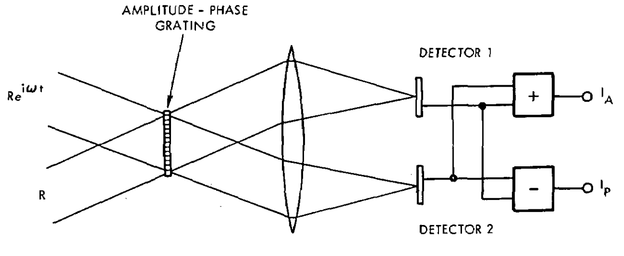

Fig. 2. Measurement technique for amplitude and phase components in hologram reconstruction.

of the beams with an acoustooptic device. The outputs of the detectors monitoring the beam powers are added to form a signal

$$I_A = (c/2)(P_{W2} + P_{W1}) = cP_{in} \exp\left(-\frac{2\alpha d}{\cos\theta}\right) \left[\cosh\left(\frac{\alpha_1 d}{\cos\theta}\right) + \sinh\left(\frac{\alpha_1 d}{\cos\theta}\right) \cos\omega t\right], \quad (7)$$

where c is a constant. I_A represents a signal that corresponds to the amplitude component of the grating only. Subtracting the outputs yields

$$I_P = (c/2)(P_{W2} - P_{W1}) = cP_{in} \exp\left(-\frac{2\alpha d}{\cos\theta}\right) \left[\sin\left(\frac{2\pi n_1 d}{\lambda \cos\theta}\right) \sin\omega t\right] \quad (8)$$

which corresponds to the phase component. After some algebra we find that

$$\frac{2\pi n_1 d}{\lambda \cos\theta} = \sin^{-1}\left[\frac{2(I_P)_{\mathrm{RMS}}{}^2}{(\bar{I}_A)^2 - 2(I_A - \bar{I}_A)_{\mathrm{RMS}}{}^2}\right]^{1/2}, \quad (9)$$

$$\frac{\alpha_1 d}{\cos\theta} = \sinh^{-1}\left[\frac{2(I_A - \bar{I}_A)_{\mathrm{RMS}}{}^2}{(\bar{I}_A)^2 - 2(I_A - \bar{I}_A)_{\mathrm{RMS}}{}^2}\right]^{1/2}. \quad (10)$$

For small modulations these reduce to the simple expressions

$$\frac{2\pi n_1 d}{\lambda \cos\theta} = \frac{2^{1/2}(I_P)_{\mathrm{RMS}}}{\bar{I}_A}, \quad (11)$$

$$\frac{\alpha_1 d}{\cos\theta} = \frac{2^{1/2}(I_A - \bar{I}_A)_{\mathrm{RMS}}}{\bar{I}_A} \quad (12)$$

This approach has the advantage of freedom from errors due to registration inaccuracies, aside from the requirement for correct Bragg registration, which is necessary in any case. Furthermore, all three measured quantities, $(I_P)_{\mathrm{RMS}}$, $(I_A - \bar{I}_A)_{\mathrm{RMS}}$, and $\bar{I}_A$, may be determined in parallel and continuously monitored, a fact that could prove useful in the measurement of the absorption spectra and anomalous dispersion curves of optically sensitive media by, for example, scanning in wavelength with a tunable dye laser. Finally, the optical gain resulting from the coherent addition of transmitted and diffracted waves results in larger signals than those obtained from the diffracted beam only, as can readily be seen by comparison of Eqs. (11) and (12) with Eq. (2) of Ref. 1. This may permit the use of lower laser power for measurement when grating destruction is a significant factor, although the precise degree of advantage incurred is somewhat uncertain and certainty depends on the particular experimental situation.

