Evolution of the Statistical Properties of Photons Passed Through a Traveling-Wave Laser Amplifier

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Abstract—**We determine the evolution of the photon statistics of a light beam as it passes through a traveling-wave laser amplifier, modeled as a birth-death-immigration (BDI) medium. The relationship between the input and output probability distributions and probability generating functions with given (but possibly varying) birth, death, and immigration rates for arbitrary input statistics is obtained. The case of constant birth, death, and immigration rates is considered in particular detail. The photon statistics at the output of a general BDI traveling-wave amplifier are always broader than those at the input, and they can take many forms. Our most general solution can be applied when the input distribution to the amplifier takes the form of a negative-binomial transform. The results are expected to be useful in calculating the performance characteristics of lightwave systems using optical amplifiers in which the object is to detect light with a broad range of statistical properties, including scattered light, spontaneous-emission light, and light emitted from a laser. In the latter case the input is Poisson, and the output distribution assumes the form of a noncentral-negative-binomial (Laguerre) distribution which is usually associated with a multimode (phase-preserving) superposition of coherent and chaotic fields.**

I. Introduction

OPTICAL-fiber and semiconductor laser amplifiers are being used increasingly in optoelectronic systems. Such amplifiers are typically operated either as traveling-wave or resonant devices, depending on the application [1], [2]. The amplification is the result of the interaction of light with a large number of atoms for which a population inversion is externally maintained.

There are several theoretical formulations of the laser amplification process that are useful for dealing with amplifiers of different configurations [3]–[5]. The population-statistical approach first used by Shimoda, Takahasi, and Townes [6] generally provides a suitable point of departure for characterizing the photon statistics associated with laser amplification. This approach has its origins in the branching-process models developed long ago for use in cosmic rays and population biology. It relies on the birth–death–immigration (BDI) process, which is well known in the theory of stochastic processes [7]–[9].

Various versions of the BDI model have been used to represent the processes of absorption, stimulated emission, and spontaneous emission taking place in a cavity [10]–[14]. In particular, Schell and Barakat [11] examined the approach to equilibrium of the photon-number distribution for a single radiation mode in a cavity, given a variety of initial photon-number distributions. They found that a Poisson initial distribution resulted in a final distribution described by the noncentral-negative-binomial with one degree of freedom.

Light amplifiers have been examined in the context of quantum optics by a number of authors. Louisell and his collaborators [15], [16] developed an early quantum model of a linear single-mode phase-insensitive intensity amplifier. The spatial propagation of the optical field through the amplifying medium was replaced by a time-dependent growth of the optical intensity. This model has provided the point of departure for a number of generalizations [3]–[5], [17]–[22]. Several of these quantum-mechanical models provided solutions for the photon-number distribution at the output of the amplifier in terms of the distribution at the input [3]–[5], [20]–[22]. The relationship between the quantum model of Louisell and the population-statistical approach of Shimoda, Takahasi, and Townes [6] has been elucidated by Shepherd and Jakeman [4]. One of the cases they consider in the context of their quantum model comprises a Poisson number of input photons coupled to the cavity by adding to the immigration. Their results, like those of Schell and Barakat [11], lead to the noncentral-negative-binomial distribution with one degree of freedom.

In this paper we study the evolution of the photon statistics of a light beam as it passes through a traveling-wave amplifier, using the population-statistical BDI approach [6]. Each of the photons in our traveling-wave configuration can be viewed as initiating its own BDI process, rather than augmenting the immigration parameter. In Section II, we obtain the relationship between the input and output probability density functions (and probability generating functions) for a BDI process with given (but possibly varying) birth, death, and immigration rates for

Manuscript received October 3, 1990; revised November 8, 1991. This work was supported in part by the National Science Foundation through the Columbia Center for Telecommunications Research.

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IEEE Log Number 9106853.

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arbitrary input statistics. The case of constant birth, death, and immigration rates is considered in Section III, which contains detailed results for two special limits: deterministic input distributions and negative-binomial-transformed input distributions. The former constitutes the least random situation achievable with our model, whereas the latter represents our most general solution. Explicit results are also provided for several intermediate cases of special importance. These include input distributions with negative-binomial photon statistics (which characterizes multimode thermal or scattered light) and with Poisson photon statistics (which characterizes pure coherent light), as well as the Poisson-driven Yule–Furry (pure birth) process. Pure coherent light at the input to the amplifier results in a noncentral-negative-binomial output distribution, but now with an *arbitrary* degrees-of-freedom parameter. The conclusion is provided in Section IV.

The noncentral-negative-binomial distribution obtained for coherent light at the input to the amplifier is essential for properly calculating the performance characteristics of a digital lightwave communication system incorporating an ideal laser and a traveling-wave optical amplifier (e.g., Er^{3+}-doped silica fiber) [23]. Performance calculations can also be carried out using the more general distributions developed here. These should enable us to determine whether optical amplifiers are useful devices for detecting other kinds of light as well, e.g., thermal (spontaneous emission) light, scattered light, and light from real semiconductor lasers (which exhibit photon statistics that are broader than Poisson).

II. BDI Model with Arbitrary Input Statistics

The relationship between the probability distributions at the input and output of a medium with given (possibly varying) birth, death, and immigration rates is provided in this section. The derivations are given in the Appendix.

We let t denote the time of traversal, or else the depth of penetration, of a medium characterized by a birth rate per particle $\lambda(t)$, a death rate per particle $\mu(t)$, and an immigration rate independent of the current number of particles $\nu(t)$. The probability distribution $p(n, t)$ that n particles are present at time (or depth) t, given that the distribution $p(n, 0)$ at $t = 0$ was $p_0(n)$, satisfies the well-known forward Kolmogorov difference-differential equation,

$$\begin{aligned}\partial p(n, t)/\partial t = {} & [(n-1)\lambda + \nu]p(n-1, t) \\ & + [(n+1)\mu]p(n+1, t) \\ & - [n(\lambda+\mu) + \nu]p(n, t) \qquad (1)\end{aligned}$$

as described in the Appendix.

