Statistical Properties of a Nonstationary Neyman–Scott Cluster Process

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***Abstract*—A recurrence relation is obtained for the counting distribution, as well as the probability density of waiting time, for a doubly stochastic Poisson point process driven by nonstationary shot noise (SNDP). For a stimulus of short duration, the counting distribution approximately reduces to the Neyman Type-A. The SNDP is an important special Neyman–Scott cluster process.**

I. Introduction

The Neyman–Scott cluster point process, originally developed in 1958 to describe the distribution of galaxies in space [1], has become an important representation for a broad range of phenomena in the physical, biological, and social sciences [2], [3]. Bartlett [4] has shown that the shot-noise driven doubly stochastic Poisson point process (SNDP) is a special but important example of a Neyman–Scott cluster process. This identity was subsequently explored by Lawrance [5]. The SNDP is a doubly stochastic Poisson point process (DSPP) [6] whose rate is a shot-noise process. It is of particular importance in electrical engineering [7], physics [8], neurophysiology [9], and geophysics [10], [11], though it was originally developed by Bartlett in connection with an ecological model.

In a recent series of papers we have examined the properties and applications of the stationary SNDP, with particular emphasis on its use in electrical engineering and physics. Explicit results have been obtained for the single-fold and multifold counting statistics, time statistics, and power spectrum [7]. The SNDP describes the photon statistics of shot-noise light. We have determined the degrees of freedom and degeneracy parameters for such light. It turns out that the excess photocount variance exhibits particlelike fluctuations, and is maximized for long counting times [8]. The interevent-time statistics [9] and counting statistics [12] have also been obtained in the presence of self-excitation (deadtime). An interesting outcome of our studies [7], [8] is that the counting distribution reduces to the Neyman Type-A distribution [13], [14] in the long counting-time limit (counting time T much longer than the characteristic time τ_p of the impulse-response function associated with the shot noise). Under certain conditions, this conclusion also applies in the presence of dead time [12]. The Neyman Type-A is a two-parameter counting distribution that characterizes the cascading of two Poisson processes.

In this note, we obtain the counting distribution, together with its mean and variance, for a nonstationary [15] SNDP (i.e., a DSPP whose stimulating rate is nonstationary shot noise). We also obtain the probability density function for the waiting time to the first event. We will show that the counting distribution again reduces to the Neyman Type-A distribution when the duration of the stimulating pulse τ_s is short. Finally a number of specific applications, in which the signal is a nonstationary pulse, will be briefly described.

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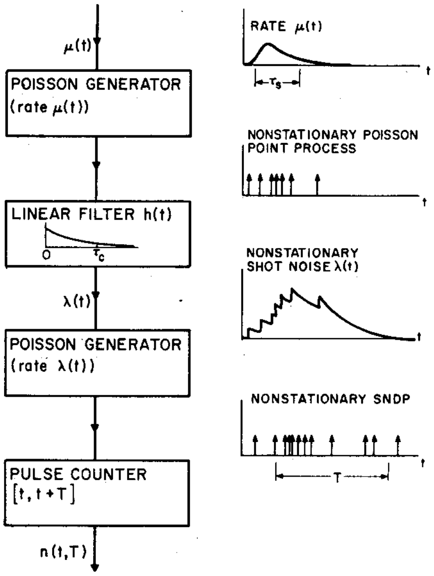

Fig. 1. Schematic representation for the generation of the nonstationary SNDP.

A schematic diagram illustrating the generation of the process is presented in Fig. 1. A nonstationary Poisson point process of rate $\mu(t)$ is passed through a linear filter of impulse response function $h(t)$. The resulting nonstationary shot-noise process $\lambda(t)$ forms the rate for a second Poisson generator, leading to the nonstationary SNDP.

II. Counting Distribution

To obtain the counting distribution over the interval $[t, t+T]$, as well as the time statistics of a DSPP, we first determine the statistics of the integrated rate

$$W(t,T) = \int_t^{t+T} \lambda(x)\, dx. \tag{1}$$

The statistics of $W(t,T)$ are found from the statistics of $\lambda(t)$. The statistical properties of nonstationary shot noise have been well documented in the literature [16], [17]. The mean and variance are

$$\langle \lambda(t) \rangle = \mu(t) \circledast h(t) \tag{2}$$

and

$$\operatorname{var}[\lambda(t)] = \mu(t) \circledast h^2(t), \tag{3}$$

respectively, and the characteristic function (cf.) is

$$\phi_{\lambda(t)}(j\omega) = \langle \exp[j\omega\lambda(t)] \rangle = \exp\{\mu(t) \circledast [e^{j\omega h(t)} - 1]\}. \tag{4}$$

