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Photocounting Array Receivers for Optical Communication Through the Lognormal Atmospheric Channel. 3: Error Bound for M-ary Equal-Energy Orthogonal Signaling

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Abstract—**A bound on the probability of error is obtained for an M-ary direct-detection optical communications system consisting of an amplitude-stabilized source, a lognormal atmospheric channel, and a photocounting detector array. Equal-energy, equiprobable, orthogonal signaling, and flat independent fading at all detectors is assumed. The result reduces to that obtained previously in the absence of fading. A comparison is made with the analogous solution for the heterodyne array receiver.**

I. INTRODUCTION

In Part 1 of this set of papers [1], we investigated the structure of various optimum and suboptimum photocounting array receivers for lognormally-faded optical radiation. In Part 2 [2], a summary of receiver performance was presented (in terms of error probabilities), assuming both orthogonal and nonorthogonal binary signaling formats. We now consider the performance of a D-detector receiver, under the assumptions of M-ary signaling and independent flat lognormal fading. An upper bound to the error probability is derived for orthogonal, equal-energy, equiprobable signal sets. The result is shown to reduce to that obtained previously in the absence of fading and is compared with the analogous solution for *heterodyne* array detection.

II. THEORY

The error probability $P(\varepsilon)$ for any one of M equiprobable waveforms may be written as [2]

$$P(\varepsilon) = 1 - \int_{-\infty}^{\infty} p(L_1)\, dL_1 \int_{-\infty}^{L_1} \cdots \int_{-\infty}^{L_1} p(L_2,L_3,\cdots,L_M)\, dL_2 \cdots dL_M. \quad (1)$$

For the usual Gaussian detection problem one can sometimes obtain the statistics of the likelihood functionals L_k, and thus evaluate (1) directly. Often, however, this expression is intractable for $M > 2$, and the usual approach is to obtain bounds to this quantity. Such bounds can generally be expressed in the form

$$\mathscr{B}_1 2^{-\kappa E} \le P(\varepsilon) \le \mathscr{B}_2 2^{-\kappa E} \quad (2)$$

where the quantity $\kappa = \log_2 M$ represents the number of bits per message, E is the system reliability function, and $\mathscr{B}_1$ and $\mathscr{B}_2$ are slowly varying functions of κ and other parameters such as the log-irradiance standard deviation σ, the mean count signal-to-noise ratio γ, and the number of detectors D. (Symbol definitions are the same as in Parts 1 and 2.) The system reliability function may usually be written in the form

$$E = \max_{0 \le \rho \le 1} \left\{ \frac{\beta E_0(\rho)}{\ln 2} - \rho \right\} \quad (3)$$

where ρ is a parameter which maximizes the exponent (and thus provides the tightest bound on $P(\varepsilon)$), β is the total detected signal energy per information bit, and $E_0(\rho)$ is a function which depends on the receiver structure. Specifically we will examine the upper bound to the error probability

$$P(\varepsilon) \le \mathscr{B}_2 2^{-\kappa E}. \quad (4)$$

Using standard Chernoff-bounding techniques [3], and assuming equally likely equal-energy orthogonal signals, the upper error bound may be written as

$$P(\varepsilon) \le M^{\rho} \exp\, [\rho \xi_0(t) + \xi_1(-\rho t)]. \quad (5)$$

Under the additional assumption of independent equal-strength (flat) fading at all D detectors $[\sigma_i = \sigma_j;\ i,j = 1,\cdots,D]$, and using (I.18) from Part 1, the quantities $\xi_0(t)$ and $\xi_1(-\rho t)$ are easily identified as [4]

$$\xi_0(t) = D \ln \sum_{n=0}^{\infty} \frac{N_B{}^n \exp(-N_B)}{n!} F(n,\gamma,N_B,\sigma)^t$$

$$\xi_1(-\rho t) = D \ln \sum_{n=0}^{\infty} \frac{N_B{}^n \exp(-N_B)}{n!} F(n,\gamma,N_B,\sigma)^{1-\rho t} \quad (6)$$

where the function $F(n,\gamma,N_B,\sigma)$ is defined as

$$F(n,\gamma,N_B,\sigma) \equiv \int_0^{\infty} (Z\gamma + 1)^n \exp(-ZN_B\gamma) p(Z)\, dZ. \quad (7)$$