The probability generating function (PGF)

$$G(z, t) = \langle z^n \rangle = \sum_{n=0}^{\infty} p(n, t) z^n \qquad (2)$$

satisfies the partial differential equation

$$\begin{aligned}\partial G(z, t)/\partial t = {} & [z-1][(\lambda z - \mu) \\ & \cdot \partial G(z, t)/\partial z + \nu G(z, t)]. \qquad (3)\end{aligned}$$

This is solved, in the general case of varying birth, death, and immigration rates, by

$$G(z, t) = G_0(Z(z, t; 0)) \exp\left\{\int_0^t [Z(z, t; \tau) - 1]\nu(\tau)\, d\tau\right\} \qquad (4)$$

in terms of the initial PGF $G_0(z) = G(z, 0)$, as is also reviewed in the Appendix. We use here the definitions of the following *known* functions, obtained directly from the given forms of the rates of birth, death, and immigration as functions of t:

$$Z(z, t; \tau) = 1 + \frac{(z-1)h(\tau)}{h(t) - [z-1][k(t) - k(\tau)]} \qquad (5)$$

$$h(u) = \exp\left[\int_0^u (\mu(t) - \lambda(t))\, dt\right]$$

$$k(u) = \int_0^u [h(t)\lambda(t)]\, dt. \qquad (6)$$

The mean number of events or particles $N(t) = \langle n \rangle = \Sigma\, np(n, t) = \partial G(1, t)/\partial z$ is given by

$$N(t) = \frac{1}{h(t)}\left[N_0 + \int_0^t (h(\tau)\nu(\tau))\, d\tau\right] \qquad (7)$$

where N_0 is the mean number at the initial point $t = 0$. The function $h(\tau)/h(t)$ is seen to serve as a transfer function, directly from input to output for the initial mean number of events or particles, and cumulatively for the number of particles that immigrate within the medium.

III. The Case of Constant Birth, Death, and Immigration Rates

If we specialize now to the familiar problem of constant birth, death, and immigration rates, λ, μ, ν, we then find

$$h(t) = \exp(\mu - \lambda)t \qquad k(t) = [\lambda/(\mu - \lambda)][h(t) - 1] \qquad (8)$$

$$\begin{aligned}& \exp\left\{\int_0^t [Z(z, t; \tau) - 1]\nu\, d\tau\right\} \\ & = \{1 - [z-1]k(t)/h(t)\}^{-\nu/\lambda} \qquad (9)\end{aligned}$$

so that the PGF $G(z, t)$ at the output of this medium is given in terms of that at the input, $G_0(z) = G(z, 0)$, by

$$G(z, t) = \frac{G_0(Z(z, t; 0))}{[1 - (z-1)k(t)/h(t)]^{\nu/\lambda}} \qquad (10)$$

with

$$Z(z, t; 0) = 1 + \frac{z-1}{h(t) - (z-1)k(t)}. \qquad (11)$$

The mean count at the output for the case of constant rates is, from (7),

$$N(t) = \frac{N_0 + (\nu/\lambda)k(t)}{h(t)}. \tag{12}$$

This reduces to the form

$$\frac{N - N_1}{N_0 - N_1} = \exp[-(\mu - \lambda)t] \qquad N_1 = \frac{\nu}{\mu - \lambda}. \tag{13}$$

If $\mu > \lambda$, then the mean count $N = N(t)$ approaches N_1 as $t \to \infty$; if the initial mean count N_0 is exactly N_1, then that count remains at N_1 for all t.

For the case of constant rates, the variance at the output is given in terms of the initial mean N_0 and variance σ_0^2 by

$$\sigma^2(t) = \frac{\sigma_0^2 - N_0 + (2k(t) + h(t))N_0 + \frac{\nu}{\lambda}k(t)\left(\frac{3}{2}k(t) - h(t)\right)}{h^2(t)}. \tag{14}$$

This is based on the values $V_1 = (\nu/\lambda)k(t)$ and $V_2 = \frac{1}{2}(\nu/\lambda)k^2(t)$ for the iterated integrals defined in (A25).

To confirm that we recover the usual results for the common special cases, we note that $G(z, t)$ can be expressed as the product

$$G(z, t) = G_0(Z(z, t; 0))G_1(z, t) \tag{15}$$

where

$$G_1(z, t) = [1 - (z - 1)k(t)/h(t)]^{-\nu/\lambda} \tag{16}$$

is recognized as the PGF for a negative-binomial distribution, with mean

$$(\nu/\lambda)k(t)/h(t) = N_1\{1 - \exp[-(\mu - \lambda)t]\}$$

and degrees-of-freedom parameter ν/λ. We use the notation

$$B(n, \alpha, N) = \frac{\alpha^\alpha\Gamma(n + \alpha)}{n!\Gamma(\alpha)} \frac{N^n}{(N + \alpha)^{n+\alpha}} \tag{17}$$

for the negative-binomial PDF, with mean N and parameter α; its PGF is

$$G(z; \alpha, N) = [1 - (z - 1)(N/\alpha)]^{-\alpha}. \tag{18}$$

This distribution reduces to the Poisson in the limit of $\alpha \to \infty$ and to the Bose–Einstein distribution for $\alpha = 1$.

A. Deterministic Input Distributions

The form of the initial PGF $G_0(z)$ is particularly simple when the number of particles or events at the input to the medium is deterministic. We cite specifically the cases of zero, one, and many initial events.

