Spot Pricing of Secondary Spectrum Access in Wireless Cellular Networks

Huseyin Mutlu, Murat Alanyali, and David Starobinski

Abstract—**Recent deregulation initiatives enable cellular providers to sell excess spectrum for secondary usage. In this paper, we investigate the problem of optimal spot pricing of spectrum by a provider in the presence of both non-elastic primary users, with long-term commitments, and opportunistic, elastic secondary users. We first show that optimal pricing can be formulated as an infinite horizon average reward problem and solved using stochastic dynamic programming. Next, we investigate the design of efficient single pricing policies. We provide numerical and analytical evidences that static pricing policies do not perform well in such settings (in sharp contrast to settings where all the users are elastic). On the other hand, we prove that deterministic threshold pricing achieves optimal profit amongst all single-price policies and performs close to global optimal pricing. We characterize the profit regions of different pricing policies, as a function of the arrival rate of primary users. Under certain reasonable assumptions on the demand function, we prove that the profit region of threshold pricing is optimal and independent of the specific form of the demand function, and that it includes the profit region of static pricing. In addition, we show that the profit function of threshold pricing is unimodal in price. We determine a restricted interval in which the optimal threshold lies. These properties enable very efficient computation of the optimal threshold policy, which is far faster than that of the global optimal policy.**

Index Terms—**Management of electromagnetic spectrum, secondary markets, congestion pricing, Markov decision processes, threshold policies.**

I. Introduction

A major global effort is underway to deregulate wireless spectrum and achieve much better utilization of this scarce resource. The Secondary Markets Initiative [2] of the Federal Communications Commission (FCC), is one of the major steps towards achieving this goal. It permits leasing of spectrum licenses subject to approval by FCC. Similar regulatory efforts are also underway in the EU [3].

Consequences of the secondary markets initiative can already be felt with the emergence of secondary cellular providers, commonly called Mobile Virtual Network Operators (MVNOs) [4]. MVNOs buy spectrum and (possibly also infrastructure) from primary providers, referred to as Mobile Network Operators (MNOs). MVNOs add the value of better penetrating certain markets and offering differentiated products. A notable example of successful MVNO endeavor in the US is Virgin Mobile who has teamed up with Sprint Nextel as its MNO and recently reached a subscriber basis of over 4 millions customers [5].

In this paper, we are interested in investigating how a provider, such as an MNO, should optimally price its excess spectrum to secondary users (SUs), such as MVNOs. On the one hand, a provider must ensure that the quality of service (QoS) of its primary users (PUs), who typically have long-term contracts, is not significantly affected by the admission of SUs. This is because the presence of SUs may increase the blocking of PU calls and hence lead to a punishment in the form of loss of business due to poor service. On the other hand, the provider is interested in maximizing its profit from the admission of SUs.

Given that the amount of excess spectrum is likely to fluctuate over time due to the inherent randomness in the PU traffic, spot pricing, based on real-time channel occupancy, emerges as the solution of choice. While spot and congestion-based pricing have been extensively studied in the literature (Cf. Section II), the typical model assumed in previous work differs significantly from the setting considered herein. Chiefly, most previous work assumes that the demand functions of *all* the users are elastic to price, i.e., all the arrival rates can be regulated with price. In contrast, in our setting, only the demand function of the SUs is elastic to price, but the arrival rate of PUs is not. As we will show, this difference is salient enough to result in fundamentally different structures for optimal (or near-optimal) pricing strategies.

Our first contribution in this paper is to formalize the profit maximization problem of a cellular provider in the presence of both PUs and SUs. Based on certain reasonable statistical assumptions, we show that optimal pricing can be formulated as an infinite horizon average reward problem and solved using stochastic dynamic programming.

Our second contribution is to investigate the design of efficient single pricing policies, i.e., policies where a provider can either admit a SU and charge a fixed price or reject a SU. These policies have the major advantage of making the cost of spectrum much more predictable to SUs. We first show that *static pricing*, which always applies the same admission price to SUs independently of the channel occupancy, may perform very poorly. This result stands in sharp contrast to the case where all the users are elastic to price. On the other hand, we provide numerical evidence that threshold pricing, which applies a fixed admission price to SUs when the channel occupancy is below some threshold T and rejects them otherwise, performs very close to optimal. Further, we prove that among all the possible single-price admission policies (including randomized), threshold pricing is the optimal one.

The authors are affiliated with the Department of Electrical and Computer Engineering at Boston University, Boston, MA

This work was supported in part by the US National Science Foundation under grants CNS-0132802, CNS-0721860 and ANI-0238397.

A preliminary version of this paper appeared in the proceedings of IEEE INFOCOM 2008 [1].

Our third contribution is to characterize the profit regions of static pricing and threshold pricing. Our goal is to determine the maximum arrival rate of PUs, at which it is still possible to achieve profit from the admission of SUs. We characterize the profit regions of different pricing policies. We prove that the profit region of threshold pricing is optimal, i.e., it is identical to that of optimal pricing and larger than that of static pricing. Through numerical example, we show that the difference between the profit regions of threshold and static pricing can sometimes be very large. An interesting observation is that the profit regions of all the pricing policies depend only on the support of the demand function of the SUs, but not on its specific form. This result applies to quite general demand functions.

Our last contribution is to devise an efficient computational procedure to calculate the optimal threshold and price for threshold pricing. In particular, we prove that, for any given threshold T, the profit function is unimodal in price. This enables us to resort to well-known logarithmic search procedures to compute the optimal price. Moreover, we show that the optimal threshold is a non-decreasing function of price. By using this property, we are able to reduce the search interval for the optimal threshold, thus speeding up calculation of the optimal threshold policy. We provide numerical results showing that the optimal threshold policy can be computed considerably faster than the global optimal policy.

The rest of the paper is organized as follows. In Section II, we survey related work. Our model and notation are introduced in Section III. In Section IV, we show how to derive the optimal pricing policy and characterize the optimal prices. In Section V, we investigate single-price policies, prove the optimality of threshold pricing, and characterize the profit regions of static, threshold and optimal pricing. In Section V, we also prove unimodality of profit function of threshold pricing. Then, in Section VI, we develop an efficient method to compute the optimal price and threshold for threshold pricing. We conclude the paper in Section VII.

II. Related work

The problem we consider in this paper is related to two well studied areas in communication networks, namely, pricing and call admission control. As such, we restrict our literature review to those papers that are the most relevant. A survey of other work related to pricing in cellular networks can be found in [6].

In [7], Paschalidis and Tsitsiklis investigate dynamic, congestion-based pricing of network resources. Their model assumes that all the users are elastic to price. They show that static pricing achieve good performance in general and can even be optimal in some asymptotic traffic regimes. This result was extended in [8] and [9], in the context of large network asymptotics. In [10], Ziya et. al. show that the optimal static price is unique. In [11], static spectrum pricing strategies capturing the effects of network-wide interferences are developed.

Threshold admission control policies have been extensively studied. Refs. [12, 13] provide useful insights into the properties of such policies. The optimality of threshold pricing for certain optimization problem is proved in [14, 15]. None of these papers integrate pricing into their formulations.

Refs. [16–18] integrate pricing with admission control in cellular networks. Ref. [16] considers time-of-day pricing methods. In our work, we consider pricing strategies that operate at much shorter time-scales, based on real-time information. Ref. [17] develops and evaluates "charge-by-time" pricing algorithms, while in our work we consider charging per admission. Ref. [18] develops a stochastic dynamic programming formulation that incorporates retrials. Our main contribution with respect to this previous body of work is to go beyond numerical optimizations and attempt to prove general structural properties, applicable to very general demand functions.

Ref. [19] analyzes a model similar to ours within the context of a generic rental management optimization problem. This work considers two type of customers, namely *walk-in* and *contract* users. Walk-in users are priced according to the congestion level of the system, similar to optimal pricing of SUs in our model. Contract users, on the other hand, have fixed prices and arrival rates which are analogous to our PUs. Different than our work, [19] focuses on determining structures of the optimal policy rather than providing a simple, near-optimal alternative as done here.

III. Network Model

In this section, we introduce our network model and notation (additional notation specific to static and threshold pricing will be provided in Section V). We consider a cellular network where each cell provides access to C channels. In each cell, calls from PUs arrive according to a Poisson process with fixed rate $\lambda_p > 0$. A punishment in the amount of K monetary units is imposed if all the channels are busy and a PU call is blocked. SUs call arrivals also form a Poisson process that is independent of the PUs call arrivals process and its rate is modulated by the price charged by the provider. We thus assume that there is a *demand function* $\lambda_s(u)$ which determines the arrival rate of SU calls, where u is the applied price. The price is a function of the state of the system, i.e., a SU pays a price u_n for its call, if there are n busy channels in the cell, where $0 \leq n < C$.