The quantities N_S and N_B represent the mean signal and noise counts, respectively, $\gamma \equiv N_S/N_B$, Z is the lognormal variate, and t is a parameter with respect to which the exponent in (5) must be minimized. The optimum choice for t can be shown to be [5] $t = (1 + \rho)^{-1}$, whereupon (5) reduces to

$$P(\varepsilon) \le M^{\rho} \exp\, [(1 + \rho)\xi_0(\rho)]. \quad (8)$$

Converting (8) to the base 2, and recalling that $\log_2 e = (\ln 2)^{-1}$, we obtain

$$P(\varepsilon) \le \text{two} \left\{ -\kappa \left[-\frac{(1+\rho)D}{\kappa \ln 2} \cdot \ln \left(\sum_{n=0}^{\infty} \frac{N_B{}^n \exp(-N_B)}{n!} F(n,\gamma,N_B,\sigma)^{1/1+\rho} \right) - \rho \right] \right\} \quad (9)$$

where the expression two $\{\cdot\}$ indicates $2^{\{\cdot\}}$. The error bound is now expressed in the simplified form

$$P(\varepsilon) \le 2^{-\kappa E}. \quad (10)$$

Comparing (3), (9), and (10), and with the definition

$$\beta \equiv DN_S/\kappa \quad (11)$$

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we obtain the following relation for the quantity $E_0(\rho)$:

$$E_0(\rho) = -\frac{(1+\rho)}{N_B\gamma} \cdot \ln\left(\sum_{n=0}^{\infty} \frac{N_B^n \exp(-N_B)}{n!} F(n,\gamma,N_B,\sigma)^{1/1+\rho}\right). \quad (12)$$

We may further simplify this expression by removing the factor $\exp(-N_B)$ to obtain

$$E_0(\rho) = \frac{(1+\rho)}{\gamma} - \frac{(1+\rho)}{N_B\gamma} \ln\left(\sum_{n=0}^{\infty} \frac{N_B^n}{n!} F(n,\gamma,N_B,\sigma)^{1/1+\rho}\right). \quad (13)$$

The quantity $\gamma = N_S/N_B$ may be thought of as the signal-to-noise ratio per diversity path and $\beta = DN_S/\kappa$ as the total mean detected signal count per information bit.

III. Error Bound

In order to obtain the tightest error bound, we must optimize the system reliability E given in (3) over the parameter ρ. This optimization is frequently difficult and sometimes not possible; nevertheless, some properties of E readily follow. Taking the derivative of E, we obtain

$$\frac{\partial E}{\partial \rho} = \frac{\beta}{\ln 2} \frac{\partial E_0(\rho)}{\partial \rho} - 1 = 0 \quad (14)$$

and ρ_{opt} is therefore found from the relation

$$\frac{\partial E_0(\rho)}{\partial \rho} = \frac{\ln 2}{\beta}. \quad (15)$$

In this case the system reliability becomes

$$E = \left\{E_0(\rho)\left(\frac{\partial E_0(\rho)}{\partial \rho}\right)^{-1} - \rho\right\}\Bigg|_{\rho=\rho_{\text{opt}}} \quad (16)$$

provided that [5]

$$\left(\frac{\partial E_0(\rho)}{\partial \rho}\right)^{-1}\Bigg|_{\rho=0} \le \frac{\beta}{\ln 2} \le \left(\frac{\partial E_0(\rho)}{\partial \rho}\right)^{-1}\Bigg|_{\rho=1}. \quad (17)$$

We define the low and high endpoints of this inequality by $\beta_{\min}/\ln 2$ and $\beta_{\text{crit}}/\ln 2$, respectively, so that

$$\beta_{\min} \le \beta \le \beta_{\text{crit}}. \quad (18)$$

In the situation where $\beta > \beta_{\text{crit}}$, we set $\rho = 1$, and (3) becomes

$$E = \frac{\beta E_0(1)}{\ln 2} - 1, \qquad \beta > \beta_{\text{crit}} \quad (19)$$

which increases with increasing β.

Although the system reliability function must be evaluated numerically, some insight into its variation with a number of parameters can be obtained. Thus the exponent κE increases at least linearly with D for fixed γ, N_B, and κ as in the heterodyne case. Such an increase corresponds to increasing the number of detectors in the array and thus the receiver aperture. Furthermore, it can be shown that for $\sigma > 0$ and constant, $E_0(\rho)$ vanishes as $\gamma \to \infty$. Thus for fixed alphabet size 2^κ and number of diversity paths D, the error exponent κE will increase at less than a linear rate with β, that is, with total detected signal energy per bit. In addition, γ must be manipulated independently of β and κ in order to obtain a linear variation of κE with β. This implies an optimum value for D and thus for γ, with DN_S constant, as has been seen in the error probability curves presented in Part 2 [2].