1) Zero Initial Particles: The output from the medium when there are no particles or events at its input stems entirely from immigration. Thus, if $p_0(n) = \delta(n)$ for which $G_0(z) = 1$, we obtain

$$p(n, t) = B(n, \nu/\lambda, (\nu/\lambda)k(t)/h(t)) \tag{19}$$

which is the negative-binomial distribution mentioned above. Fig. 1 is a plot of this counting distribution for $\alpha = \nu/\lambda = 2.5$ and a mean count $N = 64$. It would be observed, for example, in a BDI medium with no initial particles, at a depth such that the total immigration νt has attained 8.20 and the births have reached $\lambda t = 3.28$ if there are no deaths. Alternatively, it would be observed at a depth for which $\nu t = 64$ if the birth and death rates are equal at $\lambda t = \mu t = 25.6$. The distribution can never be attained in that medium, however, if deaths exceed births by a factor of more than 1.0391, at which point $\nu/(\mu - \lambda)$ becomes 64.

2) One Initial Particle: Similarly, the usually cited case of a single event at the input, associated with $p_0(n) = \delta(n - 1)$ for which $G_0(z) = z$, yields

$$G(z, t) = Z(z, t; 0)G_1(z, t) = [1 + (z - 1)(1 - k)/h]/[1 - (z - 1)k/h]^{(\nu/\lambda + 1)} \tag{20}$$

where h and k are abbreviations for $h(t)$ and $k(t)$. Since a factor of z in a PGF corresponds to a shift by one count in the PDF, this results in a combination of a negative-binomial and a shifted one, as

$$p(n, t) = [1 - (1 - k)/h]B(n, \beta, \beta k/h) + [(1 - k)/h]B(n - 1, \beta, \beta k/h) \tag{21}$$

where

$$\beta = [(\nu/\lambda) + 1]$$

and

$$[1 - (1 - k)/h] = [\mu/(\mu - \lambda)][1 - \exp[-(\mu - \lambda)t]].$$

Note also that if there are no deaths ($\mu = 0$), then $1 - k = h$ and the result is merely the shifted negative-binomial $p(n, t) = B(n - 1, \beta, \beta(e^{\lambda t} - 1))$.

3) Many Initial Particles: The more general case of a deterministic input has n_0 initial particles; i.e., $p_0(n) = \delta(n - n_0)$ for which $G_0(z) = z^{n_0}$. At the output this yields

$$G(z, t) = (Z(z, t; 0))^{n_0}G_1(z, t) = [1 + (z - 1)(1 - k)/h]^{n_0}/[1 - (z - 1)k/h]^{(\nu/\lambda + n_0)} \tag{22}$$

which corresponds to a convolution of a positive-binomial PDF with a negative-binomial one:

$$p(n, t) = B(n, \beta_0, \beta_0 k/h) * b(n, n_0, n_0[1 - k]/h) \tag{23}$$

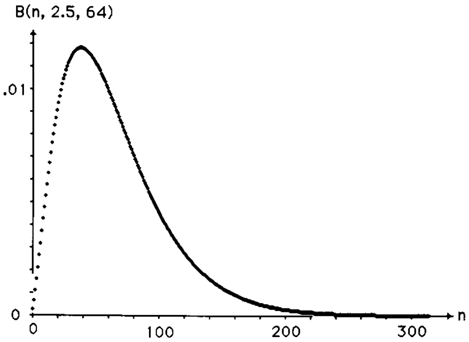

Fig. 1. The negative-binomial distribution for $\alpha = \nu/\lambda = 2.5$ and mean output count $N = 64$.

where $\beta_0 = (\nu/\lambda) + n_0$ and $b(n, m, N) = B(n, -m, N)$ is the (positive) binomial distribution (with mean N), for which the PGF is $[1 + (N/m)(z - 1)]^m$, often written $(pz + q)^m$, subject to $p + q = 1$ and with $N = mp$.

B. Negative-Binomial-Transformed Input Distribution

We can now demonstrate the most general result of this paper, which deals with the probability distribution at the output of the medium when the input distribution is a negative-binomial transform of an arbitrary continuous probability density function. We previously discussed the significance of negative-binomial transforms [24]. One particular example of interest in the current context is the noncentral-negative-binomial (NNB) distribution, which can be viewed as a modified negative-binomial transform. This provides an excellent model for the photon statistics at the output of a laser when its spontaneous emission noise is accounted for. Here we inquire into the effect on such initial counting distributions of traversing a medium in which birth, death, and immigration occurs.

The negative-binomial transform of a continuous PDF $\Phi(x, N)$ is defined by

$$B\{\Phi\} = B(n, \alpha, x) \wedge \Phi(x, N) = \varphi(n, \alpha, N) = \int_0^\infty B(n, \alpha, x)\Phi(x, N)\, dx \tag{24}$$

where $B(n, \alpha, N)$ is the negative-binomial probability distribution, with mean N and degrees-of-freedom parameter α, and the caret $\wedge$ denotes the indicated averaging over the repeated variable x. Explicitly, the transform is

$$B\{\Phi\} = \frac{\alpha^\alpha \Gamma(n + \alpha)}{n!\Gamma(\alpha)} \int_0^\infty \frac{x^n}{(x + \alpha)^{n+\alpha}} \Phi(x, N)\, dx. \tag{25}$$

This operation transforms the PDF of any continuous source distribution into a discrete counting distribution; the transform arises in many contexts [24].

For the negative-binomial transform $\varphi(n, \alpha, N)$ of $\Phi(x, N)$, the probability generating function is given in terms of an average over the original, continuous distribution $\Phi(x, N)$ as

$$G(z) = \langle z^n \rangle_n = \langle \{1 - (z - 1)(x/\alpha)\}^{-\alpha} \rangle_x = \langle G(z; \alpha, x) \rangle_x \tag{26}$$

where the notation $\langle \cdot \rangle_x$ indicates averaging with respect to x.