For both PUs and SUs, call holding times are exponentially distributed with rate μ, independently of any other events. Without loss of generality, we will assume $\mu = 1$, i.e., the mean call holding time is one unit of time.

The goal of the provider is to maximize the average profit per unit of time gained from accepting SUs. This quantity is denoted by R. We are interested in finding a pricing policy that satisfies this goal. A pricing policy is a rule that dictates which price should be advertised by the provider at any given point of time.

Under the above assumptions, the system behavior follows the dynamics of a continuous-time birth-death Markov process, and explicit expression for the average profit R can be provided as follows. First, let π_n be the steady-state probability of finding the system in state n, i.e., there are n busy channels. Next, let $\lambda_n = \lambda_s(u_n) + \lambda_p$ denote the total call arrival rate in state n and $\Lambda = (\lambda_0, \lambda_1, ..., \lambda_{C-1})$ denote the vector of arrival rates. Then, the probability of finding the system in state n, denoted by $\pi_n(\Lambda)$, can be explicitly written as follows:

$$\pi_n(\Lambda) \quad = \quad \frac{\frac{\lambda_0\lambda_1\lambda_2...\lambda_{n-1}}{n!}}{1+\frac{\lambda_0}{1!}+\frac{\lambda_0\lambda_1}{2!}+\ldots+\frac{\lambda_0\lambda_1\lambda_2...\lambda_{C-1}}{C!}}. \quad (1)$$

Due to the PASTA (Poisson Arrivals See Time Averages) property, the probability that a PU is blocked is $\pi_C(\Lambda)$. Thus, the average profit is

$$R = \sum_{n=0}^{C-1}\pi_n(\Lambda)\lambda_s(u_n)u_n - (\pi_C(\Lambda) - E(\lambda_p, C))\lambda_p K, \quad (2)$$

where $E(\lambda_p, C)$ is the blocking probability of PUs in the absence of SU arrivals. This quantity corresponds to the well-known *Erlang-B* formula

$$E(\lambda_p, C) = \frac{\frac{\lambda_p^C}{C!}}{\sum_{n=0}^{C}\frac{\lambda_p^n}{n!}}. \quad (3)$$

The first term in Eq. (2) represents the sum of the average revenues collected from SUs in each state. The second term is the average punishment due to accepted SUs. The expression $\pi_C(\Lambda)$–$E(\lambda_p, C)$ represents the increase in the blocking probability of PUs due to accepted SUs. The quantity $E(\lambda_p, C)$ acts as the normalization term to ensure that the profit is zero when all SUs are rejected.

In the sequel, we impose the following natural assumptions on the demand functions. These assumptions are required to guarantee the existence of a stationary optimal pricing policy and prove some of our structural results.

Assumption 3.1: There exists a price $u_{\max}$ for which $\lambda_s(u_{\max}) = 0$. Moreover, $\lambda_s(u)$ is a strictly decreasing, differentiable function in u over the interval $[0, u_{\max}]$ and $\lambda_s(0)$ is finite.

IV. Derivation of the Optimal Pricing Policy

In this section, we derive the *optimal pricing* policy and present properties characterizing the optimal prices.

A. Stochastic Dynamic Programming Formulation

The maximization of the profit function in Eq. (2) is a complex multi-dimensional optimization problem and becomes quickly intractable as C grows. One approach to alleviate this problem is to formulate it as an average reward stochastic *dynamic programming* (DP) problem [20, 21]. Specifically, the optimal prices u_n^* and optimal profit R^* corresponding to the optimal policy can be computed using the so-called Bellman's equations since all the states in the Markov chain are recurrent (see Proposition 7.4.1 in [21]).

Bellman's equations are usually formulated for discrete-time Markov chains. In our case, the Markov chain is continuous, but it can be discretized using a procedure called *uniformization*, where the transition rates out of each state are normalized by the maximum possible transition rate v, which in our case is given by the following expression:

$$v = \lambda_s(0) + \lambda_p + C. \quad (4)$$

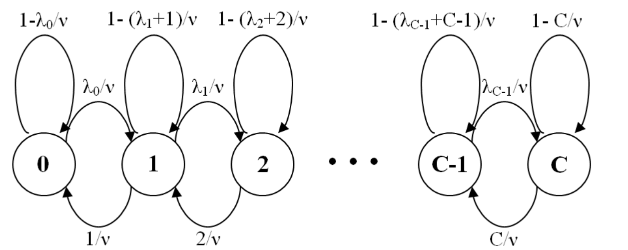

Fig. 1. Uniformized Markov Chain

The uniformized Markov chain with corresponding transition rates is shown in Fig. (1).

Bellman's equations are generally guaranteed to return the optimal solution only for a finite action (control) space $\mathbb{U}$, where $\mathbb{U}$ represents the set of all possible prices advertised by the provider. Hence, prices must be discretized. We denote the discretization step with Δu. The cardinality of the action space is thus $|\mathbb{U}| = \lceil u_{\max}/\Delta u \rceil$. On the one hand, consideration of a limited range of prices leads to a potential reduction in the profit. On the other hand, if the demand function $\lambda_s(u)$ is continuously differentiable in u, this reduction is at most linear in discretization step Δu since the profit in Eq. (2) is a smooth function of $u_0, u_1, \cdots, u_{C-1}$. Hence the alluded profit loss can be made arbitrarily small at the expense of higher computational complexity by selecting a smaller Δu. In Section VI, we describe an efficient computational procedure, applicable to threshold pricing, that scales to very large cardinality $|\mathbb{U}|$.

Equipped with the above formulation, we can now compute the optimal pricing policy using the Bellman equations:

$$\begin{aligned} J^* + h(n) = \max_{u_n \in \mathbb{U}}[\lambda_s(u_n)u_n + h(n+1)\tfrac{\lambda(u_n)}{v} \\ +h(n-1)\tfrac{n}{v} + h(n)(1 - \tfrac{\lambda(u_n)}{v} - \tfrac{n}{v})] \end{aligned} \quad (5)$$

for $n = 0, 1, 2...C-1$ and

$$J^* \quad = \quad -\lambda_p K + h(C-1)\frac{C}{v}, \quad (6)$$

whereas the optimal profit is:

$$R^* = J^* + E(\lambda_p, C)\lambda_p K. \quad (7)$$

The first term in the right-hand side (RHS) of Eq. (5) represents the profit gained at state n from the acceptance of a SU. The second and third terms are contributions to the revenue if the next transition is an arrival or departure, respectively. The last term is a consequence of the uniformization procedure. The effect of punishment due to blocked PU calls is captured by the first term in the RHS of Eq. (6). The prices maximizing the RHS of Eq. (5) represent the optimal prices.

The unknowns in the above equations are $h(n)$ and J^*. The quantities $h(n)$ denote the *relative reward* in state n with respect to state C. When the optimal policy is applied, $h(n)/v$ represents the difference between the *total revenue* gained over an infinite time horizon when starting the process from state n and that gained when starting from state C. The quantities R^* and J^* differ only by a normalization constant used to ensure the non-negativity of the profit.

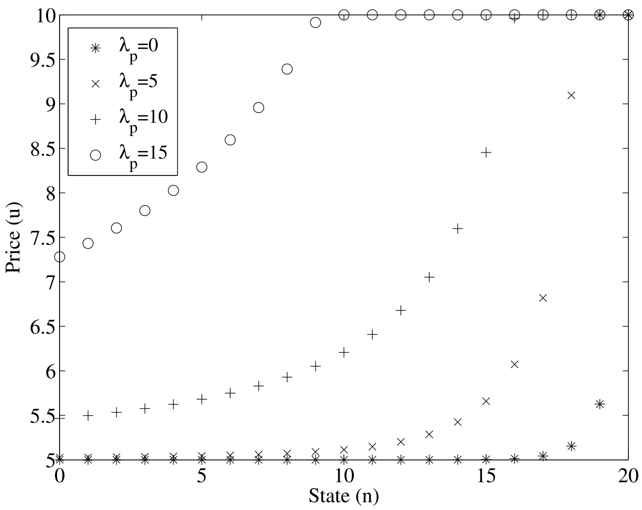

Fig. 2. Optimal prices for various PU arrival rates (λ_p) . $C = 20$, $K = 100$, $\lambda_s(u) = (10 - u)_+$ and $\Delta u = 10^{-6}$.

