Fractal Point Events in Physics, Biology, and Communication Networks

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1. OUTLINE

1. OUTLINE
2. POINT EVENTS
3. FRACTALS AND FRACTAL POINT EVENTS
4. FRACTAL PHOTONS
5. HEART RATE VARIABILITY
6. NETWORKS
7. SENSORY DETECTION
2. POINT EVENTS

COUNTING POINT EVENTS IN A FIXED TIME WINDOW \( T \)

Poisson process (zero-memory):

For this process, the counting distribution \( p(n) \) (i.e., the relative frequency or probability mass function) for the number of events \( n \) is characterized by the Poisson distribution,

\[
p(n) = \frac{(\lambda T)^n e^{-\lambda T}}{n!},
\]

whose variance-to-mean ratio \( F \) is unity, so \( \text{var}(n) \equiv \sigma^2 = \bar{n} = \lambda T \):

IMPORTANT EXAMPLES OF ANTICLUSTERED EVENTS VIA DEAD-TIME-MODIFICATION.

IMPORTANT EXAMPLES OF CLUSTERED EVENTS: NEGATIVE-BINOMIAL (NB) AND NEYMAN TYPE-A (NTA) DISTRIBUTIONS, AND VARIATIONS THEREOF.
NEURAL COUNTING AND PHOTON COUNTING


Volume 36, Number 13

PHYSICAL REVIEW LETTERS

29 March 1976

Neural Counting and Photon Counting in the Presence of Dead Time*

Malvin Carl Teich† and William J. McGill
Columbia University, New York, New York 10027
(Received 10 November 1975)

The usual stimulus-based neural counting model for audition is found to be mathematically identical to the well-known semiclassical formalism for photon counting. In particular, we explicitly demonstrate the equivalence of McGill’s noncentral negative binomial distribution and Pešina’s multimode confluent hypergeometric distribution for a coherent signal imbedded in chaotic noise. Dead-time corrections, important both in neural counting and in photon counting, are incorporated in a generalized form of this distribution. Some specific implications of these results are discussed.
Neural Counting Mechanisms and Energy Detection in Audition

WILLIAM J. McGINL
University of California San Diego, California 92037

NEYMAN TYPE-A DISTRIBUTION – BUGS

On a New Class of "Contagious" Distributions, Applicable in Entomology and Bacteriology

J. Neyman


\[
p(n) = \sum_{m=0}^{\infty} p(n|m) p(m)
\]

\[
= \sum_{m=0}^{\infty} \left( \frac{\alpha^m}{m!} e^{-\alpha m} \right) \frac{\langle m \rangle^m}{m!} e^{-\langle m \rangle m!}
\]

\[
p(0) = \exp \left[ -\langle m \rangle (1 - e^{-\alpha}) \right]
\]
Photon Counting and Energy Detection in Vision

Malvin Carl Teich and Paul R. Prucnal

Department of Electrical Engineering and Computer Science, Columbia University, New York, New York 10027


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CASCADE VARIANCE THEOREM

\[ \langle n \rangle = \langle \alpha \rangle \langle m \rangle \]

\[ \text{var}(n) = \langle \alpha \rangle^2 \text{var}(m) + \langle m \rangle \text{var}(\alpha) \]

\[ F_n \equiv \frac{\text{var}(n)}{\langle n \rangle} = \langle \alpha \rangle F_m + F_\alpha \]

for \( F_m = F_\alpha = 1 \):

\[ F_n = 1 + \langle \alpha \rangle \quad \text{(NTA)} \]
GENERATING PAIRS OF POINT EVENTS VIA SPONTANEOUS PARAMETRIC DOWN-CONVERSION: ENTANGLED PHOTONS

Conservation of energy\[ \omega_\rho = \omega_s + \omega_i \]
Conservation of momentum\[ \mathbf{k}_\rho = \mathbf{k}_s + \mathbf{k}_i \]

WHAT ARE THE COUNTING STATISTICS OF THESE POINT EVENTS?
Eighth-nerve-fiber neural-counting experiments carried out at different sound-pressure levels using windows of $T = 50$ msec and $T = 200$ msec duration.