If, therefore, the initial counting distribution at the input to the medium is a negative-binomial transform (with parameter α) of some continuous PDF $\Phi(x, N_0)$, then the initial PGF is expressed by

$$G_0(z) = G(z; \alpha, x) \wedge \Phi(x, N_0) = [1 - (x/\alpha)(z - 1)]^{-\alpha} \wedge \Phi(x, N_0). \tag{27}$$

At the output, therefore, the PGF becomes

$$G(z, t) = \frac{[1 - (x/\alpha)(Z(z, t; 0) - 1)]^{-\alpha} \wedge \Phi(x, N_0)}{[1 - (z - 1)k(t)/h(t)]^{\nu/\lambda}} \tag{28}$$

with

$$Z(z, t; 0) = 1 + \frac{z - 1}{h(t) - (z - 1)k(t)}. \tag{29}$$

This simplifies to

$$G(z, t) = \frac{[1 - (z - 1)(k/h + x/(\alpha h))]^{-\alpha} \wedge \Phi(x, N_0)}{[1 - (z - 1)k/h]^{(\nu/\lambda) - \alpha}} \tag{30}$$

where h and k are abbreviations for $h(t)$ and $k(t)$. But this is just

$$G(z, t) = G(z; \beta, \beta k/h) G[z; \alpha, (x + \alpha k)/h] \wedge \Phi(x, N_0) \tag{31}$$

where $\beta = (\nu/\lambda) - \alpha$. This, then, is the output PGF when the input PDF is a negative-binomial transform of a given distribution.

C. Interpretation

The form of (31) is merely a product of the PGF of a negative-binomial with mean $\beta k(t)/h(t)$ and parameter β with the PGF of a negative-binomial transform of $\Phi(x, N_0)$ whose kernel $B(n, \alpha, (x + \alpha k)/h)$ has had its mean value linearly transformed (rescaled by $1/h$ and shifted by $\alpha k/h$). Since a product of PGF's implies a convolution of PDF's, we have, finally, that the PDF at the output of the medium, when its input is some negative-binomial transform, is

$$p(n, t) = B(n, \beta, \beta k/h) * B(n, \alpha, (x + \alpha k)/h) \wedge \Phi(x, N_0) \tag{32}$$

where $*$ denotes discrete convolution and $\wedge$ indicates integration over x and where $\beta = (\nu/\lambda) - \alpha$.

Changing the integration variable x to $u = x + \alpha k$ or else to $v = (x + \alpha k)/h$ furnishes the alternate forms

$$p(n, t) = B(n, \beta, \beta k/h) * B(n, \alpha, u/h) \wedge \Phi(u - \alpha k, N_0) \tag{33}$$

$$p(n, t) = B(n, \beta, \beta k/h) * B(n, \alpha, v) \wedge \Phi(vh - \alpha k, N_0)h; \tag{34}$$

the latter form can be recognized as a convolution of the negative-binomial with a negative-binomial transform of a noncentral (shifted) and rescaled version of the original continuous PDF $\Phi(u, N_0)$.

We note that the convolution and the integration do not interfere with each other, so that the indicated operations are associative. We also confirm that the mean of the resultant PDF is as expected because the integration results in a distribution with mean $(N_0 + \alpha k)/h$ and the convolution adds $\beta k/h$ to that mean, yielding $\langle n \rangle = N = [N_0 + (\alpha + \beta)k]/h = [N_0 + (\nu/\lambda)k]/h$, as in (12).

The associativity allows us to study the general case by reference to the convolved-negative-binomial transform of a continuous PDF $\Phi(x, N_0)$:

$$p(n, t) = p(n, \alpha, x, \lambda, \mu, \nu, N) \wedge \Phi(x, N_0) \tag{35}$$

where

$$p(n, \alpha, x, \lambda, \mu, \nu, N) = B(n, \beta, \beta k/h) * B(n, \alpha, (x + \alpha k)/h) \tag{36}$$

with

$$\beta = (\nu/\lambda) - \alpha \qquad h = \exp[(\mu - \lambda)t]$$
$$k = [\lambda/(\mu - \lambda)]\{\exp[(\mu - \lambda)t] - 1\} \tag{37}$$

and

$$N = x \exp[-(\mu - \lambda)t] + N_1\{1 - \exp[-(\mu - \lambda)t]\} \tag{38}$$

where $N_1 = \nu/(\mu - \lambda)$. The distribution

$$p(n, \alpha, N_0, \lambda, \mu, \nu, N) = B(n, [\nu/\lambda] - \alpha, [\nu - \alpha\lambda]E) * B(n, \alpha, N_0E' + \alpha\lambda E) \tag{39}$$

with

$$E = E(t) = [1 - e^{-(\mu - \lambda)t}]/(\mu - \lambda) \quad (\text{or } E = t \text{ if } \mu = \lambda) \tag{40}$$

$$E' = dE(t)/dt = e^{-(\mu - \lambda)t} \quad (\text{or } E' = 1 \text{ if } \mu = \lambda) \tag{41}$$

$$N = N(t) = N_0E' + \nu E = \langle n \rangle \tag{42}$$

is the output counting distribution of the medium with rates λ, μ, and ν when the input is the negative-binomial $B(n, \alpha, N_0)$ with mean N_0 and parameter α. This distribution is the focus of the next section.

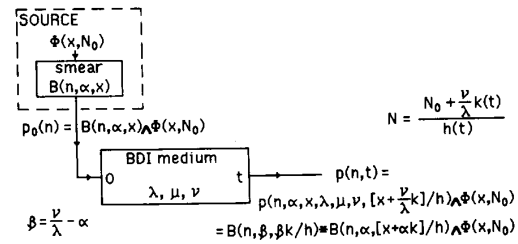

Fig. 2. The negative-binomial transform as a source for the BDI medium.

Fig. 2 illustrates the most general overall process in which a BDI medium is presented with a discrete input distribution that is itself the outcome of a compound source. The latter comprises a discrete process whose statistics are described by a negative-binomial distribution $B(n, \alpha, x)$ whose mean x undergoes a smearing process governed by a continuous distribution $\Phi(x, N_0)$, with mean N_0. As indicated in the figure, the output of the BDI medium is a convolved-negative-binomial transform of the continuous PDF $\Phi(x, N_0)$.