It must be noted that the present technique and that of Ref. 1 can be applied to planar media. For a grating described by the amplitude transmittance

$$T_a = T_o[1 + m \cos(k_1 - k_2)x] \times \exp\{i[\phi_o + \Delta\phi \cos(k_1 - k_2)x]\}, \quad (13)$$

where k_1 and k_2 are the x components of the wave vectors of the incident beams, m is the amplitude transmittance modulation, and $\Delta\phi$ is the phase modulation. An analysis similar to that above yields, in the case of small modulations,

$$\Delta\phi = \frac{2^{1/2}(I_P)_{\mathrm{RMS}}}{\bar{I}_A}, \quad (14)$$

$$m = \frac{2^{1/2}(I_A - \bar{I}_A)_{\mathrm{RMS}}}{\bar{I}_A} \quad (15)$$

These become identical to Eqs. (11) and (12) when the amplitude transmittance is a result of bulk absorption in a thin layer and when the phase variations are a result of a refractive index modulation.

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Photocounting distributions with variable dead time

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During the past several years there has been a great deal of interest in the study of photocounting distributions in the presence of photodetector dead time.[1–4] The dead time τ is defined as a fixed period of time after the registration of a photoelectron during which the photodetector cannot emit another electron. In a previous paper,[1] we obtained an expression for the dead-time-corrected counting distribution when each pulse from the photodetector is followed by a fixed dead time τ.

In this Letter we find a relationship for the counting statistics when each pulse of the process is followed by a deterministic but arbitrary value of the dead time. Thus in a counting interval t, the first pulse is followed by a dead time of duration τ_1, the second by τ_2, and so on, up to the nth recorded pulse, with the over-all restriction $\tau_1 + \tau_2 + \cdots + \tau_n \leq t$. Variable-dead-time results of this nature can be used to obtain the counting distribution in the presence of a random, rather than deterministic, dead time by averaging over the stochastic variation of the dead time. This procedure will be carried out in a subsequent publication dealing with the statistical properties of neural pulse trains encountered in vision research.[5] Such neural counting distributions have been shown to be identical in form to the photocounting distributions considered here.[6] We assume throughout that the counter is nonparalyzable and is unblocked[1,2] at the beginning of each counting interval.

We define the quantities $p_n(t',t)$ such that $p_n(t',t)dt$ is the probability that the nth registered pulse occurs in the interval $(t,t+dt)$ when counting begins at $t = t'$. $P_n(t',t)$, on the other hand, represents the probability that exactly n pulses occur in the interval (t',t). We denote with an asterisk those probabilities corresponding to an unblocked counter; those without the asterisk are assumed blocked at the beginning of the counting interval.[1,2] Since the events corresponding to $p_n^*(0,t)$ and $p_{n+1}^*(0,t)$ are mutually exclusive, $P_n^*(0,t)$ and $p_n^*(0,t)$ are related by the integral

$$P_n^*(0,t) = \int_0^t [p_n^*(0,t') - p_{n+1}^*(0,t')]dt'; \qquad (1)$$

this insures that the nth pulse occurs at some time in the interval $(0,t)$ and that the $(n+1)$st does not. For a blocked counter, the probabilities in Eq. (1) would be replaced by their blocked counterparts. Since the blocked or unblocked condition affects the presence or absence of, at most, one pulse in any counting interval $(0,t)$, the condition of the counter at the beginning of the interval becomes of minor importance as the number of counts registered in the counting interval increases.

For a Poisson process it can be shown that[2]

$$p_1^*(0,t) = \lambda \exp(-\lambda t), \qquad t \geq 0, \qquad (2a)$$

$$p_1^*(t',t) = \lambda \exp[-\lambda(t-t')], \qquad t \geq t', \qquad (2b)$$

and

$$p_1(t',t) = \lambda \exp[-\lambda(t - t' - \tau_i)], \qquad t \geq t' + \tau_i, \qquad (2c)$$

where λ is the rate parameter of the uncorrected Poisson distribution. In Eq. (2c) it is assumed that the counter is blocked by the ith pulse at t'; this accounts for the decrease in the effective counting interval from $(t-t')$ to $(t-t'-\tau_i)$. The quantity $p_2^*(0,t)$ can be obtained from the probabilities given in Eq. (2), and for a nonparalyzable counter is written as[2]

$$p_2^*(0,t) = \int_0^{(t-\tau_1)} p_1^*(0,t')p_1(t',t)dt', \qquad t \geq \tau_1, \qquad (3)$$

where the upper limit on the integral reflects the fact that the first pulse must occur at least τ_1 seconds before the end of the interval. Clearly, τ_1 will appear in $p_1(t',t)$ since it is blocked by the first pulse. Evaluating Eq. (3), we obtain

$$p_2^*(0,t) = \lambda^2(t-\tau_1)\exp[-\lambda(t-\tau_1)], \qquad t \geq \tau_1. \qquad (4)$$