The symbol $\circledast$ represents the operation of convolution. Conditions for the validity of (4) are discussed in [17, pp. 168–170]. The functions $\mu(t)$ and $h(t)$ must satisfy the condition $\omega^2\langle\lambda^2(t)\rangle = \omega^2\{\mu(t)\circledast h^2(t) + [\mu(t)\circledast h(t)]^2\} < \infty$. Actually, Snyder [17, pp. 168–170] proved a more general result for conditions of existence of the characteristic functional, from which the preceding condition is obtained as a special case.

The statistical properties of the integrated rate $W(t,T)$ can be determined by regarding it as a shot-noise process generated when Poisson pulses of rate $\mu(t)$ are filtered by a cascade of two linear filters, one with impulse response $h(t)$, followed by a second one, which is an integrator over $[t, t+T]$. The cascade is

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equivalent to a single linear filter of impulse response

$$h_T(t) = \int_t^{t+T} h(x)\,dx. \tag{5}$$

It therefore follows that the mean, variance, and cf. of $W(t,T)$ are

$$\langle W(t,T)\rangle = \mu(t) \circledast h_T(t), \tag{6}$$

$$\operatorname{var}[W(t,T)] = \mu(t) \circledast h_T^2(t), \tag{7}$$

and

$$\phi_{W(t,T)}(j\omega) = \exp\{\mu(t) \circledast [e^{j\omega h_T(t)} - 1]\}, \tag{8}$$

respectively. Because $W(t,T)$ is a shot-noise process, the condition for the validity of (8) is $\omega^2\langle W^2(t,T)\rangle < \infty$ [17, pp. 168–170].

Given the statistics of the integrated rate $W(t,T)$, we readily obtain the statistics for the SNDP. If n represents the number of events in an interval $[t, t+T]$, then [16]

$$\langle n\rangle = \langle W(t,T)\rangle = \mu(t) \circledast h_T(t) \tag{9}$$

$$\begin{aligned}\operatorname{var}(n) &= \langle W(t,T)\rangle + \operatorname{var}[W(t,T)]\\ &= (1+a)\langle n\rangle,\end{aligned} \tag{10}$$

where

$$a = [\mu(t) \circledast h_T^2(t)]/[\mu(t) \circledast h_T(t)]. \tag{11}$$

Also

$$\begin{aligned}\phi_n(j\omega) &= \phi_{W(t,T)}(e^{j\omega} - 1)\\ &= \exp\left(\mu(t) \circledast \{\exp[-(1-e^{j\omega})h_T(t)] - 1\}\right).\end{aligned} \tag{12}$$

The counting distribution $p[n(t,T)]$ can be obtained from the relation [18]

$$p(n) = \left.\frac{(-j)^n}{n!}\frac{\partial^n}{\partial\omega^n}\phi_{W(t,T)}(j\omega)\right|_{\omega=j}. \tag{13}$$

Combining (8) and (13) leads to the recurrence relation

$$(n+1)p(n+1) = \langle n\rangle \sum_{l=0}^{n} C_l p(n-l) \tag{14}$$

$$p(0) = \exp\{\mu(t) \circledast [e^{-h_T(t)} - 1]\}, \tag{15}$$

where

$$C_l = \frac{1}{l!}\{\mu(t) \circledast [h_T^{l+1}(t)e^{-h_T(t)}]\}/\{\mu(t) \circledast h_T(t)\}. \tag{16}$$

As expected, setting $\mu(t) = \mu$ in (9)–(16) reproduces our previously obtained results for the stationary SNDP.

We now consider an important limiting case, in which the rate $\mu(t)$ is an impulse

$$\mu(t) = E\delta(t), \tag{17}$$

of strength E, where $\delta(t)$ is the Dirac delta function. Substituting (17) into (9) and (10) yields a count mean and variance given by

$$\begin{aligned}\langle n\rangle &= Eh_T(t)\\ \operatorname{var}(n) &= (1+a)\langle n\rangle, \qquad a = h_T(t).\end{aligned} \tag{18}$$

Eq. (12) becomes

$$\begin{aligned}\phi_n(j\omega) &= \exp\left(E\{\exp[-(1-e^{j\omega})h_T(t)] - 1\}\right)\\ &= \exp\left(\frac{\langle n\rangle}{a}\{\exp[-a(1-e^{j\omega})] - 1\}\right).\end{aligned} \tag{19}$$

This is identically the cf. for the Neyman Type-A distribution of mean $\langle n\rangle$ and parameter a.

