Dynamic Pricing of Preemptive Service for Secondary Demand

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***Abstract*—Motivated by recent regulatory evolutions that pave the way to secondary spectrum markets, we investigate profit maximization in a loss network that accommodates calls of two classes of users: primary users (PUs) and secondary users (SUs). PUs have preemptive priority over SUs, i.e. when a PU arrives to the system and finds all channels busy, it preempts an SU. We assume that SU demand is sensitive to price whereas PU demand is inelastic. We study the optimal pricing policy of SUs to maximize the average profit by introducing a finite horizon discounted dynamic programming formulation. Our main contribution is to show that the optimal pricing policy depends only on the total number of users, i.e. the total occupancy. We also demonstrate that optimal prices increase with the total occupancy and show that the optimal pricing policy structure of the original system is not preserved for systems with price-sensitive PUs. Finally, we extend the results to non-preemptive loss systems and establish a connection with results obtained for such models in the literature.**

***Index Terms*—Dynamic programming, dynamic spectrum access, Markov decision process, stochastic control**

I. Introduction

Commercially available wireless spectrum has become drastically scarce because of the increasing use of wireless devices, such as smartphones and tablets. According to a recent study conducted on the worldwide mobile traffic in 2G, 3G and 4G networks by Ericsson, the mobile data traffic will increase tenfold between 2011 and 2016 and there will be five billion subscribers by then. Furthermore, the traffic generated by smartphones will be approximately equal to the PC network traffic [8]. The spectrum tug of war between the wireless service providers (SPs) has become so drastic that Verizon Wireless has offered $3.9 billion to buy wireless spectrum from some cable companies to increase its current spectrum [33].

While the market is in desperate need of additional spectrum, studies show that the spectrum allocated to license holders is often underutilized in space and time [26]. Federal Communication Commission (FCC) reports that spectrum utilization varies between 15% and 85% [1]. To amend woes stemming from inefficient spectrum utilization, FCC has taken a remarkable step by announcing the Secondary Market Initiative in October 2003 [9]. With the approval of FCC, licensees can lease their spectrum to secondary users. Consequently, these regulations induced the use of cognitive radio (CR) technologies which enable smart use of the spectrum through opportunistic spectrum hand-off and secondary market usage.

State-of-the-art spectrum utilization techniques are useful if the SP manages to regulate the optimal control over the secondary users. In CR systems, there are two classes of customers: primary users (PU) which have permanent license to access the spectrum and secondary users (SU) which are temporarily allowed into the system whenever the system is underutilized. In general, the PUs are long term contract customers and SUs lease excess spectrum when the system allows them [16].

A wireless SP serving both PUs and SUs has two main objectives. Firstly, the provider must attract the greatest number of SUs to increase its profit. Secondly, the provider must ensure the Quality of Service (QoS) of PUs, i.e. the performance perceived by PUs should not be affected by the presence of SUs. In this paper, our main objective is to investigate the optimal pricing policy an SP should employ in order to obtain the maximum possible profit from SUs. A pricing policy enforces the prices advertised to the SUs. SUs are considered to be *price-sensitive* users, i.e. their arrivals are regulated by the price advertised by SP. PUs, on the other hand, are considered to be *inelastic* to the price i.e., the arrivals of PUs are unaffected by pricing. The price per call from an SU is collected upon arrival.

In order to emphasize the importance of PUs, we consider the PUs as the *higher priority users* whereas the SUs are the *lower priority users*. We use a *preemption* mechanism that allows SUs in the system when there is capacity and aborts the service of an SU whenever a PU needs service, in line with the *interweave* paradigm of cognitive radios [15].

For every evicted SU, the SP has to pay a certain cost. This allows us to ensure that PUs have access to spectrum whenever they need and SUs have access to excess spectrum when available [16]. As a practical application of preemption in CR networks, FCC Block D at 700 MHz employs such a termination model where public safety services are the PUs and commercial services are the SUs as explained in Tortelier et al. [29].

Our contributions in this paper are as follows:

1) We derive the optimal pricing policy of SUs which maximizes the long-run average profit in a preemptive system with price-sensitive SUs and inelastic PUs. We formulate the problem as a two-dimensional (2D) Markov decision process (MDP) optimization problem, and prove

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that the optimal pricing policy of SUs depends only on the total number of users (PUs and SUs) in the system in the long-run.
2) We prove that the optimal prices increase with the total occupancy, through the introduction and analysis of an auxiliary one-dimensional system that has the same optimal pricing policy as the original system.
3) We provide numerical results showing that the optimal congestion-based pricing policies outperform static pricing and can be efficiently computed in networks with hundreds of channels.
4) We establish a connection with non-preemptive systems and formally demonstrate that, under the same parameters, the optimal pricing policies for preemptive and non-preemptive loss systems are identical. This result allows the application of methodologies developed for non-preemptive systems to preemptive loss systems.