We can easily obtain explicit expressions for the quantities $\beta_{\min}$ and β_{crit} in (17) and (18) by differentiating the quantity $E_0(\rho)$ given in (13)

$$\frac{\partial E_0(\rho)}{\partial \rho} = \frac{1}{\gamma} - \frac{1}{N_B\gamma} \ln\left(\sum_{n=0}^{\infty} \frac{N_B^n}{n!} F(n,\gamma,N_B,\sigma)^{1/1+\rho}\right) + \frac{1}{N_B\gamma(1+\rho)}\left[\frac{\sum_{n=0}^{\infty} \frac{N_B^n}{n!} F(n,\gamma,N_B,\sigma)^{1/1+\rho} \ln F(n,\gamma,N_B,\sigma)}{\sum_{n=0}^{\infty} \frac{N_B^n}{n!} F(n,\gamma,N_B,\sigma)^{1/1+\rho}}\right]. \quad (20)$$

Evaluating this at $\rho = 0$ yields

$$\beta_{\min} = \gamma \ln 2 \left[1 - \frac{1}{N_B} \ln\left(\sum_{n=0}^{\infty} \frac{N_B^n}{n!} F(n,\gamma,N_B,\sigma)\right) + \frac{1}{N_B}\left(\frac{\sum_{n=0}^{\infty} \frac{N_B^n}{n!} F(n,\gamma,N_B,\sigma) \ln F(n,\gamma,N_B,\sigma)}{\sum_{n=0}^{\infty} \frac{N_B^n}{n!} F(n,\gamma,N_B,\sigma)}\right)\right]^{-1}. \quad (21)$$

This quantity represents the minimum total detected energy per bit for reliable communication and is seen to be a function of γ and σ as well as of the noise level N_B. Evaluation at $\rho = 1$ yields

$$\beta_{\text{crit}} = \gamma \ln 2 \left[1 - \frac{1}{N_B} \ln\left(\sum_{n=0}^{\infty} \frac{N_B^n}{n!} F(n,\gamma,N_B,\sigma)^{1/2}\right) + \frac{1}{2N_B}\left(\frac{\sum_{n=0}^{\infty} \frac{N_B^n}{n!} F(n,\gamma,N_B,\sigma)^{1/2} \ln F(n,\gamma,N_B,\sigma)}{\sum_{n=0}^{\infty} \frac{N_B^n}{n!} F(n,\gamma,N_B,\sigma)^{1/2}}\right)\right]^{-1}. \quad (22)$$

IV. Results for $\sigma \to 0$

The function $E_0(\rho)$ given in (13) can be shown to reduce to that given by Kennedy for counting receivers with M-ary equal-energy orthogonal signals in the absence of fading [6], [7]. For $\sigma \to 0$, (7) becomes

$$\lim_{\sigma \to 0} F(n,\gamma,N_B,\sigma) = (\gamma+1)^n \exp(-\gamma N_B) \quad (23)$$

and we therefore obtain

$$\sum_{n=0}^{\infty} \frac{N_B^n}{n!} F(n,\gamma,N_B,\sigma)^{1/1+\rho} = \exp\left\{N_B(\gamma+1)^{1/1+\rho} - \frac{\gamma N_B}{1+\rho}\right\} \qquad \sigma = 0 \quad (24)$$

whereupon

$$E_0(\rho) = \frac{(1+\rho)}{\gamma} - \frac{(1+\rho)}{\gamma}(\gamma+1)^{1/1+\rho} + 1, \qquad \sigma = 0 \quad (25)$$

in agreement with Kennedy's result.

Furthermore, for $\beta_{\min} \le \beta \le \beta_{\text{crit}}$, the system reliability is obtained by means of (3) and (25), which give

$$E = \left\{\frac{\beta}{\gamma \ln 2}\left[1 + \rho + \gamma - (1 + \rho)(\gamma + 1)^{1/1+\rho}\right] - \rho\right\}\Bigg|_{\rho=\rho_{\text{opt}}}, \quad 0 \le \rho_{\text{opt}} \le 1. \quad (26)$$

