Observation of Flat Counting Distribution for Poisson Process with Linearly Swept Mean*

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Gamma-ray counting experiments performed with a gated counting time which increases linearly are reported. The resultant counting distribution is flat over a large bandpass of counts as predicted earlier for photocounting experiments, or for any Poisson process, when the mean is linearly swept.

In a previous article,[1] we have reported that an extremely flat photocounting distribution over a tunable region of counts should generally be obtained from a radiation source with triangularly modulated intensity.[2] It was pointed out that this result applies to any Poisson process whose mean can be linearly swept. The counting distribution for such a process, in which all times are sampled equally, may be written as

$$p(n)=(2N)^{-1}\left(1-e^{-2N}\sum_{k=0}^{n}\frac{(2N)^k}{k!}\right), \qquad (1)$$

for the special case of unity modulation depth ($m=1$). The quantity N represents the overall mean count. This distribution is presented in Fig. 1, with mean count values of $N=5$, 10, and 15. It is apparent that the distribution is flat for a portion of the curve down to $n=0$. The degree of flatness is quite high: for $N=15$, the curve is flat to 0.2% on the interval $0\leq n\leq 15$. For large values of N, it may be seen from (1) that only at counts approaching $2N$ does the curve begin to depart[3] from its nearly constant value $1/2N$. This arises because the summation in the second term in (1) begins to overcome the small exponential factor for n near $2N$. In this paper, we present experimental results for a γ-ray counting experiment which verifies this theoretical finding.

The experiments were performed using a 5-mCi ^{57}Co source. The γ-rays were detected by a Nuclear Enterprises NE-102 plastic scintillator and an RCA 8575 photomultiplier tube. The photomultiplier output was passed through a discriminator and amplifier so that only the 123- and 137-keV gammas were counted. The scaler was activated by an EG & G gate generator with a variable gating pulse width. A run of experiments was performed by varying the gate pulse width T and recording the number of events for each pulse width. The process was repeated, increasing the gate time T in each successive experiment. Any individual set of measurements for a given value of T yields a Poisson distribution. But the total counting distribution obtained from the entire series of experiments has been predicted to be flat over some bandpass of counts.

The overall system count rate was $\nu=258$ counts/sec. One hundred fifty individual experiments were performed, recording 40 trials in each. The minimum gate time used was $T_0=1$ msec, and in each successive experiment the gate time was augmented by 1 msec, up to 150 msec. The results for such a complete run are shown in Fig. 2. The actual number of observations is given as a function of the count number n, along with the statistical counting error which is indicated by a vertical bar. Also shown in Fig. 2 is the theoretical result (1) for the experimentally measured mean ($N=19.45$). Good agreement is obtained, indicating that a flat counting distribution is indeed generated, although with the expected statistical uncertainty given by the square root of the number of observations. The flat value attained by the theoretical curve is $1/2N$ which, for the value of N indicated, is equal to 0.0257.

The rigorously correct theoretical result for this experiment is the average of M Poisson distributions with incremented means,

$$p(n)=M^{-1}\sum_{k=1}^{M}\left[\left(2N\frac{k}{M+1}\right)^n\Big/ n!\right]\exp\left(-2N\frac{k}{M+1}\right), \qquad (2)$$

where M is the number of experiments ($M=150$) and N is the overall mean count (here, $N=75.5\nu T_0$). This expression reflects the discrete sampling process actually used in the experiment, as distinct from the result based on an infinitesimal sampling time, with the mean *continuously* swept over the range from 0 to $2N$ for an overall mean N. The latter is given by

$$p(n)=(2N)^{-1}\int_0^{2N}\frac{N^n}{n!}e^{-N}dN, \qquad (3)$$

which integrates to (1). However, the agreement between the discrete experiment and the continuous theory reported previously[1] indicates that the distinction need not be important, and that in practice a flat counting distribution is obtainable with finite sampling time. It is noted that the ratio of lowest to highest gate times (1 msec/150 msec) is much less than unity, a condition for validity of the theory. The results given here correspond closely to 100% modulation for an experiment in which the mean is triangularly modulated, because the average count for the lowest gate time (1-msec)

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[1] M. C. Teich and P. Diament, Phys. Lett. **30A**, 93 (1969).

[2] Photocounting experiments using square-wave and sinusoidal intensity modulation of a laser have been reported by G. J. Troup, IEEE J. Quantum Electron. **1**, 398 (1965); by S. Fray, F. A. Johnson, R. Jones, T. P. McLean, and E. R. Pike, Phys. Rev. **153**, 357 (1967); and by P. Pearl and G. J. Troup, Phys. Lett. **27A**, 560 (1968).

[3] It has been shown in Ref. 1 that the curve is extremely flat for large values of N. For example, with $N=50$ and $m=1$, the curve is flat to 1 part in 10^8 on the interval $0\leq n\leq 50$.