The solution of Bellman equation can be obtained by using various techniques described in the literature, such as *policy iteration* or *relative value iteration* [20, 21]. Policy iteration theoretically requires on the order of $O(|\mathbb{U}|^C)$ iterations to converge while value iteration is not guaranteed to converge within a finite number steps. However, value iteration has a lower computational complexity at each iteration. In practice, as in other infinite horizon average reward problems [22], policy iteration appears to converge faster.

For different PU arrival rates λ_p, Figure (2) shows the values of the optimal prices (computed using policy iteration), for the demand function $\lambda_s(u) = (10 - u)_+$ (where $(\cdot)_+ = \max(\cdot, 0)$), and parameters $C = 20$, $K = 100$, and $\Delta u = 10^{-6}$. The figure indicates that, as λ_p increases, the prices become higher in each state, and that SUs should not be accepted when the number of busy channels exceeds a certain threshold. More insight into this behavior will be provided in the sequel.

B. Properties of the Optimal Policy

In this section, we provide some results characterizing the optimal prices. First, we consider the ideal case of unlimited capacity.

Lemma 4.1: In the infinite capacity case (i.e., $C \to \infty$), the optimal prices for all states are equal to

$$u_\infty = \arg\max_{u \in \mathbb{U}} (\lambda_s(u)u),$$

and the corresponding profit is

$$R_\infty = \lambda_s(u_\infty)u_\infty.$$

Note that R_∞ is an upper bound on the profit achievable in any finite capacity system.

The following lemma states that in a finite capacity system, the optimal price in each state is larger than the optimal price in the infinite capacity case.

Lemma 4.2: For any $0 \le n \le C - 1$, $u^*_n \ge u_\infty$.

The next result states that the optimal prices are monotonically increasing in n.

Lemma 4.3: For any $0 \le n \le C - 1$, $u^*_{n+1} \ge u^*_n$.

Proofs of these properties follow similar methods to those used in [7]. The main difference lies in taking into consideration the effects of PU arrivals and punishments. These proofs can be found in [23].

A consequence of the above properties is that the optimal price for any state lies between u_∞ and $u_{\max}$. This fact can be exploited to reduce the size of the action space $\mathbb{U}$ when computing the optimal prices using Eq. (5).

V. Single-Price Policies

In this section, we investigate the design of single-price policies. In each state, these policies can either admit a SU and charge a fixed price u or reject a SU (which is equivalent to ask for a price $u_{\max}$ or higher). For such policies the objective is to optimize the value of u as well as the *admission policy* i.e., the decision of whether or not to admit a SU that is willing to pay the price. These policies are attractive because they allow a provider to advertise a single-price. They are also computationally easier to derive. Moreover, compared to optimal pricing, they provide more insight into the structure of good pricing policies.

A simple single-price policy is the so-called *static pricing* where SU calls are always applied the same admission price, unless all the channels are busy. For the cases where the demand functions of all the users are elastic to price and punishments are not imposed, static pricing is known to perform well and to be even asymptotically optimal in several regimes [7–9]. However, in this section, we show that, in the presence of inelastic users (PU) and punishments for blocked PU calls, the performance of static pricing degrades significantly.

Instead, we show next that among all single-price policies (including randomized), a deterministic *threshold pricing* policy performs optimally. In threshold pricing, SU calls are admitted and charged a price u when the channel occupancy is smaller than some threshold T and rejected otherwise. We also provide numerical evidence showing that threshold pricing performs very close to the optimal.

A. Optimality of Threshold Pricing

Theorem 5.1: For any price u (including the optimal one), a threshold admission policy is optimal among all single-price policies.

Proof: Let us redefine the system such that punishment in the amount of u units are imposed for each rejected SU call instead of rewarding u for an accepted one. Optimizing such a system follows same methods and results the same optimal policy. This new formulation of the problem is identical to the well-known MINOBJ problem analyzed in [14] where SU and PU calls are analogous to new and handover calls, respectively. It is shown in [14] that a threshold admission policy is the optimal solution for the MINOBJ problem and, thus, the same result applies to our setting. Note that the analogy is valid when $K > u$. If this is not the case, then the admission policy

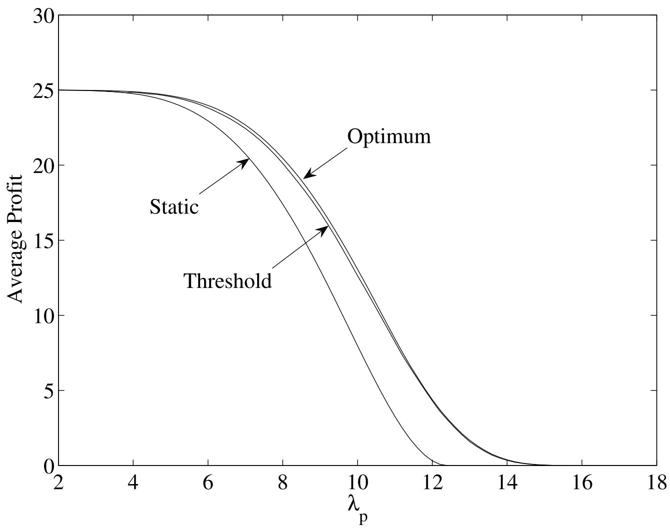

Fig. 3. Average profit vs primary load (λ_p) for different pricing policies. System parameters: $C = 20$, $K = 100$, $\lambda_s(u) = (10 - u)_+$ and $\Delta u = 10^{-6}$.

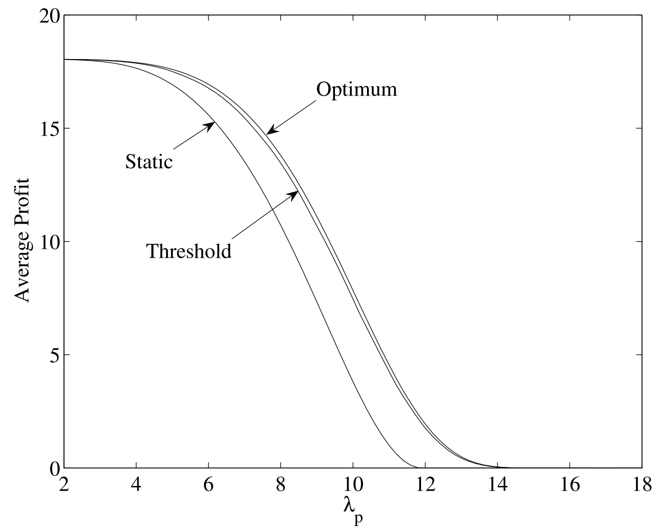

Fig. 4. Average profit vs primary load (λ_p) for different pricing policies. System parameters: $C = 20$, $K = 100$, $\lambda_s(u) = (10e^{-0.04u^2} - 0.1)_+$ and $\Delta u = 10^{-6}$.

is obvious, namely, always admit SU calls.

■

Figure (3) compares the average profits achieved by the optimal, static, and threshold policies for a linear demand function $\lambda_s(u) = (10-u)_+$ (we explain in Section VI how to compute the optimal price and threshold). Figure (4) makes the same comparison for the following popular non-linear demand function [24]

$$\lambda_s(u) = (Ae^{-\gamma u^2} - \epsilon)_+ \ , \tag{8}$$

where A and γ are scaling factors, and $\epsilon > 0$ is a small constant introduced to enforce Assumption 3.1. Both figures show that threshold pricing performs close to optimal while static pricing performs significantly worse. Furthermore, we observe that beyond a certain value of λ_p, static pricing stops generating profit while threshold pricing continues doing so.

We next provide some intuition on why threshold pricing performs so well. In Section V-D, we will show that the maximum value of λ_p, denoted $\lambda_{p,max}$, for which threshold pricing achieves positive profit is the same as the maximum value of λ_p for which optimal pricing achieves positive profit. Furthermore, we know that when $\lambda_p \to 0$, both static pricing and threshold pricing perform very well. This regime is equivalent to the case where all the users are elastic to price, a model studied in [7]. There it is shown that static pricing is optimal in certain asymptotic regimes. These results obviously extend to threshold pricing since it is the optimal single price policy. The arguments above explain the near-optimal performance of threshold pricing for the cases $\lambda_p \to 0$ and $\lambda_p \to \lambda_{p,max}$. Thus, one can expect that in between these two extremes, the profit of threshold pricing will not differ much from the optimal profit.

