Two-Photon Counting Statistics for Laser and Chaotic Radiation*

M. C. TEICH AND PAUL DIAMENT

Department of Electrical Engineering, Columbia University, New York, New York 10027

(Received 13 September 1968)

The relation between the two-quantum photocounting distribution $p(n, T)$ and the intensity fluctuations of the radiation incident on the detector is obtained and compared with the one-quantum results. The actual statistics are evaluated for several cases of interest, including a chaotic source, an amplitude-stabilized wave, and the output of a single-mode laser (Van der Pol oscillator). The two-quantum count rate $\langle n \rangle / T$ is found to depend on the mean-square intensity of the radiation, in contrast to the one-quantum count rate which is proportional to the mean intensity. The effects of photon correlations in the radiation beam become apparent since the two-quantum distributions manifest a distinctly more positive second derivative than the corresponding one-quantum distributions. Both the low- and the high-count probabilities are therefore increased at the expense of counts near the mean. The quantum-theoretical treatment is found to be equivalent to the semiclassical treatment for density operators possessing a positive-definite weight function in the P-representation. Some possible experiments to verify the theory are discussed and shown to be feasible.

I. INTRODUCTION

Photoelectron counting statistics, and their relation to the intensity distribution of radiation incident on a photodetector, have been investigated intensively in the past several years.[1–13] Analyses of counting distributions have served to verify proposed theories of the behavior of the laser, particularly the changes in the statistical properties of the emitted radiation as it ranges from below to above the threshold of oscillation. Photocounting measurements can provide such further information about the radiation source as its spectral density and its higher-order correlation functions.[14]

For the ordinary single-quantum detector, the perturbation-theory formalism representing the photoelectric interaction of radiation and matter is carried only to first order. If this interaction is calculated to second order, the theory predicts that the ordinary effect will be absent and that double-quantum photoelectric emission will occur, provided that the incident photon energy $\hbar\omega$ and the work function of the photoelectric material $e\phi$ satisfy the relation

$$\tfrac{1}{2}e\phi < \hbar\omega < e\phi. \tag{1}$$

A discussion of this two-photon effect, which has been observed in several materials to date, has recently been given by Teich and Wolga.[15,16] It is assumed that single-photon emission from the Fermi tail[17] is negligible.

The present paper develops the relationships between intensity fluctuations of a radiation source and the resultant photocounting statistics for a two-quantum detector, and compares the results with those of the single-quantum case. For several cases of importance to experiments with thermal sources and lasers, actual photocounting distributions are presented. These include narrow-band Gaussian noise, amplitude-stabilized fields, combinations of these, and the output of a Van der Pol oscillator.

Knowledge of two-quantum photocounting distributions is of intrinsic interest for understanding the double-quantum detector. It can also provide information about the correlation functions of the radiation field;[18,19] in fact the mth factorial moment of the double-quantum photocounting distribution reflects a $2m$th order correlation function, while for the single-quantum case, it corresponds to only an mth order correlation function. In particular, the one-quantum counting rate is described by a first-order correlation function, while the rate for a two-quantum process is given by a second-order correlation function.[19] Furthermore, when the density operator of the field[14] is not

* This work was supported in part by the National Science Foundation under Grant No. NSF-GK-3620.

1 L. Mandel, Proc. Phys. Soc. (London) **74**, 233 (1959).
2 L. Mandel, in *Progress in Optics*, E. Wolf, Ed. (North-Holland Publishing Company, Amsterdam, 1963), Vol. 2, p. 181.
3 L. Mandel, E. C. G. Sudarshan, and E. Wolf, Proc. Phys. Soc. (London) **84**, 435 (1964).
4 J. A. Armstrong and A. W. Smith, in *Progress in Optics*, E. Wolf, Ed. (North-Holland Publishing Company, Amsterdam, 1967), Vol. 6, p. 213.
5 A. W. Smith and J. A. Armstrong, Phys. Rev. Letters **16**, 1169 (1966).
6 R. F. Chang, R. W. Detenbeck, V. Korenman, C. O. Alley, Jr., and U. Hochuli, Phys. Letters **25A**, 272 (1967).
7 C. Freed and H. A. Haus, Phys. Rev. **141**, 287 (1966).
8 C. Freed and H. A. Haus, IEEE J. Quantum Electron. **QE-2**, 190 (1966).
9 F. T. Arecchi, A. Berné, and P. Bulamacchi, Phys. Rev. Letters **16**, 32 (1966).
10 P. L. Kelley and W. H. Kleiner, Phys. Rev. **136**, A316 (1964).
11 R. J. Glauber, in *Physics of Quantum Electronics*, P. L. Kelley, B. Lax, and P. E. Tannenwald, Eds. (McGraw-Hill Book Company, New York, 1966), p. 788.
12 R. J. Glauber, in *Quantum Optics and Electronics*, C. deWitt, A. Blandin, and C. Cohen-Tannoudji, Eds. (Gordon and Breach Science Publishers, Inc., New York, 1965), 1st ed., p. 65.
13 B. R. Mollow, Phys. Rev. **168**, 1896 (1968).
14 R. J. Glauber, Phys. Rev. **130**, 2529 (1963); Phys. Rev. **131**, 2766 (1963).
15 M. C. Teich and G. J. Wolga, Phys. Rev. **171**, 809 (1968).
16 M. C. Teich, J. M. Schroeer, and G. J. Wolga, Phys. Rev. Letters **13**, 611 (1964).
17 M. C. Teich and G. J. Wolga, J. Opt. Soc. Am. **57**, 542 (1967).
18 L. Mandel and E. Wolf, Rev. Mod. Phys. **37**, 231 (1965).
19 M. C. Teich and G. J. Wolga, Phys. Rev. Letters **16**, 625 (1966).

factorable into a product of single-mode density operators, the two-photon process may provide information which can not be obtained from single-photon counting.[20] It would therefore be of interest to compare the results of the following two-quantum photocounting theory with experiment.