The rest of this paper is organized as follows. Related work is discussed in Section II. We introduce our model and statistical assumptions in Section III. In Section IV, we derive the structure of the optimal pricing policy and prove the monotonicity of prices. In Section V, we describe the application of our results to the problem of optimal admission control of SUs. We also study systems with price-sensitive PUs and show, via numerical examples, that the optimal pricing policies in those cases depend on the individual number of PUs and SUs. We extend our results to non-preemptive loss systems in Section VI. We conclude the paper in Section VII.

II. Related Work

In this section, we present a literature review which can be grouped under three main categories: cognitive radios, preemption, and pricing.

Cognitive radios support three main approaches for spectrum access by SUs: underlay, overlay, and interweave [15]. While the underlay and overlay approaches permit concurrent communication by PUs and SUs, the interweave approach allows SUs to communicate only over spectrum holes temporarily left available by PUs. The preemption model considered in our work is consistent with the interweave approach. Thus, the performance of PUs is not impacted by SUs.

While preemption models aligned with the interweave approach already exist in the literature on cognitive radios [24, 37], these models focus on admission control rather than optimal pricing as done in our work. Specifically these papers analyze channel reservation schemes that allow a trade-off between blocking and eviction of SUs. Such a trade-off can also be achieved in our model by tuning the cost that the SP must pay to preempted calls. Increasing the cost lowers the eviction probability at the expense of raising the blocking probability, since fewer SUs are admitted.

Work on control policies for systems with preemption can also be found in the operations research literature. However, none of those works considers pricing. One of the earliest works on preemption is the work of Helly, which proposes approaches on the control of two class traffic with different priorities and limited capacity [17]. Garay and Gopal investigate the use of preemption control in high speed networks and analyze call preemption [12]. Next, Xu and Shanthikumar examine a first-come first-served non-identical multi-server system and determine its optimal admission control policy using duality and preemption [34]. Brouns and van der Wal study a single server queue with two different classes of users with identical service rates [6]. Brouns extends these results to a multi-server system where there are no preemption costs [5]. Zhao et al. use preemption in order to provide differentiated services in a parallel multi-class loss network [36]. Finally, Ulukus et al. consider a system with two classes, non-identical service rates and different priorities [31]. They study admission and preemption control that maximize the expected discounted revenue and prove that preemption is only optimal when the system is full. They provide monotonicity results and a loose threshold type of admission control policy.

There also exists a rich body of work in the field of pricing for queues. However, our work appears to be the first to consider pricing in conjunction with preemption. The seminal work of Naor [23] introduces the idea of using pricing as a queue control mechanism to achieve social optimality. Chen and Frank [7] investigate pricing for an M/M/1 queue, while Low in [19] extends it further to consider a multi-server queue with restricted waiting space. Altman et al. consider admission control for a multi-class loss system in [3]. Yildirim and Hasenbein [35] combine pricing and admission control for queues experiencing batch arrivals. The work of Feinberg and Yang in [10] considers admission control subject to several objectives, such as bias optimality. Finally, the work of Giloni et al. [14] investigates the problem pricing and admission control with several customer classes with different rewards and/or waiting costs with prices/arrival rates that are not necessarily monotone.

Paschalidis and Tsitsiklis analyze congestion-dependent pricing of a multi-class system, where all classes of users are sensitive to price [25]. Mutlu et al. investigate the optimal dynamic pricing policy of a system consisting of inelastic PUs and price-sensitive SUs [21]. Gans and Savin characterize a system consisting of two types of users within the context of a rental management problem which resemble the PUs and SUs in our model [11].

In summary, our work differs from the preceding work by focusing on pricing control in preemptive systems. Previous work on pricing does not consider preemption, whereas previous work that utilizes preemption does not consider pricing. Furthermore, our model incorporates dynamic pricing of SUs (i.e., pricing is based on the current state of the system), while a significant amount of the literature on pricing assume that prices are not congestion-dependent.

III. Model Description

In this section, we describe our model and statistical assumptions. We assume that there are C identical and parallel channels (i.e. the system capacity is C) which are allocated for the use of the calls of users of two classes: PUs and SUs.

We assume that each call requests the same amount of bandwidth corresponding to a single channel. PUs have *preemptive* *priority* over the SUs. In our case, a channel is allocated to a higher priority call even if a lower priority call is in progress. When the lower priority call is preempted, it is withdrawn from the system permanently. Note that PUs and SUs join and leave the system individually, rather than in batch.