B. Properties of Threshold Pricing

Having showed that threshold pricing is the optimal single-price policy, we next derive an expression for the profit obtained with this policy, denoted by $R_T(\lambda_s)$. Note that, the profit function is defined as a function of λ_s rather than u. This considerably simplifies the notation and proofs in the rest of the paper.

We start by computing the blocking probabilities for the PUs and SUs:

$$\begin{aligned} B_{PU}(\lambda_s, T) &= \pi_C \qquad\qquad (9)\\ &= \frac{\frac{(\lambda_s+\lambda_p)^T \lambda_p^{C-T}}{C!}}{\sum_{n=0}^{T-1} \frac{(\lambda_s+\lambda_p)^n}{n!} + (\lambda_s+\lambda_p)^T \sum_{n=T}^{C} \frac{\lambda_p^{n-T}}{n!}}; \end{aligned}$$

$$\begin{aligned} B_{SU}(\lambda_s, T) &= \sum_{n=T}^{C} \pi_n \qquad\qquad (10)\\ &= \frac{(\lambda_s+\lambda_p)^T \sum_{n=T}^{C} \frac{\lambda_p^{n-T}}{n!}}{\sum_{n=0}^{T-1} \frac{(\lambda_s+\lambda_p)^n}{n!} + (\lambda_s+\lambda_p)^T \sum_{n=T}^{C} \frac{\lambda_p^{n-T}}{n!}}. \end{aligned}$$

Note that, arrival rate until congestion level reaches T channels is $\lambda_s+\lambda_p$ and just λ_p afterwards. Finally, we can provide an explicit expression for $R_T(\lambda_s)$ as follows:

$$\begin{aligned} R_T(\lambda_s) = (1-B_{SU}(\lambda_s, T))\lambda_s u(\lambda_s) - B_{PU}(\lambda_s, T)\lambda_p K \\ +E(\lambda_p, C)\lambda_p K. \end{aligned} \tag{11}$$

where $u(\lambda_s)$ is the inverse function of $\lambda_s(u)$. The first term in Eq. (11) is the revenue collected from SU calls. The second term is a result of the punishment due to blocked PU calls. The last term is the normalization term which is used to ensure that profit is zero when there are no SUs (see Eq. (3)).

Next, we derive an important property of the blocking probabilities B_{PU} and B_{SU}, that will be exploited in the next section. Specifically, we show that the ratio of these blocking probabilities depends only on the PU's call arrival rate λ_p and threshold T but *not* on the price or the demand function of the SU.

Lemma 5.2: The ratio $\frac{B_{PU}(\lambda_s,T)}{B_{SU}(\lambda_s,T)}$ is independent of u and λ_s.

Proof:

$$\frac{B_{PU}(\lambda_s,T)}{B_{SU}(\lambda_s,T)} = \frac{\frac{\lambda_p^{C-T}}{C!}}{\sum_{n=T}^{C}\frac{\lambda_p^{n-T}}{n!}} \tag{12}$$

which is independent of u and λ_s. ∎

C. Unimodality of the Profit Function

In this section, we show that for each given threshold T, the profit function of threshold pricing R_T is *unimodal* in u (a function is unimodal over a certain interval, if it has a single maximum over that interval).

First, we define the following function which represents instantaneous profit rate when SU arrival rate is λ_s

$$Q(\lambda_s) = \lambda_s u(\lambda_s). \tag{13}$$

Unimodality of R_T requires the following mild assumption on the demand function:

Assumption 5.3: The function $Q(\lambda_s)$ is concave i.e., $Q''(\lambda_s) \le 0$ where the derivative is taken with respect to λ_s.

Assumption 5.3 implies that the marginal instantaneous profit is decreasing with respect to user demand. It ensures a well-behaved demand function [25]. This assumption is widely made in the literature [7, 10, 25] and is satisfied by several types of functions, such as linear and exponentially decaying demand functions.

The proof of our theorem will be based on the following lemma, which is proved in the Appendix.

Lemma 5.4: For all $\lambda_s > 0$,

$$\frac{B'_{SU}(\lambda_s,T)}{1-B_{SU}(\lambda_s,T)} > \frac{-B''_{SU}(\lambda_s,T)}{2B'_{SU}(\lambda_s,T)},$$

where the derivatives are taken with respect to λ_s.

We can now state our theorem:

Theorem 5.5: For a fixed threshold T, the function R_T is unimodal with respect to the price u over the interval $[0, u_{\max}]$.

Proof: We will prove that $R_T(\lambda_s)$ is unimodal with respect to λ_s. Since by Assumption 3.1 the function $\lambda_s(u)$ is strictly decreasing, this will also prove the unimodality of R_T with respect to u.

We refer to any value of λ_s at which the derivative of $R_T(\lambda_s)$ is equal to zero as a critical point. We will denote such a point with λ_s^*, i.e., $R'_T(\lambda_s^*) = 0$. To prove the theorem, we will show that $R''_T(\lambda_s^*) < 0$, for any λ_s^*. This means that there can be at most one critical point and it must be a maximum.

Let $X = \frac{B_{PU}(\lambda_s,T)}{B_{SU}(\lambda_s,T)}$ (recall Lemma 5.2). Then, we can rewrite the profit function and its first and second derivatives as follows:

$$\begin{aligned} R_T(\lambda_s) &= (1-B_{SU}(\lambda_s,T))Q(\lambda_s) \\ &\quad - XB_{SU}(\lambda_s,T)\lambda_p K + E(\lambda_p,C)\lambda_p K; \end{aligned} \tag{14}$$

$$\begin{aligned} R'_T(\lambda_s) &= (1-B_{SU}(\lambda_s,T))Q'(\lambda_s) \\ &\quad - B'_{SU}(\lambda_s,T)(Q(\lambda_s)+X\lambda_p K); \end{aligned} \tag{15}$$

$$\begin{aligned} R''_T(\lambda_s) &= (1-B_{SU}(\lambda_s,T))Q''(\lambda_s) \\ &\quad - B''_{SU}(\lambda_s,T)(Q(\lambda_s)+X\lambda_p K) \\ &\quad - 2B'_{SU}(\lambda_s,T)Q'(\lambda_s). \end{aligned} \tag{16}$$

Since $R'_T(\lambda_s^*) = 0$, we obtain from Eq. (15):

$$\frac{Q'(\lambda_s^*)}{Q(\lambda_s^*)+X\lambda_p K} = \frac{B'_{SU}(\lambda_s^*,T)}{1-B_{SU}(\lambda_s^*,T)} \tag{17}$$

From Assumption 5.3 and Eq. (16), a sufficient condition for $R''_T(\lambda_s^*) < 0$ is

$$\frac{Q'(\lambda_s^*)}{Q(\lambda_s^*)+X\lambda_p K} > \frac{-B''_{SU}(\lambda_s^*,T)}{2B'_{SU}(\lambda_s^*,T)}, \tag{18}$$

which holds true by Lemma 5.4 and Eq. (17). ∎

D. Characterization of the Profit Regions of Static and Threshold Pricing

In this section, we characterize the profit regions of different pricing policies. Specifically, we are interested in determining the maximum value of λ_p, denoted by $\lambda_{p,max}$, for which each of these policies still achieves a positive profit. The results in this section also require Assumption 5.3.

We prove that there exists a range of values of λ_p (which can be very large) for which threshold pricing achieves a positive profit while static pricing does not. Moreover, we show that profit region of threshold pricing is equal to optimal policy's profit region. Remarkably, the value of $\lambda_{p,\max}$ for all policies depend only on $u_{\max}$, but is independent of the demand function otherwise.

We first establish the condition for which static pricing stops generating profit (i.e., blocks all SU calls).

Lemma 5.6: Static pricing with optimally selected price generates profit if and only if

$$u_{\max} > (E(\lambda_p, C-1) - E(\lambda_p, C))\lambda_p K. \tag{19}$$

Proof: Profit is generated if optimal SU arrival rate is nonzero. We prove the lemma by showing that if condition (19) is satisfied then $\lambda_s^* > 0$ and if it is not then $\lambda_s^* = 0$, i.e., there is no SU arrival and no profit is generated.

Next, we analyze the expression

$$\frac{\partial R_C(\lambda_s)}{\partial \lambda_s}\Big|_{\lambda_s=0^+} \tag{20}$$

where $R_C(\lambda_s)$ represents profit function of static pricing which is same as Eq. (11) when $T = C$. The notation 0^+ is used to mean that the derivative is taken to the right of 0. If $\frac{\partial R_C(\lambda_s)}{\partial \lambda_s}\big|_{\lambda_s=0^+} > 0$ then there exists $\lambda_s > 0$ that generates profit. (i.e., $\lambda_s^* \neq 0$).