II. THEORY

Consider a quasimonochromatic, linearly polarized beam of radiation, such as that produced by a laser or by a thermal source, incident on a two-quantum detector. If the radiation fluctuates in intensity, the probability of emission will vary with time. Within the framework of the semiclassical theory of the interaction of radiation and matter, the probability per unit time of photoemission is expressible as[15] $\beta I^2(t)$, where $I(t)$ is the short-time average of the light intensity over a few cycles of the optical frequency or, more rigorously, the product of the analytic signal component of the electric field with its complex conjugate.[18] This result is obtainable through a calculation similar to that presented by Mandel *et al.*,[3] but using second-order perturbation theory. The quadratic dependence on intensity for the two-quantum detector contrasts with the usual detection probability $\alpha I(t)$ for a single-quantum process.

The factor β in the detection probability is an efficiency for the two-quantum process and includes an area dependence. It is assumed that the radiation is uniformly distributed over the photodetector area A. To make correspondence with the two-quantum yield Λ defined in an earlier publication,[15] note that $\beta I^2 = \Lambda P/e$, where $P(t)$ is the radiation power, so that $\beta \equiv \Lambda A/eI$.

With βI^2 replacing the usual αI, the argument that the probability of emission of a photoelectron within an infinitesimal time interval is independent of the emission at other times[7] applies as well to the two-quantum detector, yielding

$$p(n, T, t) = \langle M^n e^{-M}/n!\rangle, \tag{2}$$

with

$$M = \int_t^{t+T} \beta I^2(t')\,dt', \tag{3}$$

for the probability $p(n, T, t)$ that n photoelectrons are emitted within the time interval $(t, t+T)$. This is an ensemble average, over the intensity fluctuation statistics, of the underlying Poisson process. It has been assumed that the lifetime of the intermediate state of the two-photon absorption is much shorter than the coherence time, a condition that is generally well satisfied.[19] Except for the replacement of the one-quantum average $\int \alpha I dt$ by the two-quantum average $\int \beta I^2 dt$, the expression for $p(n, T, t)$ is the usual one.[3,18]

Correspondence with the quantum-theoretical treatment may be made in terms of the factorial moments of the photocounting distribution (2). For the two-quantum case, these are

$$\langle n!/(n-m)!\rangle = \langle M^m\rangle = \left\langle \left[\int_t^{t+T} \beta I^2(t')\,dt'\right]^m \right\rangle$$
$$= \beta^m \int_t^{t+T} \cdots \int_t^{t+T} \langle I(t_1)I(t_1)I(t_2)I(t_2)\cdots I(t_m)I(t_m)\rangle\, dt_1\cdots dt_m, \tag{4}$$

where the average on the left is over the photoemission statistics while that on the right is with respect to the light fluctuation statistics. The last form in (4) is presented for direct comparison with the quantum-theoretical expression for the mth factorial moment, given by

$$\langle n!/(n-m)!\rangle = \beta^m \int_t^{t+T} \cdots \int_t^{t+T} G^{(2m)}(t_1t_1\cdots t_mt_m, t_mt_m\cdots t_1t_1)\, dt_1\cdots dt_m. \tag{5}$$

Here, $G^{(2m)}$ is the $2m$th order correlation function of the radiation field and is expressed by

$$G^{(2m)} = \mathrm{Tr}[\rho E^-(t_1)E^-(t_1)\cdots E^-(t_m)E^-(t_m)E^+(t_m)E^+(t_m)\cdots E^+(t_1)E^+(t_1)]. \tag{6}$$

The quantity ρ represents the density operator for the field, and E^- and E^+ are the negative- and positive-frequency portions of the electric field operator E, respectively. Reference to the spatial coordinates in the correlation function has been omitted since $G^{(2m)}$ is assumed to be constant over the surface of the photocathode. It may be seen that the $2m$th order correlation function replaces the mth order one appearing in the one-quantum expression for the factorial moments as given by Glauber.[11]

If the electromagnetic field has a density operator which possesses a P-representation with a positive definite weight function, an analysis similar to Glauber's, but for the two-quantum case, produces a counting distribution expressible in the form of (2, 3). The attenuated form of a classical field,[11] which is appropriate for usual radiation sources, has such a weight function and the quantum-theoretical result corresponds to

[20] P. Lambropoulos, Phys. Rev. **168**, 1418 (1968).

that obtained from the semiclassical theory, as given in (2, 3).

The evaluation of the statistical mean in (2) requires a knowledge of all the higher-order joint probability density functions of the radiation in the general case.[21] The extreme cases of very short and very long observation times T can, however, be treated readily. If the counting interval T is short compared to the inverse bandwidth of the light fluctuations, then the parameter M of the Poisson process reduces to

$$M=\int_t^{t+T} \beta I^2(t')dt'=\beta I^2(t)T. \qquad (7)$$

For simplicity, we assume that the radiation field is stationary and ergodic, so that the probability $p(n, T, t)$ is independent of t and may be written as $p(n, T)$. The counting distribution for the two-quantum detector then becomes

$$p(n, T)=[(\beta T)^n/n!]\int_0^\infty I^{2n}\exp(-\beta TI^2)P(I)dI, \qquad [2Q] \quad (8)$$

where $P(I)$ is the probability density function for the light intensity. This result is to be compared with the one-quantum counting distribution, given by

$$p(n, T)=[(\alpha T)^n/n!]\int_0^\infty I^n\exp(-\alpha TI)P(I)dI. \qquad [1Q] \quad (9)$$

The bracketed symbols [1Q], [2Q] are used throughout to distinguish the one- and two-quantum cases.