Regardless of the class type, call durations are independent and exponentially distributed with mean μ^{-1} unless terminated prematurely. The exponential assumption is needed for analytical tractability, similar to other analytical work on cognitive radios [24, 37], and can be justified in certain environments [13]. We model our system as a finite state 2D continuous-time MDP. The rest of the system description is as follows:

States: The state of the system is in the form (x, y) where $x \geq 0$ is the number of PU calls in the system and $y \geq 0$ is the number of SU calls in the system, and both x and y are integers.

Rewards and costs: $u(x, y)$ is the reward per SU call at state (x, y). The reward is collected upon arrival. As defined in the work of Paschalidis and Tsitsiklis [25], a congestion-dependent pricing policy is the set of rules which determines the price advertised by the SP at any given time, depending on the current state. We denote the pricing policy as $\mathbf{u}$, and it is defined for the states within the range of the capacity limit C, i.e. $0 \leq x + y < C$. The prices at each state are chosen from an interval $\mathbb{U} = [0, u_{max}]$ where a definition of u_{max} is provided below. We discretize $\mathbb{U}$ with a step size Δu in order to obtain a finite control space. Then, the number of possible prices becomes $|\mathbb{U}| = \lfloor u_{max}/\Delta u \rfloor + 1$. From now on, we will use the discrete version of $\mathbb{U}$. If a PU call arrives and finds all the channels busy, then the system preempts an SU call given that an SU call is present in the system. The preemption mechanism is active only when all channels are busy. Whenever an SU call is preempted, the SP pays a cost K per preempted call, which is greater than the maximum price that can be chosen from $\mathbb{U}$. A call is blocked only if an arriving user finds all channels busy and preemption is not possible. For PUs, this corresponds to the case when all the calls in the system belong to PUs. For an incoming SU to be blocked, it is sufficient to have all C channels busy. A blocked call receives a busy signal and is dropped. Blocking calls of any class is free of charge.

Arrival rates: PU calls arrive according to a Poisson process with a constant rate $\lambda_1 > 0$. SU calls, however, arrive according to a Poisson process and pay a fee $u(x, y)$ upon arrival when the state is (x, y). The average arrival rate of SUs at state (x, y) is related to the price $u(x, y)$ via a *demand function* $\lambda_2(u(x, y)) \geq 0$. We denote the maximum average arrival rate over all prices by $\lambda_{2,max}$. We will use the following assumptions in all of our formulations:

Assumption 1 *There exists a price u_{max} for which $\lambda_2(u) = 0$ when $u \geq u_{max}$.*

Assumption 2 *$\lambda_2(u)$ is a strictly decreasing function of u over the interval $[0, u_{max}]$.*

Assumption 2 implies that the inverse function of $\lambda_2(u)$ exists and that the maximum possible arrival rate of SUs corresponds to the lowest possible price, i.e. $\lambda_{2,max} = \lambda_2(0)$. This assumption is only needed for the proof of Theorem IV.2.

The objective of the SP is to maximize the average profit collected from SUs per unit time. The corresponding optimal pricing policy is denoted $\mathbf{u}^*$.

IV. Model Analysis and Characterization of the Optimal Pricing Policy

In this section, we first provide an expression for the average profit rate of SUs given a policy $\mathbf{u}$. Then, we present a finite horizon discounted return maximization problem formulation to compute the optimal pricing policy. Afterwards, we determine the structure of the optimal pricing policy that maximizes the discounted profit and extend our findings to the infinite horizon.

A. Formulation of the Profit Maximization Problem

In this section, we first introduce state space definitions and then develop a formula to calculate the average profit rate collected from SUs.

We start the formulation by defining state spaces. The entire state space is denoted as follows:

$$S = \{(x, y) \mid x + y \leq C\ ,\ x, y \geq 0\}\,.$$

Let $S_1 \subset S$ be the sub-space of states where all the channels are busy and at least one SU call is present in the system. According to our system description, S_1 denotes the states at which an SU can be preempted and is formally defined as the following:

$$S_1 = \{(x, y) \mid x + y = C\ ,\ x \geq 0\ ,\ y > 0\}\,.$$

Lastly, we define $S_2 \subset S$ which corresponds to all states where an SU arrival may enter the system, i.e.,

$$S_2 = \{(x, y) \mid x + y < C\ ,\ x, y \geq 0\}\,.$$

We denote $\pi_{\mathbf{u}}(x, y)$ to be the steady state probability that the system is in state (x, y) under the pricing policy $\mathbf{u}$. Note that $\mathbf{u}$ represents an arbitrary pricing policy that may not be necessarily optimal. The average profit rate under policy $\mathbf{u}$, is expressed as follows:

$$J_{\mathbf{u}} = \sum_{(x,y)\in S_2} \lambda_2(u(x, y))\ u(x, y)\ \pi_{\mathbf{u}}(x, y) - K\lambda_1 \sum_{(x,y)\in S_1} \pi_{\mathbf{u}}(x, y). \tag{1}$$

The first term in Eq. (1) represents the cumulative average revenue collected from SUs. Since a reward is collected upon arrival, we multiply the reward with the arrival rate of SUs of the current state. Then, the resulting term is scaled with the steady state probability of the corresponding state. We repeat the same procedure for all states in S_2 and add all of them up. The second term stands for the average cost rate due to preempted SU calls. We find the sum of steady state probabilities of the preemptive states. Then, we multiply this term with K and λ_1. Subtracting the total average cost rate from the total average revenue rate gives overall average profit rate of the system at state (x, y) and under the pricing policy $\mathbf{u}$.