SOME EVIDENCE OF EVENT PAIRS: MIGHT A MULTINOMIAL MODEL REVEAL A HIDDEN NEURAL CODE?
3. FRACTALS AND FRACTAL POINT EVENTS


FORMS OF FRACTALS
- Deterministic
- Random
- Static
- Dynamical process

CUTOFFS
- Inner
- Outer

DEFINITIONS
- Scaling (both cutoffs)
- Fractal (no inner cutoff)
- Long-Range Dependence (no outer cutoff)

VIEWING THE WORLD AT MULTIPLE SCALES

Volume:

Area:

Line:

Coastline Lengths at Different Scales:

<table>
<thead>
<tr>
<th>MEASUREMENT SCALE (km)</th>
<th>MEASURED LENGTH (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>69.4</td>
<td>133</td>
</tr>
<tr>
<td>6.94</td>
<td>314</td>
</tr>
<tr>
<td>0.694</td>
<td>534</td>
</tr>
</tbody>
</table>

The dependence of the measurement outcome on the scale chosen to make that measurement is the hallmark of a FRACTAL OBJECT.

Penck (1894); Richardson (1961): $d \propto s^c$

$\therefore c = -0.30$
UBIQUITY OF FRACTAL BEHAVIOR

• Mathematics and physical sciences
  • Fractal geometry of nature
  • Noise in electronic components
  • Fabricated nonperiodic layered structures
  • Errors in telephone networks
  • Photon statistics of Čerenkov radiation
  • Earthquake patterns
  • Computer network traffic

• Neurosciences
  • Ion channels
  • Membrane voltages
  • Vesicular exocytosis and MEPCs
  • Action-potential sequences
  • Networks of cortical neurons
  • Loudness and brightness functions
  • Natural course of forgetting

• Medicine and human behavior
  • Human standing and human gait
  • Mood
  • Human heartbeat patterns
  • Movement patterns influenced by drugs
**Power-Law Shot Noise**

STEVEN B. LOWEN, STUDENT MEMBER, IEEE, AND MALVIN C. TEICH, FELLOW, IEEE

\[ h(K, t) = \begin{cases} 
  K t^{-\beta} & A < t < B \\
  0 & \text{otherwise},
\end{cases} \]

- \( A > 0 \) \( B = \infty \)
- \( A = 0 \) \( B = \infty \)
- \( A = 0 \) \( B < \infty \)
- \( A > 0 \) \( B < \infty \)

- \( 0 < \beta < \frac{1}{2} \)
- \( 2 > \alpha > 1 \)
- \( \frac{1}{2} \leq \beta < 1 \)
- \( 1 \geq \alpha > 0 \)
- \( \beta = 1 \)
- \( \beta > 1 \)
- \( 0 < \zeta < 1 \)

- \( \Pr\{X = \infty\} = 1 \) no \( S(f) \)
- \( E[X] = \infty \) no \( S(f) \)
- \( X \rightarrow \text{Gaussian} \)

- \( S(f) \) not \( 1/f^\alpha \)
- \( S(f) \) not \( 1/f^\alpha \)

- \( X \rightarrow \text{Gaussian} \)
- \( X \rightarrow \text{stable} \) no \( S(f) \)

**BOSTON UNIVERSITY**
RANDOM POINT EVENTS IN SPACE AT MULTIPLE SCALES

STARS, LIKE GALAXIES, TEND TO OCCUR IN CLUSTERS.
RANDOM POINT EVENTS IN TIME AT MULTIPLE SCALES

HOW DO THE RATE FLUCTUATIONS OF A SEQUENCE OF RANDOM POINT EVENTS
BEHAVE AS THE COUNTING WINDOW DURATION INCREASES?
SNR$_\lambda$ is observed to be essentially independent of $T$ (counting window duration).