For purposes of calculating the two-quantum photocounting statistics to be expected for various intensity distributions, we normalize to some convenient intensity I_1 and introduce a frequency ν to characterize the quantum efficiency of either the one- or two-photon detector at that intensity:

$$\nu=\alpha I_1 \quad \text{or} \quad \nu=\beta I_1^2. \qquad (10)$$

This frequency will shortly be related to observable quantities in counting experiments. The photocounting statistics (8, 9) for the two types of detector become

$$p(n, T)=[(\nu T)^n/n!]\int_0^\infty x^n\exp(-\nu Tx)P(I_1x)I_1dx \qquad [1Q] \quad (11)$$

and

$$p(n, T)=[(\nu T)^n/n!]\int_0^\infty x^{2n}\exp(-\nu Tx^2)P(I_1x)I_1dx, \qquad [2Q] \quad (12)$$

where $x=I/I_1$ is the normalized intensity and $P(I_1x)=P(I)$ is the distribution of radiation intensity.

[21] G. Bédard, J. C. Chang, and L. Mandel, Phys. Rev. **160**, 1496 (1967).

It is simpler to consider the generating function[22] for $p(n, T)$, which describes the counting statistics through a continuous parameter s, rather than the discrete one n, and also yields observable parameters of the statistics more readily than does $p(n, T)$ itself. The generating function, for which several definitions are in common use, will here be taken simply as the average of s^n over the counting distribution:

$$\langle s^n\rangle=\sum_{n=0}^{\infty} p(n, T)s^n. \qquad (13)$$

Consequently, the counting distribution $p(n, T)$ can always be recovered from $\langle s^n\rangle$ by expanding in powers of s.

Introducing this definition into (11, 12), the generating function of the counting statistics is found to be related to those of the light fluctuations for the one- and two-quantum detectors by

$$\langle s^n\rangle=\int_0^\infty \exp(-\sigma x)P(I_1x)I_1dx \qquad [1Q] \quad (14)$$

$$\langle s^n\rangle=\int_0^\infty \exp(-\sigma x^2)P(I_1x)I_1dx, \qquad [2Q] \quad (15)$$

where the quantity

$$\sigma\equiv\nu T(1-s), \qquad (16)$$

replaces s as the primary parameter. The relation between the generating function of the photocounting statistics and the distribution of light intensity is seen to be a Laplace transform for a single-quantum detector and a Gaussian transform for a double-quantum detector.

Besides simplifying the mathematical relation between the radiation and the counting statistics, the use of the generating function permits observables such as the factorial moments of the counting statistics to be displayed explicitly by merely expanding $\langle s^n\rangle$, which emerges from (14) or (15) as a function of σ, in a power series in σ. Termwise comparison with the series

$$\langle s^n\rangle=\sum_{m=0}^{\infty}[(-1)^m/m!][\sigma^m/(\nu T)^m]\langle n!/(n-m)!\rangle, \qquad (17)$$

obtained by substituting σ for s through (16) and applying the binominal theorem, yields the successive factorial moments $\langle n!/(n-m)!\rangle$ directly.

In particular, the first factorial moment or mean count is, for the two types of detector,

$$\langle n\rangle=\nu T\langle x\rangle=\alpha\langle I\rangle T \qquad [1Q] \quad (18)$$

and

$$\langle n\rangle=\nu T\langle x^2\rangle=\beta\langle I^2\rangle T. \qquad [2Q] \quad (19)$$

This basic result is in itself significant, for it shows that

[22] E. Parzen, *Modern Probability Theory and its Applications* (John Wiley & Sons, Inc., New York, 1960), p. 215.

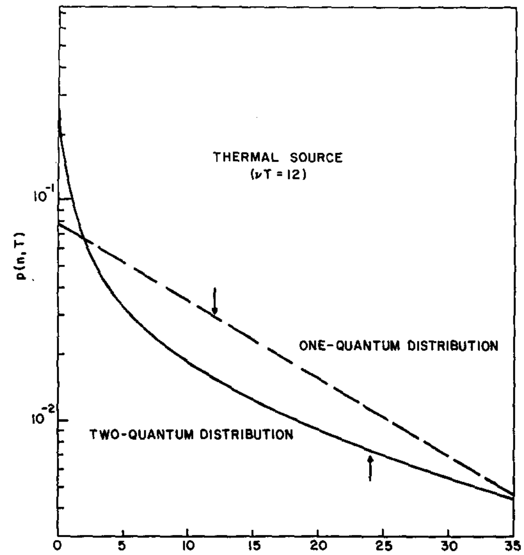

FIG. 1. One- and two-quantum photoelectron counting distributions for a Gaussian source. Arrows indicate mean counts. The count rate for the two-quantum detector is twice that for the one-quantum detector, with $\nu T=12$ in each case.

the count rate measures different moments of the intensity distribution in the two types of detector. If the source statistics, in particular the variance, can be varied without changing the mean intensity, the count rate will respond to the variation when the two-quantum detector is used, but only higher moments than the mean will be affected in the case of a one-quantum detector. This property will be made more explicit below, after the photocounting statistics corresponding to various light intensity distributions are considered in detail.

III. PHOTOCOUNTING STATISTICS

The photoelectric counting statistics to be expected from both a single- and a double-quantum detector illuminated by radiation of various intensity distributions $P(I)$ may be obtained by evaluating the Laplace and Gaussian transforms (14, 15). We consider below several cases of interest, which have already been treated in the one-quantum case,[4] and give the results for both cases for comparison.

A. Narrow-band Gaussian Noise

Chaotic sources, including thermal radiators and the output from a laser below threshold, are characterized by a Gaussian amplitude distribution of the electric field, and hence an exponential distribution of the intensity.[14,18] If I_1 is the mean intensity, then

$$I_1P(I_1x)=e^{-x}, \tag{20}$$

and the generating functions of the counting distributions for one- and two-quantum detectors are, from (14, 15),

$$\langle s^n\rangle=1/(1+\sigma) \qquad [1\mathrm{Q}] \tag{21}$$

and

$$\langle s^n\rangle=\pi^{1/2}(\tfrac{1}{2}\sigma^{-1/2})\exp[1/(4\sigma)]\operatorname{erfc}(\tfrac{1}{2}\sigma^{-1/2}). \qquad [2\mathrm{Q}] \tag{22}$$