In Theorem 5.5 we show that R_C is *unimodal* with respect to u and λ_s under Assumption 5.3. Due to the unimodality of R_C, if $\frac{\partial R_C(\lambda_s)}{\partial \lambda_s}\big|_{\lambda_s=0^+} \le 0$ then $\lambda_s^* = 0$.

Note that, in the case of static pricing

$$B_{SU}(\lambda_s, C) = B_{PU}(\lambda_s, C) = E(\lambda_s+\lambda_p, C).$$

It can be verified algebraically (Lemma 2.1 in [26]) that

$$\begin{aligned} \frac{\partial E(\lambda_s+\lambda_p,C)}{\partial \lambda_s} &= (1-E(\lambda_s+\lambda_p,C)) \\ &\quad \cdot (E(\lambda_s+\lambda_p,C-1) - E(\lambda_s+\lambda_p,C)), \end{aligned} \tag{21}$$

and by using this equation we can evaluate $\frac{\partial R_C(\lambda_s)}{\partial \lambda_s}$ at $\lambda_s = 0^+$ as

$$\begin{aligned}\frac{\partial R_C(\lambda_s)}{\partial \lambda_s}\Big|_{\lambda_s=0^+} &= (1 - E(\lambda_p, C)) \\ &\cdot(u_{max} - (E(\lambda_p, C-1) - E(\lambda_p, C))\lambda_p K).\end{aligned} \tag{22}$$

Note that $E(\lambda_p, C) < 1$. Therefore, the sign of Eq. (22) depends only on the second term of the product.

■

An interesting corollary from this lemma is that if $K = 0$ (i.e., there is no punishment), then static pricing policy will accept SUs for all values of λ_p, and hence achieves the maximum profit region. This result indicates that the non-optimality of static pricing is due to both the presence of non-elastic PUs and punishments.

Next, we conduct a similar analysis for threshold pricing. We consider the case $T = 1$. We show in the analysis of optimal pricing profit region that $T = 1$ is the optimal threshold when the PU arrival rate is close to $\lambda_{p,max}$.

Lemma 5.7: Threshold pricing with $T = 1$ and optimally selected price generates profit if and only if

$$u_{\max} > E(\lambda_p, C)K. \tag{23}$$

Proof: It can be shown that

$$\frac{\partial B_{PU}(\lambda_s,1)}{\partial \lambda_s} = (1 - B_{SU}(\lambda_s, 1))B_{PU}(\lambda_s, 1)\frac{1}{\lambda_s+\lambda_p}, \tag{24}$$

$$\frac{\partial B_{SU}(\lambda_s,1)}{\partial \lambda_s} = (1 - B_{SU}(\lambda_s, 1))B_{SU}(\lambda_s, 1)\frac{1}{\lambda_s+\lambda_p}. \tag{25}$$

By using Eqs. (24) and (25), we can evaluate $\frac{\partial R_1(\lambda_s)}{\partial \lambda_s}$ at $\lambda_s = 0^+$ as

$$\begin{aligned}\frac{\partial R_1(\lambda_s)}{\partial \lambda_s}\Big|_{\lambda_s=0^+} &= (1 - B_{SU}(0, 1)) \\ &\cdot (u_{max} - B_{PU}(0, 1)K).\end{aligned} \tag{26}$$

Note that, $B_{PU}(0, 1) = E(\lambda_p, C)$. The result follows based on arguments similar to those of the previous lemma.

■

We next show that the profit region of threshold is larger than that of static pricing. We do so by showing that the RHS of Eq. (23) is larger than that of Eq. (19).

Lemma 5.8: For $C > 1$ and $\lambda_p > 0$

$$(E(\lambda_p, C-1) - E(\lambda_p, C))\lambda_p > E(\lambda_p, C). \tag{27}$$

Proof: Manipulating expressions we obtain that Eq. (27) holds if only if

$$E(\lambda_p, C) > \frac{\lambda_p - C}{\lambda_p}$$

which is equivalent to

$$1 - E(\lambda_p, C) < \frac{C}{\lambda_p}.$$

Note that at this point the claim is proved for $C > \lambda_p$. Further manipulation yields

$$1 - E(\lambda_p, C) < \frac{1}{\lambda_p}\frac{\sum_{n=0}^{C-1}\frac{\lambda_p^n}{n!}}{\sum_{n=1}^{C}\frac{\lambda_p^{n-1}}{n!}} \tag{28}$$

$$= \frac{1}{\lambda_p}\frac{1 + \sum_{n=1}^{C-1}\frac{\lambda_p^n}{n!}}{1 + C^{-1}\sum_{n=2}^{C}\frac{\lambda_p^{n-1}C}{n!}} \tag{29}$$

$$< \frac{1}{\lambda_p}\frac{1 + \sum_{n=1}^{C-1}\frac{\lambda_p^n}{n!}}{1 + C^{-1}\sum_{n=1}^{C-1}\frac{\lambda_p^n}{n!}} < \frac{C}{\lambda_p}. \tag{30}$$

■

Finally, we state our theorem which is a result of the previously stated lemmas.

Theorem 5.9: If Eq. (27) holds then for any demand function $\lambda_s(u)$ there exists values of λ_p for which static pricing blocks all SU calls but threshold and optimal pricing do not.

Proof: By Lemma 5.8, there exist a λ_p for which the following is true:

$$(E(\lambda_p, C-1) - E(\lambda_p, C))\lambda_p K \geq u_{max} > E(\lambda_p, C)K \tag{31}$$

and by Lemmas 5.6 and 5.7 the result follows.

■

E. Profit Region of Threshold Pricing is Optimal

In this is section we show that the profit region of threshold pricing is the same as profit region of optimal pricing. We start by characterizing the profit region of optimal pricing. For this purpose, we need the following additional lemma which is proven in the appendix.

Lemma 5.10: If $u_{\max} \leq E(\lambda_p, C)K$ then $\frac{\partial R(\Lambda)}{\partial \lambda_{s,0}} \leq 0$ where $R(\Lambda)$ is profit function for optimal pricing defined in Eq. (2) and $\lambda_{s,0} = \lambda_s(u_0)$ i.e., SU arrival rate when system is empty.

Lemma 5.11: Optimal pricing generates profit if and only if

$$u_{\max} > E(\lambda_p, C)K. \tag{32}$$

Proof: Lemma 5.10 indicates that the profit function is decreasing in $\lambda_{s,0}$ when $u_{max} \leq E(\lambda_p, C)K$ which means that the optimal value of $\lambda_{s,0}$ is 0 i.e., $u_0^* = u_{max}$. By lemma 4.3 we know that optimal prices are increasing with occupancy. Therefore the optimal prices for all states are equal to u_{max}. This means that there is no SU arrival and consequently no profit is generated when Eq. (32) is not satisfied. By lemma 5.7 we know that when $u_{max} > E(\lambda_p, C)K$ threshold pricing generates profit. Since optimal pricing must be better or equal to threshold pricing, we conclude that optimal pricing generates profit when Eq. (32) is satisfied. ■

This lemma also shows that when system is critical loaded (i.e., PU arrival rate is very close to $\lambda_{p,max}$) the optimal threshold for threshold pricing is $T = 1$ for any demand function which satisfies Assumption 5.3.

The following theorem states the optimality of threshold in terms of achieving the maximum profit regime. This theorem is an immediate consequence of Lemmas (5.7) and (5.11).

Theorem 5.12: The profit region of threshold pricing is optimal.

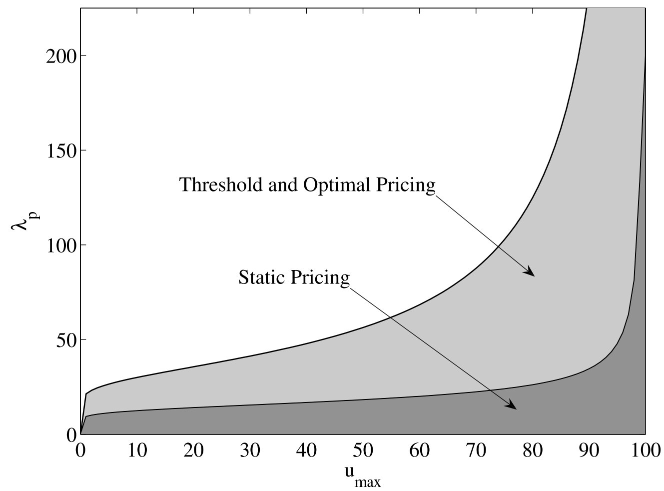

Fig. 5. Profit regions for different values of u_{max} where $C = 20$ and $K = 100$. If a (λ_p, u_{max}) pair lies in the dark-grey area all three pricing policies generate profit. If it lies in the light-grey area only threshold and optimal pricing generate profit.