D. Negative-Binomial Input Distribution

The output distribution for the BDI medium of depth t such that the mean count becomes N when the input distribution is the negative-binomial $B(n, \alpha, N_0)$ is denoted $p(n, \alpha, N_0, \lambda, \mu, \nu, N)$. The input–output process is depicted schematically in Fig. 3, in which a box represents a BDI medium with parameters λ, μ, ν, and depth t. The top figure represents the general case of an arbitrary input particle distribution $p_0(n)$ at $t = 0$, which becomes the output distribution $p(n, t)$ at the depth t. The three middle examples represent particular cases of special input distributions, a deterministic one, a Poisson, and a Bose–Einstein, each of which is a limiting version of the negative-binomial input distribution. The output distributions are indicated as special cases of the general result. The bottom illustration defines that general case in which the negative-binomial distribution $B(n, \alpha, N_0)$ is converted into $p(n, \alpha, N_0, \lambda, \mu, \nu, N)$.

Fig. 4 shows the development of the counting distribution at increasing depths in such a BDI medium with parameters ($\lambda, \mu, \nu = 7, 6, 8$ in relative units) such that births exceed deaths and the medium amplifies the mean, which, initially, was $N_0 = 8$, as $N(t) = 16e^t - 8$. The degrees-of-freedom parameter of the input distribution is $\alpha = 7.5$. The counting distributions are presented for $\lambda t = 0$, the input negative binomial with mean $N = 8$, then for $\lambda t = 1$ with mean $N = 10.457$, then for $\lambda t = 2$ with mean $N = 13.291$, and finally for $\lambda t = 4$ with mean $N = 20.333$. The plots show clearly how the initial distribution broadens and flattens indefinitely as the amplifying medium is traversed.

An inversion of the birth and death rates of the BDI medium makes the death rate exceed the birth rate, in which case the medium acts as an attenuator rather than an amplifier. The counting distribution then approaches a

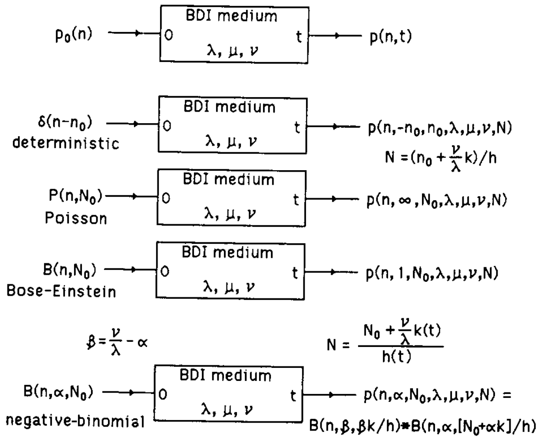

Fig. 3. Counting distributions at the output of a BDI medium for various input distributions.

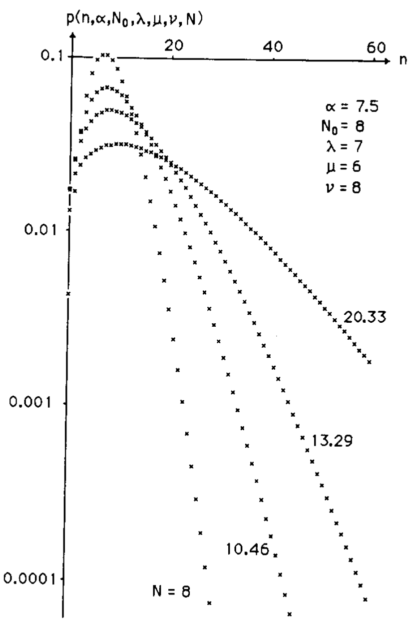

Fig. 4. Evolution of the initial negative-binomial distribution in a traveling-wave laser amplifier.

steady state, as presented in Figs. 5 and 6. For Fig. 5, the initial mean count is $N_0 = 20$ and the parameters (λ, μ, ν = 5, 6, 8) make the mean approach $N = 8$ from above, as $N(t) = 8 + 12e^{-t}$. The degrees-of-freedom parameter is again $\alpha = 7.5$. The distributions are presented at successive depths of μt = 0, 2, 4, 8, and ∞, for which the mean counts are, respectively, N = 20, 16.598, 14.161, 11.163, and 8. The peak of the distribution first drops and then rises again as the PDF sharpens on its way to the equilibrium distribution.

Parameters (λ, μ, ν = 5, 6, 20) in the case of Fig. 6 are chosen again with deaths exceeding births, but now making the mean approach its steady-state value from below, starting at $N_0 = 8$ and rising to $N = 20$ as $N(t) = 20 - 12e^{-t}$. With the same degrees of freedom and the distributions shown again at depths μt = 0, 2, 4, 8, and ∞, the mean values grow from $N = 8$ through 11.402, 13.839, and 16.837 on the way to $N = 20$. In this case, the peaks fall and the PDF's broaden as they approach the limiting distribution.

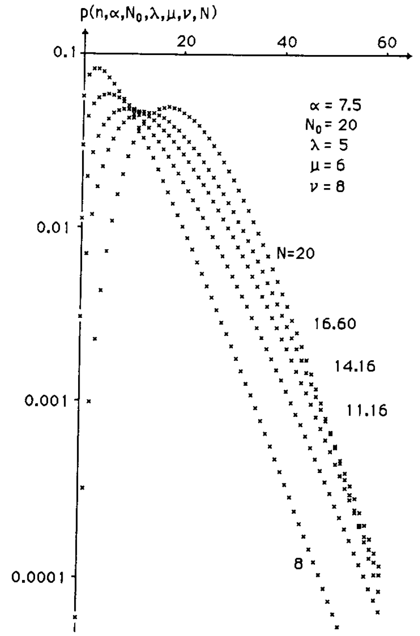

Fig. 5. Downward approach to equilibrium in an attenuating medium.