The factorial moments are hence given by

$$\langle n!/(n-m)!\rangle=(\nu T)^m m! \qquad [1\mathrm{Q}] \tag{23}$$

and

$$\langle n!/(n-m)!\rangle=(\nu T)^m(2m)!. \qquad [2\mathrm{Q}] \tag{24}$$

The actual photocount statistics for the two detectors are correspondingly,

$$p(n,T)=(\nu T)^n/(1+\nu T)^{n+1} \qquad [1\mathrm{Q}] \tag{25}$$

and

$$p(n,T)=[(2n)!/n!][\pi^{1/2}/2(\nu T)^{1/2}]$$
$$\times\exp[1/(4\nu T)]\mathrm{i}^{2n}\operatorname{erfc}[1/2(\nu T)^{1/2}]. \qquad [2\mathrm{Q}] \tag{26}$$

The function $\mathrm{i}^n\operatorname{erfc} x$ is the nth repeated integral of the error function,[23] the salient properties of which are indicated in the Appendix. The photocounting statistics for both cases are plotted in Fig. 1. The former, (25) is the well-known Bose–Einstein or geometric distribution.

B. Amplitude Stabilized Source

If only phase fluctuations are present in the radiation, so that the intensity is strictly constant at I_1, then the intensity distribution is

$$P(I)=\delta(I-I_1), \tag{27}$$

and both the Laplace and the Gaussian transform yield

$$\langle s^n\rangle=e^{-\sigma}, \tag{28}$$

corresponding to a Poisson distribution for both the single- and the double-quantum cases. The factorial moments are then

$$\langle n!/(n-m)!\rangle=(\nu T)^m. \tag{29}$$

The counting statistics,

$$p(n,T)=e^{-\nu T}(\nu T)^n/n!, \tag{30}$$

differ from a one- to a two-quantum detector only in that their different efficiencies assign them different values of ν. The Poisson counting distribution should be observed when the source is a well-stabilized laser very far above threshold.

C. Amplitude Stabilized Signal Plus Narrow-Band Gaussian Noise

A possible model for radiation from a laser operating above threshold is Gaussian noise superimposed on a signal of stable amplitude.[7–9,24] This model is a good approximation to the nonlinear oscillator representation

[23] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (National Bureau of Standards, U.S. Govt. Printing Office, Washington, 1964), p. 297.
[24] P. J. Magill and R. P. Soni, Phys. Rev. Letters **16**, 911 (1966).

for output intensities which are at least five times threshold.[4] Provided that the detector bandwidth overlaps both the signal and noise spectra, the resultant intensity distribution, normalized to the noise intensity I_N, is[24]

$$I_N P(I_N x) = \exp[-(x+y)] I_0(2[xy]^{1/2}). \qquad (31)$$

Here $y = I_C/I_N$ is the ratio of the average coherent signal and noise intensities, and $I_0(x)$ is the modified Bessel function. The corresponding counting statistics for the single-quantum detector are expressed by

$$\langle s^n \rangle = \exp[-y\sigma/(1+\sigma)]/(1+\sigma)$$
$$= \sum_{n=0}^{\infty} (-1)^n \sigma^n L_n(-y), \qquad [1Q] \qquad (32)$$

where the expansion in terms of Laguerre polynomials $L_n(x)$ is given for comparison with the two-quantum case, which yields

$$\langle s^n \rangle = \sum_{n=0}^{\infty} (-1)^n \sigma^n [(2n)!/n!] L_{2n}(-y). \qquad [2Q] \qquad (33)$$

The factorial moments are given by

$$\langle n!/(n-m)! \rangle = (\nu T)^m m! L_m(-y) \qquad [1Q] \qquad (34)$$

and

$$\langle n!/(n-m)! \rangle = (\nu T)^m (2m)! L_{2m}(-y). \qquad [2Q] \qquad (35)$$

Note that although the normalization has here been made to I_N, so that $\nu = \alpha I_N$ or βI_N^2, for large S/N ratios the results revert to those of the noiseless case, with normalization to I_C, since the leading term in $[m! L_m(-y)]$ is y^m. The counting distributions are

$$p(n, T) = [(\nu T)^n/(1+\nu T)^{n+1}]$$
$$\times \exp[-y\nu T/(1+\nu T)] L_n[-y/(1+\nu T)] \qquad [1Q] \qquad (36)$$

and

$$p(n, T) = (e^{-y}/n!)[\pi^{1/2}/2(\nu T)^{1/2}] \exp[1/(4\nu T)]$$
$$\times \sum_{m=0}^{\infty} [(2n+m)!/m!m!] \mathrm{i}^{2n+m}\mathrm{erfc}[1/2(\nu T)^{1/2}]$$
$$\times [y/(\nu T)^{1/2}]^m. \qquad [2Q] \qquad (37)$$

D. Amplitude Stabilized Signal Plus Independent Noise

If the Gaussian noise superimposed on the stable signal is well separated in frequency from the signal, and the beat frequency does not lie within the detector bandwidth, then only the noise fluctuations contribute to the intensity variations. This case arises when radiation from nonlasing modes, at frequencies far from those of the lasing mode, is incident on the detector.[4] The effective intensity distribution is then

$$P(I) = (1/I_N) \exp[-(I-I_C)/I_N], \qquad I \geq I_C, \qquad (38)$$

and zero otherwise. The corresponding generating functions for the counting statistics are

$$\langle s^n \rangle = e^{-\sigma y}/(1+\sigma) \qquad [1Q] \qquad (39)$$

and

$$\langle s^n \rangle = e^y (\pi^{1/2}/2\sigma^{1/2}) \exp[1/(4\sigma)] \,\mathrm{erfc}(y\sigma^{1/2} + \tfrac{1}{2}\sigma^{-1/2}), \qquad [2Q] \qquad (40)$$

with y and σ defined as in the previous case. The factorial moments are, respectively,

$$\langle n!/(n-m)! \rangle = (\nu T)^m e^y \Gamma(m+1, y) \qquad [1Q] \qquad (41)$$

and

$$\langle n!/(n-m)! \rangle = (\nu T)^m e^y \Gamma(2m+1, y), \qquad [2Q] \qquad (42)$$

where $\Gamma(n, x)$ is the incomplete gamma function. The counting statistics are given by

$$p(n, T) = [(\nu T)^n/(1+\nu T)^{n+1}](e^y/n!)$$
$$\times \Gamma[n+1, (1+\nu T)y] \qquad [1Q] \qquad (43)$$

and

$$p(n, T) = [(2n)!/n!] e^y [\pi^{1/2}/2(\nu T)^{1/2}] \exp[1/(4\nu T)]$$
$$\times \sum_{m=0}^{2n} (y^m/m!) \mathrm{i}^{2n-m}\mathrm{erfc}[y(\nu T)^{1/2} + \tfrac{1}{2}(\nu T)^{-1/2}]. \qquad [2Q] \qquad (44)$$