In Fiq. (5) we show profit regions for different pricing policies. The policies generate profit when (λ_p,u_{max}) pair lies in the shaded regions of Fiq. (5). For example if $\lambda_p = 50$ and $u_{max} = 70$, threshold pricing generates profit while static pricing does not. The difference between profit regions can be quite high depending on the value of u_{max}. As u_{max} gets closer to K, $\lambda_{p,max}$ goes to infinity because SU calls and PU calls worth the same and it does not make sense to reject SU calls.

VI. Efficient Computation of the Optimal Threshold and Price

Our numerical results in the previous section have showed that threshold pricing performs close to optimal and far better than static pricing. In this section, we show another benefit of threshold pricing, namely, low computational complexity.

In Section V-C we showed that the profit function of threshold pricing is unimodal. This allows us to exploit efficient logarithmic search techniques, such as Fibonacci search [27], to find the optimal price for a given threshold in $O(\log |\mathbb{U}|)$ iterations. This result is significant because the discretization step Δu should be chosen very small in order to minimize the loss of profit due to price discretization.

The optimal threshold and price of the threshold policy can be computed by finding optimal price and comparing corresponding profits for all possible threshold values. This process requires overall $O(C \log |\mathbb{U}|)$ iterations. In the rest of this section we show that the search intervals for the optimal price and threshold can be greatly restricted. Having smaller search intervals significantly speeds up the optimization.

The following lemma shows that the optimal price for threshold pricing is higher than or equal to u_∞. Hence, we can restrict the search range for the optimal price to the interval $[u_\infty, u_{\max}]$.

Lemma 6.1: For any given threshold T, $u^*(T) \geq u_\infty$, where $u^*(T)$ is the optimal price when the threshold is set to T.

Proof: Let $u_- < u_\infty$. From Assumption 3.1 on the demand function, we know that $\lambda_s(u_-) > \lambda_s(u_\infty)$ and $Q(\lambda_s(u_-)) < Q(\lambda_s(u_\infty))$. Moreover, for any T we have $B_{SU}(\lambda_s(u_\infty), T) < B_{SU}(\lambda_s(u_-), T)$ and $B_{PU}(\lambda_s(u_\infty), T) < B_{PU}(\lambda_s(u_-), T)$. These inequalities lead to the following conclusion:

$$(1-B_{SU}(\lambda_s(u_-), T))Q(\lambda_s(u_-)) - B_{PU}(\lambda_s(u_-), T)\lambda_p K$$
$$< (1-B_{SU}(\lambda_s(u_\infty), T))Q(\lambda_s(u_\infty)) - B_{PU}(\lambda_s(u_\infty), T)\lambda_p K.$$

Thus, for any T (including T^*) the profit decreases as prices go below u_∞. Therefore, for any T, $u^*(T) \geq u_\infty$. ∎

Next, we prove that optimal threshold is $T_\infty \leq T^* \leq C$ where

$$T_\infty = \arg \max_{0 \leq T \leq C} (R_T(\lambda_s(u_\infty))$$

i.e., T_∞ is the optimal threshold when price is set to u_∞. This statement enables us to reduce search interval for the optimal threshold. Thus speeding up the computation time of threshold pricing policy. For the proof we need the following lemma.

Lemma 6.2: Assume λ_s and u are independent variables. The optimal threshold T^* is a non-increasing function of λ_s assuming u is fixed. It is also non-decreasing function of u assuming λ_s is fixed.

Proof: This lemma is obtained by applying Theorem 2 of [19] (see Section II). In order to apply results of [19] to our threshold model we assume that there are no *walk-in* users. We consider PUs and SUs as two different *contract* user classes for which prices are set to K and u and arrival rates are λ_p and λ_s, respectively. Theorem 2 of [19] states that in such a setting if we increase u (λ_s is fixed) the corresponding optimal threshold for SUs will increase. Moreover, if we reduce λ_s (u is fixed) the threshold will increase again. ∎

Now, we can state the theorem

Theorem 6.3: Optimal threshold is in the range

$$T_\infty \leq T^* \leq C. \tag{33}$$

Proof: Assume u is increased in the amount of α units to $u + \alpha$. Let $\beta = \lambda_s(u) - \lambda_s(u + \alpha)$ be the corresponding decrease in demand. By using Lemma 6.2 we can claim

$$T^*(\lambda_s, u) \leq T^*(\lambda_s, u + \alpha) \leq T^*(\lambda_s - \beta, u + \alpha) \tag{34}$$

Eq. (34) means if we increase u, the corresponding optimal threshold will not decrease.

Since optimal price, u^*, is higher than u_∞, we can conclude that $T_\infty \leq T^*$. ∎

Using the above theorem and lemmas, we can thus compute T^* and $u^*(T^*)$ very easily, namely, we first compute the optimal price $u^*(T)$ (within the range $[u_\infty, u_{\max}]$) for each threshold $T \in [T_\infty, C]$, using a logarithmic search procedure. Then, we establish the optimal threshold

$$T^* = \arg \max_{T_\infty \leq T \leq C} R_T(\lambda_s(u^*(T))),$$

with corresponding optimal price $u^*(T^*)$.

C	$t_{run}^{OP}/t_{run}^{TP}$	R_{OP}	R_{TP}	R_{SP}
250	5.9	3.8	3.1	0
500	11.1	42.1	39.7	15.0
750	18.7	111.6	108.4	75.5
1000	27.7	188.6	185.7	155.3

TABLE I
REVENUES OF OPTIMAL PRICING (OP), TP, AND SP AND RATIO OF RUN TIMES OF OP AND TP. SYSTEM PARAMETERS: $\lambda_p = \frac{9}{10}C$, $K = 100$, $\lambda_s(u) = \frac{C}{250}(10e^{-(\frac{u}{5}-1)^2} - 10^{-1})_+$ FOR $u \geq 5$, AND $\Delta u = 10^{-6}$.

In Table II, we present a numerical comparison of the time required to compute the threshold policy and optimal policy. For computation of the optimal policy *policy iteration* is used due to reasons mentioned in section IV. Moreover, the policy iteration procedure is also speeded up by taking advantage of unimodality. Both algorithms were developed in MATLAB and run on a Pentium M 1.7GHz PC. These numerical results demonstrate the practical importance of the optimization method developed in this section.

VII. Concluding Remarks

In this paper we have investigated the problem of devising efficient pricing policies for secondary spectrum usage. Specifically, we have formalized the problem of profit maximization for the usage of wireless spectrum in the presence of both primary and secondary users (PUs and SUs). We have provided a stochastic dynamic programming formulation of the problem and shown how to derive the optimal stationary pricing policy using policy iteration or relative value iteration.

A drawback of the optimal policy is to charge SUs different prices over time, depending on the channel occupancy. This makes the cost of spectrum access much less predictable and could potentially reduce demand. Thus, we have investigated the design of simple, yet efficient, single-price policies. We have provided numerical and analytical evidences that static pricing policies do not perform well in such settings (in contrast to settings where all the users are elastic). On the other hand, we have proven that deterministic threshold pricing achieves the optimal profit amongst all the single-price policies and performs close to global optimal pricing for a variety of demand functions.

Under certain reasonable assumptions on the demand function, we have proven that the profit function of threshold pricing is unimodal in the price. We have also showed that the optimal threshold lies within a specific interval. By taking advantage of these properties we showed that the optimal threshold policy is computationally more efficient than the optimal policy.

Lastly, we have introduced the new concept of *profit region*. The profit region indicates the range of arrival rate of PUs for which a secondary market is viable (i.e., positive profit is achievable by selling spectrum to secondary users). The success of secondary market initiatives hinges on the design of simple pricing policies that maximize the profit region. In this work, we have proven that the profit region of threshold pricing is optimal (again, under appropriate assumptions on the demand function). Furthermore, this profit region depends only on the support of the demand function and, in particular, the parameter $u_{\max}$, but not on the specific form of the demand function.