For Poisson: $\text{var}(n) = \sigma_n^2 = \bar{n}$

$\text{SNR} = \frac{\bar{n}}{\sigma_n} = \sqrt{n}$

Rate $\lambda \equiv n/T$ so $\bar{\lambda} = \frac{\bar{n}}{T}$ and $\sigma_{\bar{\lambda}} = \frac{\sigma_n}{T}$

Thus, $\text{SNR}_{\bar{\lambda}} \equiv \frac{\bar{\lambda}}{\sigma_{\bar{\lambda}}} = \text{SNR}_n \propto T^{1/2}$

Hence, $\text{SNR}_{\bar{\lambda}} \uparrow$ as $T \uparrow$

---

SNR increases with $T$, but far more slowly than for the shuffled intervals (Poisson data).

SNR$_\lambda$ is essentially independent of $T$ for the miniature end-plate current (MEPC) data.


SNR₁ is nearly independent of $T$ for the human heart rate.

<table>
<thead>
<tr>
<th>$T$</th>
<th>RATE ESTIMATE</th>
<th>NORMAL, SHUFFLED INTERVALS</th>
<th>$T$</th>
<th>RATE ESTIMATE</th>
<th>NORMAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 sec</td>
<td><img src="image1" alt="Graph" /></td>
<td><img src="image2" alt="Graph" /></td>
<td>10 sec</td>
<td><img src="image3" alt="Graph" /></td>
<td><img src="image4" alt="Graph" /></td>
</tr>
<tr>
<td>50 sec</td>
<td><img src="image5" alt="Graph" /></td>
<td><img src="image6" alt="Graph" /></td>
<td>100 sec</td>
<td><img src="image7" alt="Graph" /></td>
<td><img src="image8" alt="Graph" /></td>
</tr>
</tbody>
</table>
Random point events in time often exhibit self-scaling that takes the form of power-law behavior in their statistics (e.g., spectrum and count variance-to-mean ratio): **FRACTAL BEHAVIOR OF POINT EVENTS**
NORMALIZED VARIANCE $F(T)$ AND
NORMALIZED HAAR-WAVELET VARIANCE $A(T)$

a) CONSTRUCTION OF NORMALIZED VARIANCE $F(T)$

b) CONSTRUCTION OF NORMALIZED HAAR-WAVELET VARIANCE $A(T)$
POWER-LAW SCALE INVARIANCE

\[
\frac{S(f)}{S(2f)} = \frac{k_1 f^{-\alpha}}{k_1 (2f)^{-\alpha}} = 2^\alpha
\]

INDEPENDENT OF \( f \)

SCALE INVARIANT

\[
\frac{A(T)}{A(T/2)} = \frac{k_2 T^\alpha}{k_2 (T/2)^\alpha} = 2^\alpha
\]

INDEPENDENT OF \( T \)

SCALE INVARIANT

\( \alpha = \) SCALING EXPONENT

\[ \square = \text{SCALING RANGE} \]
4. FRactal Photons

BU Photonics Center
Quantum Photonics Lab – Room 733
LASER
FREQUENCY-DOUBLED Nd$^{3+}$:YVO$_4$ LASER OPERATING AT 532 nm

LED
COMMERCIAL BLUE LED WITH CENTRAL WAVELENGTH OF 430 nm
ORIGINS OF FRACTAL BEHAVIOR

• Empirical power-law behavior
• Diffusion
• Convergence to stable (Lévy) distributions
• Lognormal distribution
• Self-organized criticality
• Highly optimized tolerance
• Scale-free networks
• Superposition of relaxation processes
FRACTAL-BASED POINT-PROCESSES – MODELS

Fractal Gaussian Noise

- A) STOCHASTIC RATE $\lambda(t)$
- B) INTEGRATE-AND-RESET SAMPLE FUNCTION
- C) DOUBLY STOCHASTIC POISSON-PROCESS SAMPLE FUNCTION