The availability of the parameter y, the S/N ratio, for fitting theoretical statistics to observations make this and the preceding model a significant improvement over the single-parameter models of laser radiation.

E. Nonlinear Oscillator

Armstrong and Smith,[4] and more recently Chang *et al.*[6] have shown through photocounting experiments that the behavior of a single-mode laser is well described throughout the range from well below to near to well above threshold by Risken's intensity distribution,[25]

$$P(I) = (2/\pi^{1/2}) \{\exp - [(I/I_1) - w]^2 / I_1(1 + \mathrm{erf} w)\}. \qquad (45)$$

Since $I \geq 0$, this is a truncated Gaussian. The parameter w describes the state of excitation of the laser; it is negative below threshold, zero at threshold, and positive above threshold.[4,25] The normalization is such that, far above threshold, the average intensity is wI_1. Throughout its range of excitation, the average laser output is given by

$$\langle I \rangle = I_1 \left[w + \frac{\exp(-w^2)/\pi^{1/2}}{(1+\mathrm{erf} w)} \right]. \qquad (46)$$

Far above threshold, for large positive w ($w > 2$), the intensity distribution is Gaussian, with mean wI_1 and

[25] H. Risken, Z. Physik **186**, 85 (1965); V. Arzt, H. Haken, H. Risken, H. Sauerman C. Schmid, and W. Weidlich, Z. Physik **197**, 207 (1966).

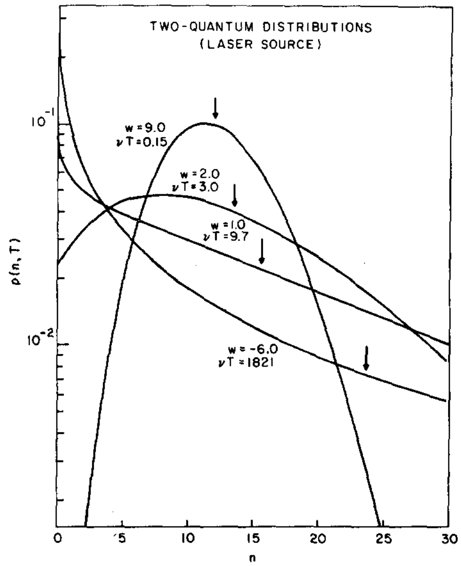

FIG. 2. Double-quantum counting distributions for a laser source (Risken intensity distribution) ranging from well below to well above the threshold of oscillation. The curves are normalized to the same mean intensity. Arrows indicate the mean counts, which vary with changing radiation statistics.

standard deviation $I_1/\sqrt{2}$. At threshold ($w=0$), the distribution is the positive-argument half of a Gaussian peaked at $I=0$, with a mean intensity of $I_1/\pi^{1/2}$. Far below threshold, for large negative w ($w<-3$), the distribution reverts essentially to the exponential intensity distribution of Gaussian noise, but with a slight correction factor introduced by the nonlinear source; the average output is then $I_1/2(-w)$. The Risken distribution, derived from a nonlinear Fokker–Planck equation,[25,26] accounts for the nonlinearity of the laser, applies near threshold as well as at extremes of its state of excitation, and has the particular virtue of describing the intensity variations through a single parameter w.

The generating functions for the counting statistics to be expected from a nonlinear source so described are, for the single- and double-quantum detectors, respectively,

$$\langle s^n\rangle = \exp[(\sigma/2)-w]^2 \operatorname{erfc}[(\sigma/2)-w]/[\exp(w^2)\times \operatorname{erfc}(-w)] \quad [1Q] \quad (47)$$

and

$$\langle s^n\rangle = u\exp(u^2)\operatorname{erfc}(-u)/w\exp(w^2)\operatorname{erfc}(-w),\quad u=w/(1+\sigma)^{1/2}. \quad [2Q] \quad (48)$$

The factorial moments are, correspondingly,

$$\langle n!/(n-m)!\rangle = (\nu T)^m m!\, \mathrm{i}^m\operatorname{erfc}(-w)/\operatorname{erfc}(-w) \quad [1Q] \quad (49)$$

and

$$\langle n!/(n-m)!\rangle = (\nu T)^m (2m)!\, \mathrm{i}^{2m}\operatorname{erfc}(-w)/\operatorname{erfc}(-w). \quad [2Q] \quad (50)$$

The actual photocounting distributions are given by

$$p(n,T) = (\nu T)^n \frac{\exp[(\nu T/2)-w]^2\, \mathrm{i}^n\operatorname{erfc}[(\nu T/2)-w]}{\exp(w^2)\operatorname{erfc}(-w)} \quad [1Q] \quad (51)$$

and

$$p(n,T) = [(2n)!/n!][\nu T/(1+\nu T)]^n \times [v\exp(v^2)\, \mathrm{i}^{2n}\operatorname{erfc}(-v)/w\exp(w^2)\operatorname{erfc}(-w)], \quad [2Q] \quad (52)$$

with

$$v = w/(1+\nu T)^{1/2}. \quad (53)$$

Representative two-quantum counting distributions are presented in Fig. 2 for a range of laser excitation from below to above threshold. Figures 3 and 4 present comparisons of one- and two-quantum counting distributions for parameters selected as explained in the next section.