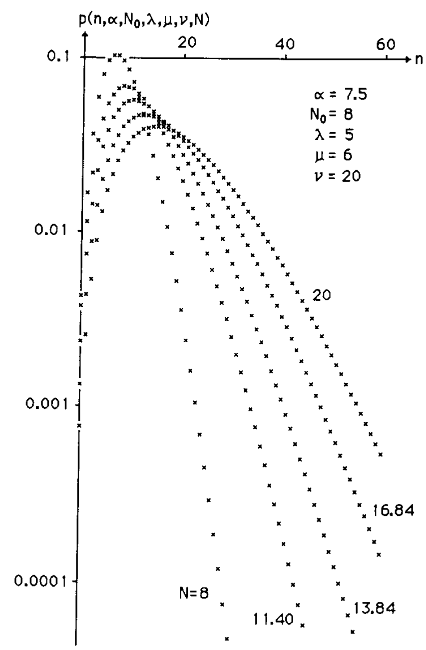

Fig. 6. Upward approach to equilibrium in an attenuating medium.

E. Poisson Input Distribution

The special case of a Poisson-distributed input merits particular attention because it provides results for the simplest model of laser light (when spontaneous emission from the laser is ignored). It is the limit of the previous case as the degrees-of-freedom parameter α becomes infinite. Since the PGF for the Poisson distribution is $G_0(z) = \exp[N_0(z-1)]$, the output PGF given in (10)–(11) reduces to

$$G(z, t) = \frac{\exp\{(N_0/h)(z-1)/[1-(k/h)(z-1)]\}}{[1-(k/h)(z-1)]^{\nu/\lambda}}. \tag{43}$$

This is recognized as the generating function for the generalized Laguerre polynomials, so that the factorial moments are given by

$$\langle n!/(n-m)!\rangle = m!(k/h)^m L_m^{(\nu/\lambda-1)}(-N_0/k). \tag{44}$$

The PDF itself is known as the noncentral-negative-binomial (or Laguerre) distribution

$$p(n, t) = \frac{s^n \exp[-su/(s+1)]}{(s+1)^{n+\nu/\lambda}} L_n^{(\nu/\lambda-1)}(-u/(s+1)) \tag{45}$$

where the abbreviations $s = k/h$ and $u = N_0/k$ have been used. This distribution, also denoted [24] $L(n, k/h, \nu/\lambda, N(t))$, often arises in the statistical description of superimposed coherent and chaotic fields [25], [26]. Its mean is $N(t) = N_0/h + (\nu/\lambda)k/h$ and its variance is

$$\sigma^2(t) = (N_0/h)[2(k/h)+1] + (\nu/\lambda)(k/h)[(k/h)+1]. \tag{46}$$

In terms of our general notation, this PDF is $p(n, \infty, N_0, \lambda, \mu, \nu, N)$, as indicated in Fig. 3.

Fig. 7 presents the development of the initially Poisson distribution in a BDI medium with parameters (λ, μ, ν = 5, 6, 8) from an initial mean value $N_0 = 20$ down to the equilibrium state with mean $N(\infty) = 8$, again at depths given by $\mu t = 0, 2, 4, 8$, and ∞. Note the initial drop and later rise of the peak of the distribution as it broadens.

F. Poisson-Driven Yule–Furry Distribution

A special case of the noncentral-negative-binomial distribution deserves special mention. This is the case of a pure-birth process ($\mu = 0$ and $\nu = 0$) that captures the essence of the BDI process in a simple way. In this instance, there is simple, exponential amplification of the mean, as $N(t) = N_0 e^{\lambda t}$, and the PDF reduces to the Poisson-driven Yule–Furry distribution

$$p(n, t) = L(n, e^{\lambda t}-1, 0, N_0 e^{\lambda t})$$
$$= (1-e^{-\lambda t})^n e^{-N_0} L_n^{(-1)}(-N_0/(e^{\lambda t}-1)). \tag{47}$$

Fig. 8 shows this distribution's development in a pure-birth medium at depths $\lambda t = 0, 0.125, 0.25$, and 0.5 for an initial mean count of $N_0 = 20$ and successive mean values 22.663, 25.681, and 32.974.

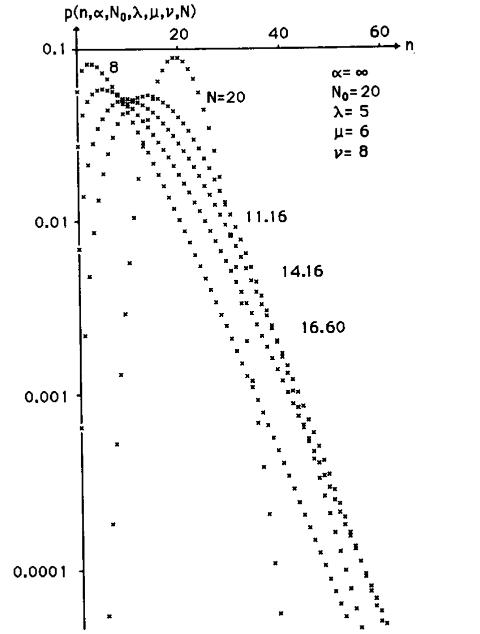

Fig. 7. Evolution to equilibrium in Poisson-driven BDI medium.

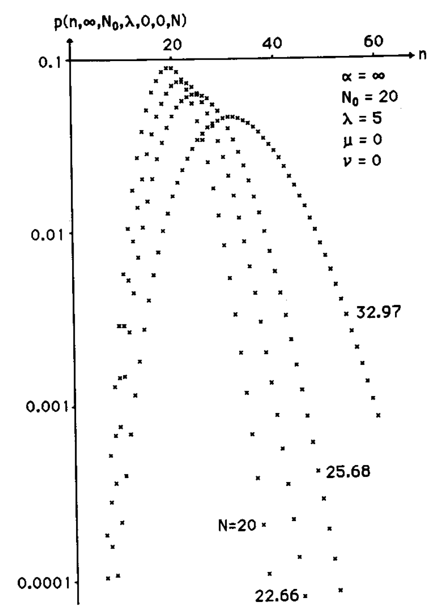

Fig. 8. Evolution of the Poisson-driven Yule–Furry distribution with increasing depth.