Fractal Renewal Process

- INTEREVENT INTERVAL DENSITY $p(t)$
  - ABRUPT EXPONENTIAL

Alternating Fractal Renewal Process

- M ALTERNATING RENEWAL PROCESSES $X_n(t)$
- BINOMIAL NOISE $X_b(t)$

Fractal Shot-Noise-Driven Poisson

- RATE $\mu$
- POISSON
- LINEAR FILTER $h(t)$
- SHOT NOISE $X(t)$
- SHOT-NOISE-DRIVEN POISSON POINT PROCESS $dN(t)$

Kolmogorov
Mandelbrot
Gauss
Bartlett
5. HEART RATE VARIABILITY

CAN BE STUDIED VIA:
COUNTS -- Duration of time window selected by experimenter affects observation
or
TIME INTERVALS: More exhaustive since all information is retained

After Lowen & Teich, Fractal-Based Point Processes (Wiley, 2005).
CONGESTIVE HEART FAILURE

INABILITY OF HEART TO INCREASE CARDIAC OUTPUT IN PROPORTION TO METABOLIC DEMANDS

Symptom complex:
Many different presentations and etiologies

Typical symptoms:
• Shortness of breath
• Swelling in legs
• General fatigue and weakness

Clinical diagnostics:
• Ascultate heart
• Carotid pulse
• Electrocardiogram
• Chest radiograph

Collaborators:
➢ Steven Lowen, Harvard Medical School
➢ Conor Heneghan, University College Dublin
➢ Robert Turcott, Stanford Medical School
➢ Markus Feurstein, Wirtschaftsuniversität Wien
➢ Stefan Thurner, Allgemeines Krankenhaus Wien
ELECTROCARDIOGRAM

a)

b)

\[ 0 \quad T \quad 2T \quad 3T \quad \text{TIME } t \]

\[ \tau_1 \quad \tau_2 \quad \tau_3 \quad \ldots \]
NORMALIZED HAAR-WAVELET VARIANCE

\[ A(T) \propto T^{\alpha_A} \]

\( \alpha_A = \) scaling exponent

SCALE-INDEPENDENT

INTERVAL-BASED MEASURES: SPECTRUM

\[ S_\tau(f) \propto f^{-\alpha_{st}} \]

\( \alpha_{st} = \) scaling exponent

SCALE-INDEPENDENT

INTERVAL-BASED HAAR WAVELET
EXAMINES ALL SCALES
MITIGATES AGAINST NONSTATIONARITIES

\( m = \text{scale index}; 2^m = \text{scale} \)

\[
W^\text{wav}_{\psi, \tau} (m,i) = \sum_k 2^{-m/2} \psi(2^{-m} k - i) \tau_k
\]

\[
\sigma^2_{\text{wav}} \equiv \text{Var} \left[ W^\text{wav}_{\psi, \tau} (m,i) \right] = 2^{-m} \sum_k \sum_l \psi(2^{-m} k - i) \psi(2^{-m} l - i) R_{\tau}(l - k)
\]

\[
A_{\tau}(k) \equiv \text{Var} \left[ W^\text{wav}_{\psi, \tau} (m,i) \right] / \text{Var} [\tau]
\]

INTERVAL-BASED MEASURES: NIWV

After Lowen & Teich, Fractal-Based Point Processes (Wiley, 2005).
\[ \sigma_{\text{wav}}^2(T) \propto T^{\alpha_{\text{Ar}}} \]
\[ \alpha_{\text{Ar}} = \text{scaling exponent} \]

**SCALE-INDEPENDENT**


IDENTIFYING PATIENTS WITH CARDIAC DYSFUNCTION

MEASURES OF STATISTICAL SIGNIFICANCE

• *p* VALUE, *d’*, AND VARIANTS (rely on Gaussian assumption)