IV. DISCUSSION AND CONCLUSIONS

For all the radiation sources investigated, with the exception of the wave of perfectly stabilized amplitude, the two-quantum photocounting distributions have been found to be broader and flatter than the corresponding one-quantum distributions. This is expected, since two photons are required for each absorption in the double-quantum detector. The consequences of photon bunching will therefore be accentuated, giving rise to a

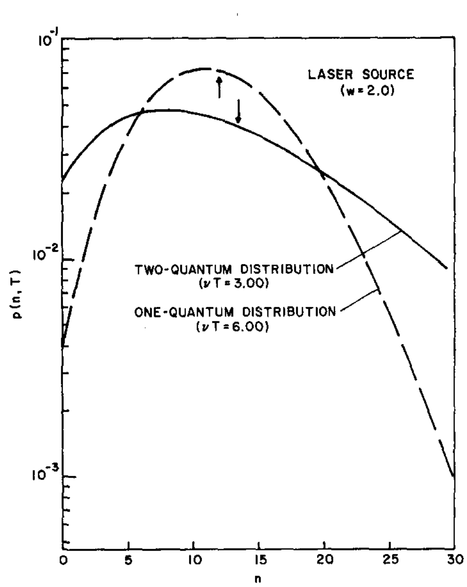

FIG. 3. Comparison of one- and two-quantum photocounting distributions for a laser somewhat-above threshold.

[26] R. D. Hempstead and M. Lax, Phys. Rev. 161, 350 (1967).

heavier representation of intervals of time containing either fewer or more photoelectrons relative to the mean. The curvature (second derivative) of the two-quantum probability distribution is therefore more positive in all cases than for the corresponding one-quantum cases. This may be seen in Figs. 1, 3, 4, where the single- and double-quantum distributions are compared for the thermal source and for the Van der Pol oscillator. For third-[27] or higher-order photoemission, this effect would be even more pronounced.

It is observed that the two-quantum distribution approaches the one-quantum distribution as the nonlinear oscillator is driven far above threshold. With increasing laser excitation, represented by larger w in the Risken distribution, the intensity fluctuations of the radiation source become smaller, and the two distributions become more alike, as may be seen in Fig. 4. For the case of the ideal amplitude-stabilized wave, there is neither intensity fluctuation nor photon bunching, and the probability distributions of both become Poisson. Far below threshold, the two-quantum distributions from a laser and a thermal source become identical, as is shown by a comparison of Figs. 1 and 2.

In plotting the results for the laser source, we have fixed the single- and double-quantum mean counts at the same value $\langle n\rangle = 12$ with the laser far above threshold ($w \gg 1$), and have maintained the average laser output intensity at a constant value (in an experiment this may be done by external means, e.g., attenuators) as the excitation parameter w is varied. This procedure manifests the change in the two-quantum count rate resulting from the variation in radiation statistics as w is varied, by countering the differences between the single- and double-quantum count rates arising from the I vs I^2 dependence. For this purpose, the value of νT must vary and is indicated for each curve in the figures. The arrows indicate the mean of the distribution, which is seen to vary with the radiation statistics, as mentioned previously.

Figure 5 presents the ratio of two-quantum to one-quantum count rates for a laser source as a function of the laser-excitation parameter w, under conditions that maintain a constant mean intensity, as above. In terms of the radiation statistics, this ratio is the reduced second moment $\langle x^2\rangle / \langle x\rangle^2$. In terms of the observable photocounting statistics such as those obtained in Sec. III, $\langle x^2\rangle = \langle n\rangle / \nu T$ for a two-quantum detector and $\langle x\rangle = \langle n\rangle / \nu T$ for the one-quantum case; i.e., the first factorial moments normalized to νT. Physically, the ratio reflects the changes in the form of the intensity distribution $P(I)$, independent of effects due to variations in the intensity level. For large negative values of w, i.e., far below threshold, the count rate is seen to be a factor of 2 greater than that for the amplitude stabilized wave, as it is in the case of the Gaussian source (see Fig. 1). This has been described previously as arising from correlated photon arrivals.[19] For $w \gg 1$, the count rate is the same as that for the amplitude stabilized wave, for which photon arrival times are uncorrelated.

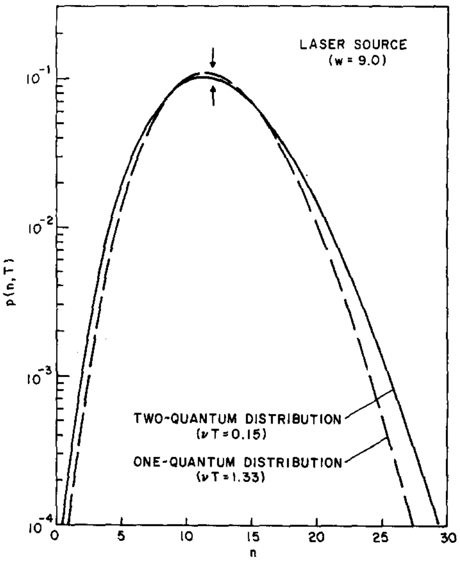

FIG. 4. Comparison of one- and two-quantum photocounting distributions for a laser well above threshold. The distributions are similar to a Poisson distribution.

Furthermore, the ratio plotted in Fig. 5 precisely gives the dependence of Titulaer and Glauber's coherence parameter[28,19] g_2 (for sources possessing first-order coherence) on the laser excitation parameter w for the Van der Pol oscillator. The curve is similar to the reduced second factorial moment of the one-quantum distribution, H_2, used by Smith and Armstrong,[5] but only count rates are involved here, not higher moments of the photocounting distribution. It may be noted that the semiclassical theory yields mth factorial moments normalized to $(\nu T)^m$ for the two-quantum case equal to the identically normalized $2m$th factorial moments of the one-quantum case. In principle, therefore, the statistics of the radiation are contained in the full set of moments of the single-photon counting distribution, to the extent that the semiclassical treatment is adequate.