IV. Conclusion

The photon statistics at the output of a general BDI traveling-wave amplifier are always broader than those at the input and they can take many forms. Interesting insights are offered by some of the special cases we have considered. In particular, when a Poisson number of photons is presented to the input of the amplifier, the output distribution assumes the form of a noncentral-negative-binomial (Laguerre) distribution [25], [26]. These statistics are usually associated with a multimode (phase-preserving) superposition of coherent and chaotic fields. However, they also emerge from the population-dynamics approach [3], [4], [11], [14], [23] as a result of the proportionality of the stimulated and spontaneous emission terms in the BDI (forward Kolmogorov) equations. The results obtained here are applicable to a cascade of optical amplifiers as well as a single amplifier, even in the presence of intervening loss, provided that the normalized bandwidth is the same for all amplifiers [27], [28].

In addition to being used in modeling the evolution of the photon statistics of a source of light as it passes through an optical amplifier, the particle-based traveling-wave model considered here may also be useful for describing the effects of random media (such as the clear-air turbulent atmosphere [29]) on the statistical properties of a light source. Models of the kind presented here result in greater fluctuations at the output than otherwise-equivalent intensity-based models [24]. This is because the light source is modeled in terms of its photon statistics, comprising both wave and particle fluctuations, rather than simply wave fluctuations as in intensity-based models.

Finally, we note that it would be useful to incorporate the results of amplifier saturation in the general case; there are a number of ways in which this might be carried out [30]–[33].

Appendix
Derivation of PDF and PGF Input–Output Relations

This Appendix reviews the development of the relationship between the probability density functions (PDF's) and probability generating functions (PGF's) at the input and output of a BDI medium with given (possibly varying) birth, death, and immigration rates.

A. Difference-Differential Equation for the PDF

Adopting the model of stochastic distributions of events or particles, we let t denote the time of traversal, or else the depth of penetration, of a medium characterized by a birth rate per particle $\lambda(t)$, a death rate per particle $\mu(t)$, and an immigration rate independent of the current number of particles $\nu(t)$. We seek the probability distribution $p(n, t)$ that n particles are present at time (or depth) t, given that the distribution $p(n, 0)$ at $t = 0$ was $p_0(n)$. Special cases include that of constant λ, μ, and ν and the usual problem of one particle (event) at the starting point, $p_0(n) = \delta(n - 1)$, where δ denotes the Kronecker delta (unity at zero argument, zero otherwise).

Within an infinitesimal interval dt, we assume that the number of particles can only remain the same or change by one; these mutually exclusive possibilities occur with the probabilities defined by the birth, death, and immigration rates. Hence, there can be n particles at the end of the interval if there were $n - 1$ at its start and either birth or immigration occurred, or there were $n + 1$ previously and a death happened, or else there was no change in number during dt, if no birth, death, or immigration took place. The governing difference-differential equation is, then,

$$\begin{aligned} p(n, t + dt) = {} & p(n - 1, t)[(n - 1)\lambda\, dt] \\ & + p(n - 1, t)[\nu\, dt] \\ & + p(n + 1, t)[(n + 1)\mu\, dt] \\ & + p(n, t)[1 - (n\lambda + n\mu + \nu)\, dt] \end{aligned} \quad \text{(A1)}$$

which becomes, as $dt \to 0$,

$$\begin{aligned} \partial p(n, t)/\partial t = {} & [(n - 1)\lambda + \nu]p(n - 1, t) \\ & + [(n + 1)\mu]p(n + 1, t) \\ & - [n(\lambda + \mu) + \nu]p(n, t). \end{aligned} \quad \text{(A2)}$$

We seek the solution $p(n, t)$ of this difference-differential equation (often called the forward Kolmogorov equation), given the initial distribution $p(n, 0) = p_0(n)$, for given but varying parameters $\lambda(t)$, $\mu(t)$, and $\nu(t)$. This furnishes the evolution of the PDF in the medium.

B. Partial Differential Equation for the PGF

The equation is reduced to a simpler partial differential equation for the probability generating function (PGF)

$$G(z, t) = \sum p(n, t) z^n \quad \text{(A3)}$$

by multiplying the original equation by z^n and summing over all n; we can avoid dealing with "missing" terms at $n = 0$ by considering all sums to range from $-\infty$ to ∞ but with $p(n, t) \equiv 0$ for $n < 0$. Thus, replacing the summation variables $n \pm 1$ in (A2) by m where appropriate, we get

$$\begin{aligned} (\partial/\partial t) & \sum p(n, t) z^n \\ & = \lambda z \sum m p(m, t) z^m + \nu z \sum p(m, t) z^m \\ & \quad + (\mu/z) \sum m p(m, t) z^m \\ & \quad - [\lambda + \mu] \sum n p(n, t) z^n - \nu \sum p(n, t) z^n \end{aligned} \quad \text{(A4)}$$

or, since $\sum m p(m, t) z^m = z\, \partial G/\partial z$,

$$\begin{aligned} \partial G(z, t)/\partial t = {} & \lambda z^2\, \partial G/\partial z + \nu z G + \mu\, \partial G/\partial z \\ & - (\lambda + \mu) z\, \partial G/\partial z - \nu G \end{aligned} \quad \text{(A5)}$$

or, finally,

$$\begin{aligned} \partial G(z, t)/\partial t = {} & (z - 1)[(\lambda z - \mu) \\ & \cdot \partial G(z, t)/\partial z + \nu G(z, t)]. \end{aligned} \quad \text{(A6)}$$