If the rate $\mu(t)$ has a time course τ_s that is very short in duration compared with the sum of the width τ_p, of the impulse response function $h(t)$, and the counting time T ($\tau_s \ll \tau_p + T$), then the mean and variance of the distribution are approximately given by (18) with $E = \int\mu(t)\,dt$. Moreover, if $\tau_s \ll [\tau_p + T]/N$ then, for an exponentially decaying $h(t)$, the probability distribution $p(n)$ may be approximated by a Neyman Type-A distribution for all $n < N$. This may be shown by noting that since $h(t)$ is a decaying function of width τ_p, $h_T(t)$ will have an approximate width $T + \tau_p$. If τ_s, the width of $\mu(t)$, is much shorter than $T + \tau_p$, then in (9) and (11), convolution with $\mu(t)$ may be approximately replaced by multiplication by $E = \int\mu(t)\,dt$. Eq. (18) is then reproduced. Furthermore, in (16), the width of $h_T^l(t)$ is approximately $(T+\tau_p)/l$. The product $h_T^l(t)\exp(-h_T(t))$ has approximately the same width, $(T+\tau_p)/l$. Therefore, if $\tau_s \ll [\tau_p + T]/N$, in the computation of C_l, for $l < N$, $\mu(t)$ may be replaced by a delta function. Consequently, $p(n)$, which is computed from (14), may be approximated by a Neyman Type-A distribution for $n < N$.

We conclude that an SNDP, stimulated by a rate of sufficiently short duration, yields approximately the Neyman Type-A counting distribution with parameter

$$a = h_T(t) = \int_t^{t+T} h(x)\,dx. \tag{20}$$

If $h(t)$ is a decaying function of time, the probability distribution of the number of counts over the interval $[t, t+T]$ remains Neyman Type-A as t is increased, but has a decaying mean and a decaying parameter. Eventually, for large t, the distribution approaches the Poisson distribution. This limiting case will remain valid as long as the width of the function $\mu(t)$ is much less than that of the impulse response $h(t)$.

Finally, we note that if the rate $\mu(t)$ consists of a superposition of a uniform with a pulsed rate $\mu(t)$, the overall counting distribution in the limit of very short τ_s will also be approximately Neyman Type-A.

III. Waiting Time to the First Event

For a DSPP, the distribution of the waiting time τ (the time interval from an arbitrary time t to the time at which the first event occurs) is [18]

$$\begin{aligned}p_t(\tau) &= \left\langle \lambda(t+\tau)\exp\left[-\int_t^{t+\tau}\lambda(t')\,dt'\right]\right\rangle\\ &= -\frac{\partial}{\partial\tau}\phi_{W(t,\tau)}(-1).\end{aligned} \tag{21}$$

Using (5) and (8), we obtain

$$p_t(\tau) = \{\mu(t) \circledast [h(t+\tau)e^{-h_\tau(t)}]\} \cdot \exp\{\mu(t) \circledast [e^{-h_\tau(t)} - 1]\}. \tag{22}$$

For a stationary SNDP, (22) reduces to the previously obtained result [7].

For a nonstationary SNDP whose primary rate $\mu(t)$ is of duration very short compared to τ_p, we can use (17) to obtain

$$p_0(\tau) \approx Eh(\tau)e^{-h_\tau(0)}\exp\{E[e^{-h_\tau(0)} - 1]\}, \tag{23}$$

when the starting time is $t = 0$. This may be shown by noting that if $h(t)$ has a width τ_p, $h_\tau(t)$ has a width $\tau_p + \tau$, and therefore both functions $[e^{-h_\tau(t)} - 1]$ and $h(t+\tau)e^{-h_\tau(t)}$ in (22) have widths $\tau_p + \tau$. If τ_s, the width of $\mu(t)$, is much shorter than τ_p, it is also much shorter than $\tau_p + \tau$ and therefore the convolutions in (22) may be approximated by

$$\mu(t) \circledast [h(t)e^{-h_\tau(t)}] \approx Eh(t)e^{-h_\tau(t)} \tag{24}$$

and

$$\mu(t) \circledast [e^{-h_\tau(t)} - 1] \approx E[e^{-h_\tau(t)} - 1]. \tag{25}$$

The validity of (23) is subject to the validity of the approximations in (24) and (25), which will hold for τ_s sufficiently smaller than τ_p.