For counting intervals T much *longer* than the coherence time of the intensity fluctuations, and for low-brightness sources, the counting distribution will be Poisson, independent of the radiation probability density. As in the case of the one-quantum detector, the fluctuations average out to some constant value and no further ensemble average is necessary. Thus a broad-linewidth thermal source and a well-stabilized laser far

[27] E. M. Logothetis and P. L. Hartman, Phys. Rev. Letters **18**, 581 (1967).

[28] U. M. Titulaer and R. J. Glauber, Phys. Rev. **140**, B676 (1965).

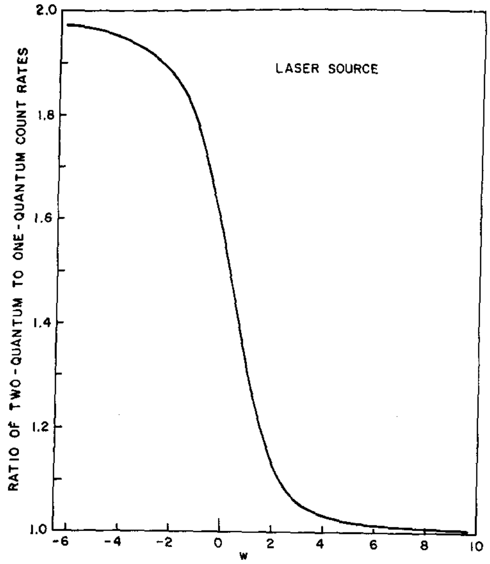

FIG. 5. Ratio of two-quantum to one-quantum count rates vs laser excitation parameter w, with mean intensity fixed. The curve is therefore the reduced second moment of the radiation intensity distribution. The enhancement over the stabilized laser reaches a factor of 2 well below threshold, as for a thermal source.

above threshold will both give rise to a Poisson distribution for the two-quantum detector, as well as for the single-quantum detector. For the two-quantum detector with T large compared to the coherence time, however, the fluctuating component will contribute to the mean-square intensity, while the single-quantum detector ignores the fluctuations over long periods.

It may also be pointed out, as did Wolf and Mehta[29] for the one-quantum case, that the complete probability density of the fluctuating radiation intensity can, at least in principle, be determined by observing the counting distribution of two-quantum photoelectrons. For the one-quantum case, the well-known inverse Laplace transform is required to obtain the intensity distribution from the counting statistics. In the double-quantum case, an inversion of the Gaussian transform is to be performed. This can, however, be reduced to the familiar inverse Laplace transform as well, by a simple change of variable from I to $I^{1/2}$. Although the inversions will often not be feasible in practice on the basis of a reasonable number of experimentally obtained moments,[4] the theoretical invertibility indicates that, in this restricted sense, the two-quantum photoelectric counting statistics faithfully reflect those of the fluctuating incident radiation, provided that the counting intervals are short compared to the coherence time.

Finally, we should point out that experiments to verify the theory and distributions set forth in this paper are presently feasible. Values for the double-quantum photoelectric yield Λ range from $\sim 10^{-15} I$ A/W for metals to $\sim 10^{-8} I$ A/W for some organic single crystals.[15] Here, I is the incident radiation intensity in W/cm^2. The output of a single-mode He–Ne laser operating at 1.15 μm, focused to a 10 μm spot, can provide an intensity $I \sim 10^3$ W/cm^2 near threshold. If this radiation impinges on a Cs_3Sb photocathode[30,31] which has a work function $e\phi \simeq 2.05$ eV and a yield of $\sim 10^{-11} I$ A/W, a value $\nu = \beta I^2 = \Lambda P/e \sim 10^8$ sec^{-1} results. We observe that the condition expressed by (1) is satisfied. Therefore, with a counting interval of $T \simeq 1$ μsec, sufficiently short compared to the coherence time, an experiment may easily be performed. Alternatively, a stable beam from a laser operating above threshold could be subjected to random scattering,[32] and a chaotic source of much narrower spectral width than occurs in any natural source could be obtained thereby. This would permit observation times short compared to the inverse radiation bandwidth for the source, and yet still realistic for measurements.

APPENDIX

The function $i^n \mathrm{erfc}\, x$ appears repeatedly in expressions for two-quantum photoelectron distributions and moments, and also for the one-quantum case when the intensity fluctuations may be described by a Van der Pol oscillator model. The properties of this function which are most useful for utilization and interpretation of the results presented in this paper are summarized here.

The function, which is tabulated,[23] is the nth iterated integral of the familiar complementary error function:

$$i^n \mathrm{erfc} x = \int_x^\infty i^{n-1} \mathrm{erfc} t \, dt \tag{A1}$$

$$i^0 \mathrm{erfc} x = \mathrm{erfc} x, \qquad i^{-1} \mathrm{erfc} x = (2/\pi^{1/2}) e^{-x^2}. \tag{A2}$$

It is also expressible as a single integral,

$$i^n \mathrm{erfc} x = [(2/\pi^{1/2})/n!] \int_x^\infty (t-x)^n \exp(-t^2) dt \tag{A3}$$

and satisfies the recurrence relation

$$2n\, i^n \mathrm{erfc} x + 2x i^{n-1} \mathrm{erfc} x = i^{n-2} \mathrm{erfc} x, \tag{A4}$$

as well as the differential equation

$$(d^2y/dx^2) + 2x(dy/dx) - 2ny = 0. \tag{A5}$$

[29] E. Wolf and C. L. Mehta, Phys. Rev. Letters **13**, 705 (1964).

[30] H. Sonnenberg, H. Heffner, and W. Spicer, Appl. Phys. Letters **5**, 95 (1964).

[31] R. A. Soref (unpublished).

[32] W. Martienssen and E. Spiller, Am. J. Phys. **32**, 919 (1964); Phys. Rev. Letters **16**, 531 (1966).