In a similar manner, $p_3^*(0,t)$ can be written as

$$p_3^*(0,t) = \int_{\tau_1}^{(t-\tau_2)} p_2^*(0,t')p_1(t',t)\,dt', \qquad (5)$$

where τ_2 is the dead time blocking $p_1(t',t)$ and the lower limit reflects the fact that the second pulse cannot occur prior to $t' = \tau_1$. Eq. (5) thus yields

$$p_3^*(0,t) = \{\lambda^3[t-(\tau_1+\tau_2)]^2/2!\} \times \exp\{-\lambda[t-(\tau_1+\tau_2)]\}, \; t \geq \tau_1+\tau_2. \qquad (6)$$

In general, therefore, $p_n^*(0,t)$ is obtained from $p_{n-1}^*(0,t)$ by the relation

$$p_n^*(0,t) = \int_{(\tau_1+\tau_2+\ldots+\tau_{n-2})}^{(t-\tau_{n-1})} p_{n-1}^*(0,t')p_1(t',t)dt', \qquad (7)$$

with $p_1(t',t)$ blocked by τ_{n-1}, resulting in

$$p_n^*(0,t) = \{\lambda^n[t-(\tau_1+\tau_2+\ldots+\tau_{n-1})]^{n-1}/(n-1)!\} \times \exp\{-\lambda[t-(\tau_1+\tau_2+\ldots+\tau_{n-1})]\}, \qquad (8)$$

for $t \geq \tau_1+\tau_2+\cdots+\tau_{n-1}$.

To obtain $P_n^*(0,t)$, Eq. (1) must be evaluated using the distribution given in Eq. (8). Equation (1) can be rewritten as

$$P_n^*(0,t) = \int_0^t p_n^*(0,t')dt' - \int_0^t p_{n+1}^*(0,t')dt', \qquad (9)$$

and since the integrals are of the same form, we seek to evaluate

$$I_n = \int_0^t \{\lambda^n[t'-(\tau_1+\tau_2+\ldots+\tau_{n-1})]^{n-1}/(n-1)!\} \times \exp\{-\lambda[t'-(\tau_1+\tau_2+\ldots+\tau_{n-1})]\}dt'. \qquad (10)$$

Using successive integration by parts, we obtain

$$P_n^*(0,t) = \sum_{k=0}^{n} p_k(n,\lambda) - \sum_{k=0}^{n-1} p_k(n-1,\lambda), \qquad n \geq 1 \qquad (11)$$

with

$$p_k(n,\lambda) = \{\lambda^k[t-(\tau_1+\tau_2+\ldots+\tau_n)]^k/k!\} \times \exp\{-\lambda[t-(\tau_1+\tau_2+\ldots+\tau_n)]\} \qquad (12)$$

for $t \geq \tau_1+\tau_2+\cdots+\tau_n$. For $\tau_1 = \tau_2 = \cdots = \tau_n \equiv \tau$, this correctly reduces to the result found previously.[1] For the unblocked counter, the probability at $n = 0$ is the same as that obtained from the uncorrected Poisson distribution.

The counting distribution presented in Eqs. (11) and (12) can be extended to yield the dead-time-corrected counting distribution for a source of arbitrary statistics; this has been done previously for the unique dead-time case.[1] Each of the $p_k(n,\lambda)$ represents the probability of registering k counts in an interval $t-(\tau_1+\tau_2+\cdots+\tau_n)$ for a process that obeys a Poisson probability law with rate λ. For a source of arbitrary statistics, in most cases it suffices to take an ensemble average of $P_n^*(0,t)$ over the statistics of the source. Since the sums involved are finite, this is equivalent to averaging the $p_k(n,\lambda)$ over the source statistics, which is nothing more than an application of the Poisson transform relation (Mandel's formula).[1,6] In short, we have obtained an expression for the variable dead-time-corrected counting distribution registered by a nonparalyzable counter for most sources whose statistics are known.

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