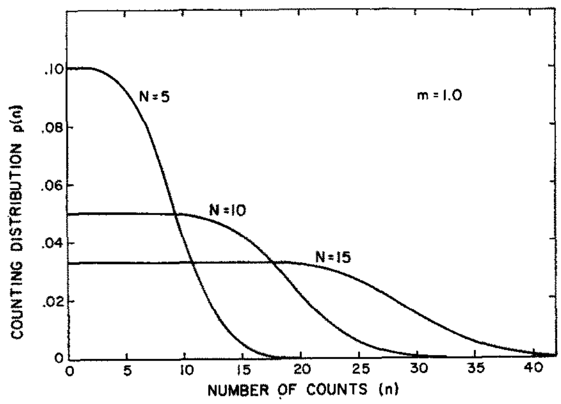

FIG. 1. Counting distribution $p(n)$ for 100% triangularly-modulated Poisson process with mean counts of $N=5$, 10, and 15.

experiment was kept near zero (it is about $\frac{1}{4}$). Thus the effect of the finite gate time,[1] which is qualitatively equivalent to a slightly reduced modulation depth and manifests itself by a decrease in the number of observations near $n=0$ [by about 12%, according to (2), for this experiment] is only barely discernible within the statistical error.

The data from this same experiment also verify the theory for the "bandpass" distribution predicted when the mean of a Poisson process is triangularly modulated to a depth m less than 100%. The continuously swept case yields the distribution[1]

$$p(n)=(N_1-N_0)^{-1}\int_{N_0}^{N_1}\frac{N^n e^{-N}}{n!}\,dN, \qquad (4)$$

when the modulation depth is $m=(N_1-N_0)/(N_1+N_0)$. This integrates to

$$p(n)=(N_1-N_0)^{-1}\times\left(\exp(-N_0)\sum_{k=0}^{n}\frac{N_0^k}{k!}-\exp(-N_1)\sum_{k=0}^{n}\frac{N_1^k}{k!}\right), \qquad (5)$$

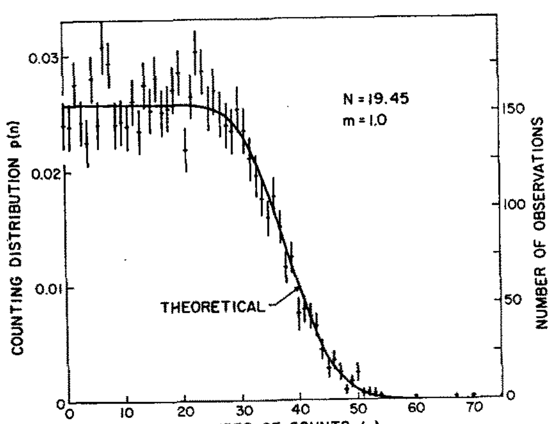

FIG. 2. Experimental and theoretical distributions for γ-ray counting with linearly swept mean. The overall mean count is $N=19.45$.

and is shown as the solid curve in Fig. 3 for a modulation depth of $m=0.76$ about an overall mean of $N=(N_1+N_0)/2=19.6$. Such partial triangular modulation is achieved in the actual experiment by considering only the data obtained for gate times between 19 msec and 133 msec. The resultant experimental data are also shown in Fig. 3 and fit the trapezoidal shape quite well, exhibiting the predicted flat range within the bandpass. Note that the rigorous result, accounting for the finite sample time, is actually

$$p(n)=(M_1-M_0+1)^{-1}\sum_{k=M_0}^{M_1}\frac{(Nk/M)^n}{n!}\exp(-Nk/M), \qquad (6)$$

where N is still the overall mean, $M=(M_1+M_0)/2$, and the modulation depth is best defined by

$$m=(M_1-M_0)/(M_1+M_0-2). \qquad (7)$$

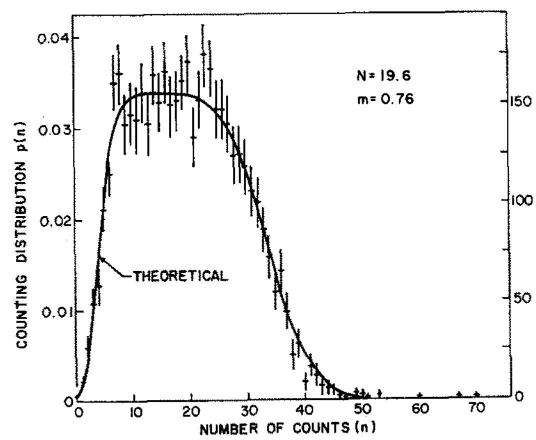

FIG. 3. Experimental and theoretical distributions for γ-ray counting with 76% triangular modulation. The overall mean count is $N=19.6$.

Comparison of the summation (6) with the integration (4) to which it tends for large (M_1-M_0) again confirms that the theory for a continuously swept mean gives accurate results, so long as sufficiently short finite sampling time increments are applied. Plots of the two theoretical curves are, in fact, barely distinguishable.

Although the sweeping was performed in this experiment by electronically varying the sampling width, it is emphasized that the bandpass feature of the counting distribution will be generated regardless of the mechanism used for linearly varying the mean. Thus, the triangular electro-optic modulation of a laser or a light bulb, or the triangular modulation of grid current in a vacuum tube containing an electron multiplier are also expected to yield a flat distribution for counting experiments performed under the specified conditions.

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