The quantity ρ_{opt}, obtained from (15) and (20) with $\sigma = 0$, is given by

$$\frac{\ln 2}{\beta} = \frac{\partial E_0(\rho)}{\partial \rho} = \frac{1}{\gamma}\left\{1 - (\gamma + 1)^{1/1+\rho}\left[1 - \frac{\ln(\gamma + 1)}{(1 + \rho)}\right]\right\}, \quad \sigma = 0. \quad (27)$$

For the case $\beta > \beta_{\text{crit}}$, in accordance with (19) we set $\rho = 1$ in (26) to obtain

$$E = \frac{\beta}{\gamma \ln 2}\left[(\gamma + 1)^{1/2} - 1\right]^2 - 1, \qquad \beta > \beta_{\text{crit}}. \quad (28)$$

Finally, setting $\sigma = 0$ in (21) and (22), respectively, yields the quiescent-atmosphere expressions

$$\beta_{\min} = \frac{\gamma \ln 2}{-\gamma + (\gamma + 1)\ln(\gamma + 1)}, \qquad \sigma = 0 \quad (29)$$

and

$$\beta_{\text{crit}} = \frac{\gamma \ln 2}{1 - (\gamma + 1)^{1/2}\left[1 - \frac{1}{2}\ln(\gamma + 1)\right]}, \qquad \sigma = 0. \quad (30)$$

We note that (16a) of [6] is incorrect: a factor of $\frac{1}{2}$ should appear before the logarithmic term in the denominator as obtained in (30).

V. Comparison with Heterodyne Detection

The quantity analogous to $E_0(\rho)$ as represented in (12) and (13) has been evaluated by Kennedy and Hoversten [8] for heterodyne detection and is given by

$$E_0^H(\rho) = -\frac{(1 + \rho)}{\alpha_p}\ln\left\{\int_0^\infty dy \exp(-y)F_r(\alpha_p, y, \sigma)^{1/1+\rho}\right\} \quad (31)$$

where $F_r(\alpha_p, y, \sigma)$ is the "frustration" function defined as [9]

$$F_r(\alpha_p, y, \sigma) = \int_0^\infty I_0(2u\sqrt{y\alpha_p}) \exp(-\alpha_p u^2) p(u)\, du. \quad (32)$$

Here $I_0(\cdot)$ is the modified Bessel function of order zero, α_p is the signal energy-to-noise power density ratio, and $p(u)$ is the log-*amplitude* density function. The frustration function is to be compared with the analogous direct-detection quantity $F(n,\gamma,N_B,\sigma)$ given in (7), where the correspondence between α_p and γ is $\alpha_p \leftrightarrow \gamma N_B$. The quantity y in (31) and (32) corresponds to the magnitude of a field sample, whereas the analog in (7) is the photoelectron count n.

The usual distinction is observed: (31) and (32) depend only on the signal-to-noise ratio, whereas (7) and (12) also depend on the mean signal energy. A detailed comparison of direct and heterodyne detection systems is difficult, however, because of the complex form of the error bound and the lack of quantitative error probability curves for the latter. Extensive numerical evaluation of the two error exponents should reveal in greater detail which of the two is larger under a given set of conditions.

The basic distinction between the two systems is that in direct detection the choice of signal set and appropriate preprocessing of the field to reduce background noise is very important, while the spatial processing is not as critical. In the heterodyne case, the basic limiting factor in performance is the spatial processing —the object is to extract as much coherent signal as possible out of the faded wavefront. The latter generally requires a much more complex and critical receiver structure. A basic advantage of the direct detection system is its ability to gate out background radiation by using very high data rates [10], [11]. Thus we can conjecture that in the presence of turbulence, as well as in its absence, and in wavelength regions where photocounting can be performed, the direct detection system will be simpler to construct and perform better than the analogous heterodyne system.

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Sampled Data Reconstruction of Deterministic Band-Limited Signals

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***Abstract*—In obtaining a discrete set of data points to represent a signal, the problem of how to obtain sufficient information about the signal can become pronounced if the rate of obtaining samples of the signal is limited. The multiple-channel interpolation scheme presented in this correspondence uses periodic samples of the output of a set of prefilters excited by a deterministic input signal to reconstruct the original input signal. The input signals considered are band limited, and the sampling rate proposed is less than the Nyquist rate. This procedure may have applications in the analysis of transient signals.**

Introduction

An interpolation scheme based upon a rate of sampling less than twice the highest frequency component of a signal has been of interest to mathematicians as well as to engineers. Shannon [1], Fogel [2], Linden [3], and Kahn and Liu [4] are some of the authors who have presented schemes for obtaining information about a signal by using samples.

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