Generally, $i^n\mathrm{erfc}x$ is a rapidly decaying function of x for any n, and of n for any x. Many of the attendant computational difficulties can be avoided by dealing instead with the function $\exp(x^2)\ i^n\mathrm{erfc}x$, which appears naturally in the counting statistics.

To assist in interpreting the results quoted in certain limiting cases of interest, we note the following limits:

$$i^0\mathrm{erfc}x \rightarrow \exp(-x^2)/(\pi^{1/2}x), \qquad x\rightarrow\infty \tag{A6}$$

$$i^0\mathrm{erfc}x \rightarrow 2, \qquad x\rightarrow -\infty. \tag{A7}$$

The corresponding limits for the higher-order functions are easily obtainable from the recurrence relation (A4).

JOURNAL OF APPLIED PHYSICS VOLUME 40, NUMBER 2 FEBRUARY 1969

Surface and Bulk Waves on Axially Magnetized Plasma Columns*

VLADISLAV BEVC

The Aerospace Corporation, Los Angeles, California 90045

(Received 3 September 1968; in final form 23 October 1968)

The dispersion relation and field solutions of slow electromagnetic waves propagating along a cold, collisionless, cylindrical plasma in an axial magnetic field of finite magnitude are studied numerically. Field distribution and gradual metamorphosis of surface waves into bulk waves along the dispersion characteristics are investigated. It is shown that both surface waves and bulk waves belong to the same set of plasma modes. Criteria concerning conditions under which surface waves occur are extended in include transitional situations. Power-flow density along the plasma may in certain cases be in opposite directions inside and outside of the plasma, both for forward surface waves as well as for backward cyclotron waves, implying transfer of energy between propagation media.

I. INTRODUCTION

Waves propagating in a cylindrical, cold, collisionless plasma in a finite axial-magnetic field have been treated numerically by several authors.[1–3] In these studies, the more appropriate dynamic analysis[4–6] has been applied in preference to the quasi-static approximation.[7,8] Other geometrical configurations such as plasma slabs of infinite extent have also been investigated.[9] The principal objective of these papers has been the determination of dispersion characteristics of discrete modes in the slow-wave (subluminous phase velocity) region,[2,3] and of perturbed waveguide modes.[1] In considering this problem, a question arises whether it represents any essential variation from the more exhaustively treated problem of magnetized plasma completely filling a metalic cylindrical waveguide,[10–13] since changes in geometrical configuration are not expected to introduce new effects. Analysis of the situation shows that there are indeed important differences. First, the plasma is now assumed to be bounded by vacuum, and the boundary is allowed to become rippled under the influence of the high-frequency electromagnetic field. This effect is represented by a surface-charge layer that causes a discontinuity of the radial electric field.[4] Second, the plasma column in vacuum is an open structure, and as such may support in addition to discrete modes a continuous eigenvalue spectrum of the radiation field.[14] It is thus quite reasonable to expect some field solutions and dispersion characteristics for this configuration to be different from those in a plasma-filled waveguide. Moreover, the configuration under study is a better model for a physical situation likely to exist in a laboratory experiment.[7,15–20] As is seen below, the field components of slow waves outside the plasma column decrease rather rapidly, and the influence of any waveguide wall removed from the plasma boundary by a distance comparable to the plasma radius does not modify significantly the field distribution; the surface rippling from a discontinuity of the radial electric field at the plasma boundary is the dominant effect of the boundary in this model.

* This research was supported by the U.S. Air Force.

[1] P. O. Berrett and C. C. Johnson, IEEE Trans. Nucl. Sci. **NS-11,** 34–40 (1964).
[2] M. Camus, in *Proc. 5th Intern. Congr. Microwave Tubes, Paris,* 1964 (Academic Press, Inc., New York, 1965).
[3] G. L. Yip and S. R. Seshadri, Can. J. Phys. **45,** 2889 (1967).
[4] W. C. Hahn, Gen. Elec. Rev. **42,** 258 (1939).
[5] A. A. Th. M. Van Trier, Appl. Sci. Res. **B-3,** 305 (1954).
[6] H. Suhl and L. R. Walker, Bell System Tech. J. **33,** 579 (1954).
[7] A. W. Trivelpiece, *Slow Wave Propagation in Plasma Waveguides* (San Francisco Press, San Francisco, 1967).
[8] L. D. Smullin and P. Chorney, in *Proc. Symp. Electron. Waveguides* (Interscience Publishers Inc., New York, 1958) p. 229.
[9] A. J. Weil, Pennsylvania State Univ., Sci. Rept. 314, NASA Grant NsG, 134 (February 1968).
[10] V. Bevc and T. E. Everhart, Univ. of California Electron. Res. Lab., Berkeley, Rept. 362, AF 33(616)-6139 (July 1961).
[11] V. Bevc and T. E. Everhart, J. Electron. Control (GB) **13,** 185 (1962).
[12] M. Camus and J. Le Mezec, in *Proc. Symp. at Copenhagen, on Electromagnetic Theory and Antennas,* 1962 (The Macmillan Company, New York, 1963), Pt. I, p. 323.
[13] R. Likuski, Rept. No. A1-TDR-64-157, Univ. of Illinois, Elec. Eng. Res. Lab., Urbana, Illinois (1964).
[14] R. E. Collin, *Field Theory of Guided Waves* (McGraw-Hill Book Co., Inc., New York, 1960), Chap. 11.
[15] G. Bittner, Z. Angew. Phys. **10,** 117 (1958).
[16] A. W. Trivelpiece and R. W. Gould, J. Appl. Phys. **30,** 1784–1793 (1959).
[17] H. Riedel, Z. Angew. Phys. **13,** 360 (1961).
[18] R. L. Ferrairi and A. Reddish, Nachr. Fachberichte **22,** 2 (1961).
[19] C. B. Wharton and J. H. Malmberg, Proc. Seventh Intern. Conf. on Phenomena in Ionized Gases, Beograd, **2,** 256 (1965).
[20] A. Bouchoule and M. Weinfeld, *Eighth Intern. Conf. Phenomena Ionized Gases, Vienna, Contributed Papers* (1967), p. 381.