C. Solution for the PGF

To solve this, we integrate along the path $z = z(t)$ from an as yet undetermined initial point $(z_0, 0)$ to a fixed but arbitrary point (z_1, t_1), where $z(t)$ is defined by

$$dz/dt = -(z - 1)(\lambda z - \mu). \quad \text{(A7)}$$

Along this path, the partial differential equation becomes the ordinary one

$$dG/dt = (z - 1)\nu G \quad \text{(A8)}$$

for $G = G(z(t), t)$; this is solved by

$$G(z_1, t_1) = G(z_0, 0) \exp\left[\int_0^{t_1} (z(t) - 1)\nu(t)\, dt\right]. \quad \text{(A9)}$$

To obtain the path $z = z(t)$, we define

$$h(u) = \exp\left[\int_0^u (\mu(t) - \lambda(t))\, dt\right] \quad \text{(A10)}$$

which is a known function. By using $\mu(t) - \lambda(t) = (1/h(t))\, dh(t)/dt$, this allows reduction of the equation (A7) for the path from its alternate form

$$d(z - 1)/dt = -\lambda(z - 1)^2 + (\mu - \lambda)(z - 1) \quad \text{(A11)}$$

to

$$h(t)\, d(z - 1)/dt = -\lambda(t)(z - 1)^2 h(t) + (z - 1)\, dh(t)/dt, \quad \text{(A12)}$$

on multiplying by $h(t)$. The last equation has the form

$$\frac{h(\tau)}{z(\tau) - 1} - \frac{h(t)}{z(t) - 1} = k(\tau) - k(t) \quad \text{(A13)}$$

in which the right side is known; we therefore define another function

$$k(u) = \int_0^u (h(t)\lambda(t))\, dt \quad \text{(A14)}$$

which is also known, and integrate (A13) from t to τ to get

$$\frac{h(\tau)}{z(\tau) - 1} - \frac{h(t)}{z(t) - 1} = k(\tau) - k(t) \quad \text{(A15)}$$

so that we have for any pair of values (t, τ) along the path

$$z(\tau) - 1 = \frac{(z(t) - 1)h(\tau)}{h(t) - (z(t) - 1)(k(t) - k(\tau))}. \quad \text{(A16)}$$

In particular, since $h(0) = 1$ and $k(0) = 0$, we now have $z(0) = z_0$ in terms of $z(t_1) = z_1$ as

$$z_0 = 1 + \frac{(z_1 - 1)}{h(t_1) - (z_1 - 1)k(t_1)}. \quad \text{(A17)}$$

Consequently, we have obtained the PGF $G(z, t)$ at an arbitrary point (z_1, t_1), in terms of the initial PGF $G(z_0, 0)$, as in (A9):

$$G(z_1, t_1) = G(z_0, 0) \cdot \exp\left\{\int_0^{t_1} \frac{(z_1 - 1)h(t)\nu(t)}{h(t_1) - (z_1 - 1)(k(t_1) - k(t))}\, dt\right\}. \quad \text{(A18)}$$

Realizing that (z_1, t_1) was an arbitrary point, we can set $t_1 = t$ and $z_1 = z$ in (A17) and also in (A18) after changing the dummy integration variable to τ. Defining $Z(z, t; \tau)$ as $z(\tau)$ on the path from $(z_0, 0)$ and (z, t), we have the known function of z, t, and τ:

$$Z(z, t; \tau) = 1 + \frac{(z - 1)h(\tau)}{h(t) - (z - 1)(k(t) - k(\tau))}. \quad \text{(A19)}$$

In particular, $Z(z, t; 0) = z_0$, the initial point of the path from $(z_0, 0)$ to (z, t), is now known to be

$$Z(z, t; 0) = 1 + \frac{(z - 1)}{h(t) - (z - 1)k(t)} \quad \text{(A20)}$$

so that the PGF is given by

$$G(z, t) = G(Z(z, t; 0), 0) \cdot \exp\left\{\int_0^t [Z(z, t; \tau) - 1]\nu(\tau)\, d\tau\right\}. \quad \text{(A21)}$$

D. Evolution of the Count Mean and Variance

Since $Z(1, t; \tau) = 1$, we immediately confirm that $G(1, t)$, which is the sum of the probabilities at time (or depth) t, equals the initial sum $G(1, 0) = 1$, for all t. We also find the mean number of events or particles $N(t) = \langle n \rangle = \Sigma\, np(n, t) = \partial G(1, t)/\partial z$ to be given by

$$N(t) = \frac{1}{h(t)}\left(N_0 + \int_0^t h(\tau)\nu(\tau)\, d\tau\right) \quad \text{(A22)}$$

where N_0 is the mean number at the initial point $t = 0$. This follows directly from $\partial Z(z, t; \tau)/\partial z = h(t)h(\tau)/[h(t) - (z - 1)(k(t) - k(\tau)]^2 \rightarrow h(\tau)/h(t)$ as $z \rightarrow 1$. This expression for the mean also solves the differential equation that governs its evolution,

$$dN(t)/dt = -(\mu - \lambda)N(t) + \nu \quad \text{(A23)}$$

which arises directly from the definitions of the birth, death, and immigration parameters.

In a similar manner, from $\partial^2 G(1, t)/\partial z^2$, we find the evolution of the variance in the medium with varying BDI parameters to be given by

$$\sigma^2(t) = [\sigma_0^2 - N_0 + (2k + h)(N_0 + V_1) - V_2]/h^2 \quad \text{(A24)}$$

where

$$\sigma^2 = \langle (n - \langle n \rangle)^2 \rangle$$

and

$$V_1(t) = \int_0^t h(\tau)\nu(\tau)\, d\tau \qquad V_2(t) = \int_0^t h(\tau)k(\tau)\nu(\tau)\, d\tau \quad \text{(A25)}$$

are known iterated integrals of the BDI parameters.

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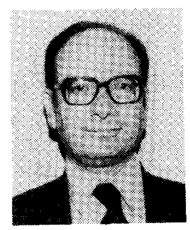

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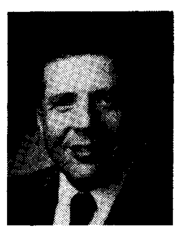

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