• SENSITIVITY/SPECIFICITY MEASURES OF CLINICAL SIGNIFICANCE (distribution free)

SENSITIVITY ≡ proportion of heart-failure patients that are properly identified

e.g., Hypothesis that all normal patients are so identified ≡ 100% SPECIFICITY

• ROC CURVES & AREA UNDER ROC
ROC CURVES & AREA UNDER ROC

SCALE-DEPENDENT $\sigma_{\text{wav}} (32)$

SCALE-INDEPENDENT $\alpha_{A\tau}$

![ROC Curve Diagram]
ROC-AREA CURVES: NORMAL & CHF DATA

DATA LENGTH ANALYZED $L$
PHYSIOLOGICAL ORIGIN OF FRACTAL BEHAVIOR

6. NETWORKS

EXAMPLES OF SCALE-FREE NETWORKS

• Cellular metabolic networks
• Air transportation
• Internet
  (as of 2005, >100,000 separate networks, >100 million hosts, millions of routers, billions of web locations, tens of billions of catalogued documents)
• Web
• Scientific collaborations (linked by joint publications)
• Scientific papers (linked by citations)
• People (connected by professional associations or friendships)
• Businesses (linked by joint ventures)

SUCH NETWORKS ARE ROBUST AGAINST ACCIDENTAL FAILURES BECAUSE RANDOM BREAKDOWNS SELECTIVELY AFFECT THE MOST PLENTIFUL NODES, WHICH ARE THE LEAST CONNECTED
SURROGATE DATA ANALYSIS

**Deletion**
- a) ORIGINAL POINT PROCESS
- b) DECIMATION
- c) RANDOM DELETION
- d) DEAD-TIME DELETION

**Displacement**
- a) ORIGINAL POINT-PROCESS REALIZATION $dN_1(t)$
- b) DISPLACED POINT-PROCESS REALIZATION $dN_R(t)$

**Shuffling**
- a) ORIGINAL POINT PROCESS REALIZATION $dN_1(t)$
- b) SHUFFLED POINT PROCESS REALIZATION $dN_R(i)$

**Exponentialization**
- a) ORIGINAL POINT PROCESS REALIZATION $dN_1(t)$
- b) INTERVAL-TRANSFORMED POINT PROCESS REALIZATION $dN_R(i)$
APPLICATIONS: COMPUTER NETWORK TRAFFIC

1 million consecutive Ethernet packet arrivals over 29 min (data set BC-pOct89)

Snapshot of 11,000 ISPs from CAIDA (April/May 2003)

Erlang

Palm

BC-pOct89 PERIODOGRAM

ESTIMATED RATE SPECTRUM $\hat{S}_n(f, T)$

DATA

SHUFFLED

EXPONENTIALIZED

COUNTING TIME $T$ (sec)

$10^{-4}$ $10^{-2}$ $10^{0}$ $10^{2}$

ESTIMATED NHV $\hat{A}(T)$

$10^{0}$ $10^{1}$ $10^{2}$ $10^{3}$

FREQUENCY $f$ (Hz)

$10^{-3}$ $10^{-2}$ $10^{-1}$ $10^{0}$ $10^{1}$
IDENTIFYING THE NETWORK-TRAFFIC POINT PROCESS
TABLEAU OF NINE STATISTICAL MEASURES

CLASSIC BC-pOct89 DATA SET
1-million Consecutive Ethernet Packet Arrivals at Main Ethernet Cable

SIMULATED FRAC TAL NEYMAN-SCOTT POINT PROCESS
(Rectangular Fractal-Shot-Noise-Driven Point Process: RFSNDP)
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7. SENSORY DETECTION

SENSORY TRANSMISSION
AND DETECTION
M. C. Teich & W. J. McGill

http://people.bu.edu/teich
Alerting Signals and Detection in a Sensory Network

William J. McGill and Malvin C. Teich

Columbia University

SUPPORT GRATEFULLY ACKNOWLEDGED