\infty$) and $T = \infty$, we have

$$L_{X_j}(s) = \left\langle \exp\left\{ \int_{-\infty}^{\infty} X_{j-1}(t) \times \left[\exp\left(-\int_{-\infty}^{\infty} h_j(\tau - t) s(\tau)\, d\tau \right) - 1 \right] dt \right\} \right\rangle. \tag{A5}$$

APPENDIX B: DERIVATION OF THE COUNTING DISTRIBUTION AT THE mth STAGE

Examining (13), we perform the following substitutions:

$$\theta_{m-1}(t,s) = -s h_{m-1,T}(-t), \quad m \geqslant 2 \tag{B1a}$$

and

$$\theta_j(t,s) = \int_{-\infty}^{\infty} h_j(\tau - t) [\exp(\theta_{j+1}(\tau,s)) - 1]\, d\tau, \quad j = 1,2,\ldots,m-2. \tag{B1b}$$

Then (13) becomes

$$Q_{W_m}(s) = \exp\left\{ \mu \int_{-\infty}^{\infty} [\exp(\theta_1(t,s)) - 1]\, dt \right\}. \tag{B2}$$

Taking the $(n+1)$st derivative, with respect to s, on both sides of (B2) yields

$$\frac{\partial^{n+1}}{\partial s^{n+1}} Q_{W_m}(s) = \mu \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^{n-k}}{\partial s^{n-k}} Q_{W_m}(s) \times \frac{\partial^{k+1}}{\partial s^{k+1}} \int_{-\infty}^{\infty} \exp(\theta_1(t,s))\, dt. \tag{B3}$$

Using (16), together with the substitution

$$I^{(k+1)} = \frac{\partial^{k+1}}{\partial s^{k+1}} \int_{-\infty}^{\infty} \exp(\theta_1(t,s))\, dt \Big|_{s=1}, \tag{B4}$$

leads to a recurrence relation for the counting distribution at the mth stage, given by

$$(n+1) p_m(n+1) = \mu \sum_{k=0}^{n} \frac{(-1)^k}{k!} p_m(n-k) I^{(k+1)},$$

$$p_m(0) = Q_{W_m}(s)|_{s=1} = \exp\left\{ \mu \int_{-\infty}^{\infty} [\exp(\theta_1(t,s)) - 1]\, dt \right\} \Big|_{s=1}. \tag{B5}$$

Equation (B4) can be rewritten as

$$\begin{aligned} I^{(k+1)} &= \frac{\partial^{k+1}}{\partial s^{k+1}} \int_{-\infty}^{\infty} \exp(\theta_1(t,s))\, dt \Big|_{s=1} \\ &= \int_{-\infty}^{\infty} \frac{\partial^{k+1}}{\partial s^{k+1}} \exp(\theta_1(t,s))\, dt \Big|_{s=1} \\ &= \int_{-\infty}^{\infty} D_1^{(k+1)}(t)\, dt. \end{aligned} \tag{B6}$$

We have assumed that the order of integration and differentiation can be interchanged, and we have used the substitution

$$D_1^{(k+1)}(t) = \frac{\partial^{k+1}}{\partial s^{k+1}} \exp(\theta_1(t,s))\Big|_{s=1}. \tag{B7}$$

The $(k+1)$st derivative of the exponential function of (B1b) for $j = 1$ yields

$$\frac{\partial^{k+1}}{\partial s^{k+1}} \exp(\theta_1(t,s)) = \sum_{r=0}^{k} \binom{k}{r} \frac{\partial^{k-r}}{\partial s^{k-r}} \exp(\theta_1(t,s)) \times \int_{-\infty}^{\infty} h_1(\tau - t) \frac{\partial^{r+1}}{\partial s^{r+1}} \exp(\theta_2(\tau,s))\, d\tau. \tag{B8}$$

Substituting (B8) into (B7) gives rise to

$$D_1^{(k+1)}(t) = \sum_{r=0}^{k} \binom{k}{r} D_1^{(k-r)}(t) \times \int_{-\infty}^{\infty} h_1(\tau - t) D_2^{(r+1)}(\tau)\, d\tau. \tag{B9}$$

Similarly

$$D_j^{(k+1)}(t) = \sum_{r=0}^{k} \binom{k}{r} D_j^{(k-r)}(t) \int_{-\infty}^{\infty} h_j(\tau - t) D_{j+1}^{(r+1)}(\tau)\, d\tau, \quad j = 1,2,\dots,m-2, \tag{B10}$$

and

$$D_{m-1}^{(k)}(t) = \frac{\partial^k}{\partial s^k} \exp(\theta_{m-1}(t,s))\Big|_{s=1} = \{ -h_{m-1,T}(-t)\}^k \exp\{ -h_{m-1,T}(-t)\}. \tag{B11}$$

APPENDIX C: DERIVATION OF THE AUTOCOVARIANCE FUNCTION

From the Fourier transform (F.T.) of (30b) and (30c), we obtain

$$\widetilde{U}_k(\omega,t_j) = [H^*(\omega)]^k \widetilde{U}_0(\omega,t_j), \quad j = 1,2, \tag{C1}$$

where

$$\widetilde{U}_0(\omega,t_j) = \text{F.T. of } U_0(t,t_j).$$

Similarly, the Fourier transform of (30d) and (30e) yields

$$\widetilde{V}_k(\omega,t_1,t_2) = \sum_{r=0}^{k-1} \widetilde{\psi}_r(\omega,t_1,t_2)[H^*(\omega)]^{k-r}, \tag{C2}$$

where

$$\widetilde{\psi}_r(\omega,t_1,t_2) = \widetilde{U}_r(\omega,t_1) * \widetilde{U}_r^*(\omega,t_2).$$

Taking the inverse Fourier transform of (C1) and (C2), and substituting into (30a), results in (31).

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