Appendix A
Proof of Lemma 5.4

It can be verified that

$$\frac{\partial}{\partial \lambda_s}(-\ln(1-B_{SU}(\lambda_s,T))) = \frac{B'_{SU}(\lambda_s,T)}{1-B_{SU}(\lambda_s,T)} \tag{35}$$

and

$$\frac{\partial}{\partial \lambda_s}(-\frac{1}{2}\ln(B'_{SU}(\lambda_s,T))) = -\frac{B''_{SU}(\lambda_s,T)}{2B'_{SU}(\lambda_s,T)}. \tag{36}$$

From Eqs. (35) and (36) it follows that

$$\frac{B'_{SU}(\lambda_s,T)}{1-B_{SU}(\lambda_s,T)} - \frac{-B''_{SU}(\lambda_s,T)}{2B'_{SU}(\lambda_s,T)} > 0 \tag{37}$$

if and only if

$$\frac{1}{2}\frac{\partial}{\partial \lambda_s}\big(\ln\big(\frac{B'_{SU}(\lambda_s,T)}{(1-B_{SU}(\lambda_s,T))^2}\big)\big) > 0. \tag{38}$$

Since the $\ln(\cdot)$ function is strictly increasing, Eq. (38) is equivalent to

$$\frac{\partial}{\partial \lambda_s}\big(\frac{B'_{SU}(\lambda_s,T)}{(1-B_{SU}(\lambda_s,T))^2}\big) > 0. \tag{39}$$

It can be shown that

$$\frac{B'_{SU}(\lambda_s,T)}{(1-B_{SU}(\lambda_s,T))^2} = Y\big(\frac{E'(\lambda_s+\lambda_p,T)}{(1-E(\lambda_s+\lambda_p,T))^2}\big), \tag{40}$$

where $Y = T!(\frac{1}{T!} + \frac{\lambda_p}{(T+1)!} + ... + \frac{\lambda_p^{C-T}}{C!})$. Since $Y > 0$ and it is independent of λ_s, Eq. (39) is satisfied if and only if

$$\frac{\partial}{\partial \lambda_s}\big(\frac{E'(\lambda_s+\lambda_p,T)}{(1-E(\lambda_s+\lambda_p,T))^2}\big) > 0. \tag{41}$$

Recall total arrival rate $\lambda = \lambda_s + \lambda_p$. Since taking derivative with respect to λ_s is same as taking derivative with respect to λ, Eq. (41) is equivalent to

$$\frac{\partial}{\partial \lambda}\big(\frac{E'(\lambda,T)}{(1-E(\lambda,T))^2}\big) > 0. \tag{42}$$

It can be verified that Eq. (42) is true by the proof of Proposition 6.1 in [10]. This completes the proof of the lemma.

Appendix B
Proof of Lemma 5.10

Recall the profit function (Eq. (2)) for optimal pricing which can be rearranged as the following

$$R(\Lambda) = \frac{N(\Lambda)}{D(\Lambda)} + E(\lambda_p, C)\lambda_p K \tag{43}$$

where

$$\begin{aligned} N(\Lambda) = Q(\lambda_{s,0}) + Q(\lambda_{s,1})\tfrac{\lambda_0}{1!} + Q(\lambda_{s,2})\tfrac{\lambda_0\lambda_1}{2!} + \\ \ldots + Q(\lambda_{s,C-1})\tfrac{\lambda_0\lambda_1\ldots\lambda_{C-2}}{(C-1)!} - K\lambda_p\tfrac{\lambda_0\lambda_1\ldots\lambda_{C-1}}{C!} \end{aligned} \tag{44}$$

and

$$D(\Lambda) = 1 + \tfrac{\lambda_0}{1!} + \tfrac{\lambda_0\lambda_1}{2!} + \ldots + \tfrac{\lambda_0\lambda_1\ldots\lambda_{C-1}}{C!}. \tag{45}$$

Here $\lambda_n = \lambda_{s,n} + \lambda_p$ where $\lambda_{s,n} = \lambda_s(u_n)$ i.e., the total arrival rate when system is in state n.

We start by proving that if

$$\frac{\partial R(\Lambda)}{\partial \lambda_{s,0}}\Big|_{\lambda_{s,0}=0} \leq 0 \tag{46}$$

then

$$\frac{\partial R(\Lambda)}{\partial \lambda_{s,0}} \leq 0. \tag{47}$$

The derivative of $R(\Lambda)$ with respect to $\lambda_{s,0}$ is

$$\frac{\partial R(\Lambda)}{\partial \lambda_{s,0}} = \frac{\frac{\partial N(\Lambda)}{\partial \lambda_{s,0}} D(\Lambda) - N(\Lambda)\frac{\partial D(\Lambda)}{\partial \lambda_{s,0}}}{D^2(\Lambda)}. \tag{48}$$

It can be verified that

$$\frac{\partial N(\Lambda)}{\partial \lambda_{s,0}} = Q'(\lambda_{s,0}) + \frac{N(\Lambda) - Q(\lambda_{s,0})}{\lambda_0} \tag{49}$$

and

$$\frac{\partial D(\Lambda)}{\partial \lambda_{s,0}} = \frac{D(\Lambda) - 1}{\lambda_0}. \tag{50}$$

and by substituting Eq. (49) and Eq. (50) into Eq. (48) we have

$$\frac{\partial R(\Lambda)}{\partial \lambda_{s,0}} = \frac{(Q'(\lambda_{s,0}) - \frac{Q(\lambda_{s,0})}{\lambda_0})D(\Lambda) + \frac{N(\Lambda)}{\lambda_0}}{D^2(\Lambda)}. \tag{51}$$

Note that, the sign of the RHS of Eq. (51) is determined by the numerator

$$(Q'(\lambda_{s,0}) - \frac{Q(\lambda_{s,0})}{\lambda_0})D(\Lambda) + \frac{N(\Lambda)}{\lambda_0}. \tag{52}$$

Next, we show that Eq. (52) is decreasing in $\lambda_{s,0}$. The derivative of Eq. (52) is

$$\frac{\partial}{\partial \lambda_{s,0}}((Q'(\lambda_{s,0}) - \frac{Q(\lambda_{s,0})}{\lambda_0})D(\Lambda) + \frac{N(\Lambda)}{\lambda_0}) = Q''(\lambda_{s,0})D(\Lambda). \tag{53}$$

Since $D(\Lambda) > 0$ and $Q''(\lambda_{s,0}) < 0$, due to Assumption 5.3, Eq. (52) is decreasing with respect to $\lambda_{s,0}$. Therefore if $\frac{\partial R(\Lambda)}{\partial \lambda_{s,0}}\big|_{\lambda_{s,0}=0} \leq 0$ then $\frac{\partial R(\Lambda)}{\partial \lambda_{s,0}} \leq 0$ for $\lambda_{s,0} > 0$.

The second step of our proof is to show if

$$u_{max} \leq E(\lambda_p, C)K \tag{54}$$

then

$$\frac{\partial R(\Lambda)}{\partial \lambda_{s,0}}\Big|_{\lambda_{s,0}=0} \leq 0. \tag{55}$$

The sign of RHS of Eq. (55) is defined by Eq. (52). If $\lambda_{s,0} = 0$ then $Q(0) = 0$, $Q'(0) = u_{max}$ and $\lambda_0 = \lambda_p$. By substituting these values into Eq. (52) we obtain

$$\begin{aligned}(Q'(0) - \frac{Q(0)}{\lambda_p})D(\Lambda) + \frac{N(\Lambda)}{\lambda_p} &= u_{max}D(\Lambda) + \frac{N(\Lambda)}{\lambda_p} = \\ u_{max} + u_{max}\frac{\lambda_p}{1!} + u_{max}\frac{\lambda_p\lambda_1}{2!} &\ldots + u_{max}\frac{\lambda_p\lambda_1\ldots\lambda_{C-1}}{C!} \\ +Q(\lambda_{s,1})\frac{1}{1!} + Q(\lambda_{s,2})\frac{\lambda_1}{2!} &\ldots + Q(\lambda_{s,C-1})\frac{\lambda_1\ldots\lambda_{C-2}}{(C-1)!} \\ &-K\lambda_p\frac{\lambda_1\ldots\lambda_{C-1}}{C!}.\end{aligned} \tag{56}$$