It is important to note that $p_0(\tau)$, as obtained from (21), (22), or (23), is not always a proper probability density function. In situations where $\langle \int_0^\infty \lambda(t)\,dt \rangle$ is finite, $\int_0^\infty p_0(\tau)\,d\tau \neq 1$. This is a manifestation of the finite probability (given by $\langle \exp[-\int_0^\infty \lambda(t)\,dt] \rangle$) that zero events occur in the interval $[0, \infty]$, thereby resulting in an infinite waiting time.

It is therefore appropriate to normalize (22) and (23) by the factor $\exp\{E[e^{-h_\infty(t)} - 1]\}$, in order to provide a probability density function of unit area. This peculiarity is not present in the stationary case.

In the limit of a weak-impulsive stimulus ($E \ll 1$), (23) can be approximated by

$$p_0(\tau) \approx Eh(\tau)\exp\left[-\int_0^\tau h(t)\,dt\right]. \tag{26}$$

When properly normalized, this is identical to the waiting time for a nonstationary (inhomogeneous) Poisson process of rate $\lambda(t) = h(t)$. This limit can be understood by examining Fig. 1. When the primary impulsive rate is weak, the first Poisson generator produces (with probability E) a single pulse. This pulse, in turn, gives rise to a function $h(t)$ that acts as a driving rate for the second Poisson generator. Thus the waiting time distribution will have the same shape as that of a simple nonstationary Poisson process with rate $\lambda(t) = h(t)$.

As an example, we consider a system with an exponential impulse-response function

$$h(t) = (2\alpha/\tau_p)\exp(-2t/\tau_p), \qquad t \geq 0, \tag{27}$$

excited by a short-duration impulsive stimulus. The quantity τ_p is the decay time of $h(t)$ and α is its area. The parameter α represents the average number of secondary events per primary event (or the multiplication parameter of the SNDP). Using (23), we obtain

$$p_0(\tau) \approx \frac{2N}{\tau_p}\exp\left(-\frac{2\tau}{\tau_p} - \alpha(1 - e^{-2\tau/\tau_p}) + \frac{N}{\alpha}\left\{\exp\left[-\alpha(1 - e^{-2\tau/\tau_p})\right] - 1\right\}\right), \tag{28}$$

where $N = \alpha E$ represents the average total number of events in the SNDP over the entire time interval $[0, \infty]$. To make its area unity, this function must be normalized by the factor $\exp\{E[e^{-\alpha} - 1]\}$, as discussed earlier. For E and α very small (i.e., N also very small), (28) becomes the exponential distribution

$$p_0(\tau) \approx (2N/\tau_p)\exp(-2\tau/\tau_p), \tag{29}$$

as expected.

IV. Applications

The foregoing model is expected to provide a useful representation for applications in a number of fields. An example of a physical process describable by our results is the generation of cathodoluminescence radiation by a pulsed electron beam.

A number of information transmission and processing systems, in which the signal is a nonstationary pulse, can be characterized by the treatment presented here. Examples of such systems, in which the signal is a pulse of radiation, are the image intensifier (the signal is a pulse of light, infrared radiation, or X rays), X-ray tomography (the signal is a pulse of X rays), and systems incorporating X-ray film and intensifying screens [19] (the signal in that case is a small spot of light, the spatial analog of a short pulse).

An application of importance that has been studied in great detail, is the transmission of visual information from the retina to higher visual centers through the optic nerve [9]. A series of experiments characterizing the neural counting distribution in the mammalian retinal ganglion cell has recently been conducted by Barlow, Levick, and Yoon [20]. These authors illuminated a small spot on the retina by means of a short ($\tau_s = 10$ ms) flash of (Poisson) light, and recorded the probability distribution for the number of nerve spikes in a much longer interval of time ($T = 200$ ms). They showed that the mean and variance of the Neyman Type-A counting distribution, though they did not refer to it by name, adequately accounted for the experimental observations of these quantities. From their study, they concluded that single absorbed quanta can cause multiple nerve impulses at the ganglion cell, representing the kind of two-step statistical process discussed here. Indeed, the results of our nonstationary analysis predict just such behavior when $\tau_s \ll \tau_p + T$. Since τ_p is known to be $\simeq 30$ ms from other studies [9], this condition is well satisfied in the ganglion cell experiments.

Finally, we note that applications involving a cascade of Poisson processes, driven by stationary and nonstationary signals, have been considered elsewhere [21].

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