It is enough to prove the statement for $K = \frac{u_{max}}{E(\lambda_p,C)}$ since larger values of K make the RHS of Eq. (56) only smaller. By inserting

$$K = \frac{u_{max}}{E(\lambda_p, C)} = u_{max}\frac{1 + \frac{\lambda_p}{1!} + \frac{\lambda_p^2}{2!}\ldots\frac{\lambda_p^C}{C!}}{\frac{\lambda_p^C}{C!}} \tag{57}$$

and rearranging, Eq. (56) becomes

$$\begin{aligned}u_{max}D(\Lambda) + \frac{N(\Lambda)}{\lambda_p} &= [u_{max} - u_{max}\frac{\lambda_1\ldots\lambda_{C-1}}{\lambda_p^{C-1}}] \\ &+[u_{max}\lambda_p + Q(\lambda_{s,1}) - u_{max}\frac{\lambda_1\ldots\lambda_{C-1}}{\lambda_p^{C-2}}] \\ +\sum_{i=2}^{C-1}\frac{\lambda_1\ldots\lambda_{i-1}}{i!}&[u_{max}\lambda_p + Q(\lambda_{s,i}) - u_{max}\frac{\lambda_i\ldots\lambda_{C-1}}{\lambda_p^{C-i-1}}].\end{aligned} \tag{58}$$

Next, we show that all three terms in the above equation are negative. The first term is negative because $\lambda_n \geq \lambda_p$ for $n = 1, 2, ..., C-1$. For the second and the third term the sign is defined by the following expression

$$u_{max}\lambda_p + Q(\lambda_{s,n}) - u_{max}\frac{\lambda_n\ldots\lambda_{C-1}}{\lambda_p^{C-n-1}} \tag{59}$$

for $n = 1, 2...C-1$. Since $\lambda_n \geq \lambda_p$

$$\begin{aligned}u_{max}\lambda_p &+ Q(\lambda_{s,n}) - u_{max}\frac{\lambda_n\ldots\lambda_{C-1}}{\lambda_p^{C-n-1}} \\ &\leq u_{max}\lambda_p + Q(\lambda_{s,n}) - u_{max}\lambda_n \\ &= Q(\lambda_{s,n}) - u_{max}\lambda_{s,n} \\ &= (u(\lambda_{s,n}) - u_{max})\lambda_{s,n}.\end{aligned} \tag{60}$$

Since $u(\lambda_{s,n}) \leq u_{max}$, all the terms are negative. Therefore if $u_{max} < E(\lambda_p, C)K$ then $\frac{\partial R(\Lambda)}{\partial \lambda_{s,0}}\big|_{\lambda_{s,0}=0} \leq 0$. Together with the first part of the proof this concludes the proof of the lemma.

References

[1] H. Mutlu, M. Alanyali, and D. Starobinski, "Spot pricing of secondary spectrum usage in wireless cellular networks," in *INFOCOM*, April 2008.

[2] "The Federal Communications Commission, Secondary Markets Initiative." http://wireless.fcc.gov/licensing/secondarymarkets/.

[3] "Radio Spectrum Policy Group." http://rspg.groups.eu.int/.

[4] R. Dewenter and J. Haucap, "Incentives to license mobile virtual network operators (MVNOs)," in *Proc. 34th Research Conference on Communication, Information and Internet Policy*, Oct. 2006.

[5] M. Lev-Ram, "As mobile ESPN falls, a Helio rises." http://money.cnn.com/2006/10/17/magazines/business2/espnmobile_mvno.biz2/index.htm. CNN Business 2.0 Magazine.

[6] Z. Ezziane, "Charging and pricing challenges for 3G systems," *IEEE Communications Surveys and Tutorials*, vol. 7, pp. 58–68, Fourth Quarter 2005.

[7] I. C. Paschalidis and J. N. Tsitsiklis, "Congestion-dependent pricing of network services," *IEEE/ACM Transactions on Networking*, vol. 8, pp. 171–184, April 2000.

[8] I. C. Paschalidis and Y. Liu, "Pricing in multiservice loss networks: static pricing, asymptotic optimality, and demand substitution effects," *IEEE/ACM Transactions on Networking*, vol. 10, pp. 425–438, June 2002.

[9] X. Lin and N. Shroff, "Simplification of network dynamics in large systems," *IEEE/ACM Transactions on Networking*, vol. 13, pp. 813–826, August 2005.

[10] S. Ziya, H. Ayhan, and R. Foley, "A Note on optimal pricing for finite capacity queueing systems with multiple customer classes," *Naval Research Logistics*, vol. 55, pp. 412–418, 2008.

[11] A. Al-Daoud, M. Alanyali, and D. Starobinski, "Secondary pricing of spectrum in cellular CDMA networks," in *IEEE DySPAN*, 2007.

[12] G. Haring, R. Marie, R. Puigjaner, and K. Trivedi, "Loss formulas and their application to optimization for cellular networks," *IEEE Transactions on Vehicular Technology*, vol. 50, pp. 654–673, May 2001.

[13] S.-H. Oh and D.-W. Tcha, "Prioritized channel assignment in a cellular radio network," *IEEE Transactions on Communications*, vol. 40, pp. 1259–1269, July 1992.

[14] R. Ramjee, R. Nagarajan, and D. F. Towsley, "On optimal call admission control in cellular networks," in *INFOCOM*, pp. 43–50, 1996.

[15] B. Miller, "A queueing reward system with several customer classes," *Management Science*, vol. 16, no. 3, 1969.

[16] J. Hou, J. Yang, and S. Papavassiliou, "Integration of pricing with call admission control to meet qos requirements in cellular networks," *IEEE Transactions on Parallel & Distributed Systems*, vol. 13, no. 9, pp. 898–910, 2002.

[17] I.-R. Chen, O. Yilmaz, and I.-L. Yen, "Admission control algorithms for revenue optimization with QoS guarantees in mobile wireless networks," *Wireless Personal Communication*, vol. 38, no. 3, pp. 357–376, 2006.

[18] S.-L. Hew and L. White, "Optimal integrated call admission control and dynamic pricing with handoffs and price-affected arrivals," in *Communications, 2005 Asia-Pacific Conf. on Communications*, pp. 396–400.

[19] N. Gans and S. Savin, "Pricing and capacity rationing for rentals with uncertain durations," *Management Science*, vol. 53, March 2007.

[20] S. Ross, *Introduction to Stochastic Dynamic Programming*. Academic Press, 1983.

[21] D. Bertsekas, *Dynamic Programming and Optimal Control*. Athena Scientific, 2005.

[22] V. D. Blondel and J. N. Tsitsiklis, "A survey of computational complexity results in systems and control," *Automatica*, vol. 36, no. 9, pp. 1249–1274, 2000.

[23] H. Mutlu, M. Alanyali, and D. Starobinski, "Spot pricing of secondary spectrum usage in wireless cellular networks." Boston University, Center for Information and Systems Engineering, Technical Report, http://www.bu.edu/systems/research/publications/2008/2008-IR-0030.pdf, 2008.

[24] P. Fishburn and A. Odlyzko, "Dynamic behavior of differential pricing and quality of service options for the internet," *Decision Support Systems*, vol. 23, pp. 123– 136, March 2000.

[25] S. Ziya, H. Ayhan, and R. Foley, "Relationships among three assumptions in revenue management," *Operations Research*, vol. 52, pp. 804–809, September-October 2004.

[26] F. Kelly, "Routing in circuit-switched networks: optimization, shadow prices and decentralization," *Advances in Applied Probability*, vol. 20, pp. 112–144, 1988.

[27] D. Wilde, *Optimum seeking methods*. Prentice-Hall, 1964.

Huseyin Mutlu received his B.Sc. degree from Middle East Technical University, Ankara, Turkey in 2003 and M.Sc. degree from Northeastern University, Boston, MA in 2005 both in electrical engineering. Currently, he is a Ph.D. candidate at Boston University, Electrical and Computer Engineering Department.

His research interests are primarily in pricing and management of wireless communication systems.

Murat Alanyali received his Ph.D. degree in Electrical and Computer Engineering from the University of Illinois at Urbana-Champaign in 1996. He held positions at Bell Laboratories, Holmdel, NJ, during 1996-1997, and at the Department of Electrical and Electronics Engineering at Bilkent University, Ankara, Turkey, from 1998 to 2002. He is presently an Associate Professor of Electrical and Computer Engineering at Boston University. His research interests are in communication networks and distributed algorithms. Dr. Alanyali is a recipient of the National Science Foundation (NSF) CAREER award in 2003.

David Starobinski received his Ph.D. in Electrical Engineering (1999) from the Technion-Israel Institute of Technology. In 1999-2000, he was a visiting post-doctoral researcher in the EECS department at UC Berkeley. In 2007-2008, he was on sabbatical at EPFL (Switzerland), where he was an invited Professor. Since September 2000, he has been with Boston University, where he is now an Associate Professor.

Dr. Starobinski received a CAREER award from the U.S. National Science Foundation and an Early Career Principal Investigator (ECPI) award from the U.S. Department of Energy. His research interests are in the modeling and performance evaluation of high-speed, wireless, and sensor networks.