The optimal pricing policy $\mathbf{u}^*$ is the policy which maximizes Eq. (1) and it yields optimal profit J^*. Finding the optimal price decisions at every state using the given equation is a multifaceted optimization problem. To solve this problem, we will formulate it as a *stochastic dynamic programming (DP) problem* [4] which determines the optimal pricing policy of the system.

B. Characteristics of the Optimal Pricing Policy

In this section, we introduce a theorem which states that the optimal pricing policy depends only on the total occupancy. To demonstrate this, we first present a finite horizon expected discounted return DP formulation. We obtain some properties of the finite horizon discounted profit and use these results to characterize the optimal pricing policy of the original system. First, we present some concepts such as discounting and uniformization.

Discounting: Our system has an exponential discount rate with parameter $\alpha \geq 0$ which implies that the reward gained in the present is more valuable than future rewards. The discount rate is considered to be the rate by which the process vanishes as explained in Walrand's work [32].

In order to implement the DP formulation, we need to find the discrete-time equivalent of the continuous-time Markov chain model of the system using a technique called *uniformization* [18].

Uniformization: Our current model is a continuous-time MDP. To convert the system to its discrete-time equivalent, we apply a uniformization method whereby every rate coming out of a state is normalized by the maximum transition rate possible [4, Vol. II, p. 258]. The maximum transition rate is given by $v = \lambda_1 + \lambda_{2,max} + C\mu + \alpha$. Without loss of generality, we set $v = 1$. We scale every rate of the continuous-time MDP with v which gives the probability of every transition. The events and the corresponding probabilities of a system at state (x, y) and price u are as follows:

- a PU arrival occurs with a probability of $\lambda_1/v = \lambda_1$
- an SU arrival occurs with a probability of $\lambda_2(u)/v = \lambda_2(u)$
- a PU departure occurs with a probability of $x\mu/v = x\mu$
- an SU departure occurs with a probability of $y\mu/v = y\mu$
- the process stays at the same state (x, y) with probability $(1-(\lambda_1+\lambda_2(u)+x\mu+y\mu+\alpha)/v) = (1-\lambda_1-\lambda_2(u)-x\mu-y\mu-\alpha)$
- the process vanishes with a probability of $\alpha/v = \alpha$.

Criterion: We aim to maximize the total expected discounted profit of the SP over a finite horizon. We are interested in finding an optimal pricing policy $\mathbf{u}^*$ which achieves this goal. Note that, for the sake of simplicity, we use the same notation for the optimal pricing policy in both the average return and discounted cases. The optimal policy itself is obviously different in each case, but the structure remains the same as detailed in the sequel.

In the DP formulation, we reverse the time index and define n as the number of observation points left until the end of the time horizon. The price decision for an SU at state (x, y) and time period n is defined as u. We define the profit function at this point.

Definition IV.1 *$V_n(x, y)$ is the maximal expected discounted profit for the system in the current state (x, y) at time period n.*

The corresponding finite horizon DP optimality equations are as follows:

For $n = 0$**:**

$$V_0(x, y) = 0 \qquad \text{for} \quad x, y \geq 0$$

For $n \geq 1$**:**

$$\begin{aligned} V_n(x, y) = \max_{u \in \mathbb{U}} \Big\{ & \lambda_1 V_{n-1}(x+1, y)\mathbf{1}\{x+y<C\} && (2)\\ & + \lambda_1 (V_{n-1}(x+1, y-1) - K) \\ & \quad \cdot \mathbf{1}\{x+y=C\}\mathbf{1}\{y>0\} && (3)\\ & + \lambda_1 V_{n-1}(x, y)\mathbf{1}\{x+y=C\}\mathbf{1}\{y=0\} && (4)\\ & + \lambda_2(u)(V_{n-1}(x, y+1) + u)\mathbf{1}\{x+y<C\} && (5)\\ & + x\mu V_{n-1}(x-1, y) && (6)\\ & + y\mu V_{n-1}(x, y-1) && (7)\\ & + (1-\lambda_1-\lambda_2(u)-x\mu-y\mu-\alpha) \\ & \quad \cdot V_{n-1}(x, y)\mathbf{1}\{x+y<C\} && (8)\\ & + (1-\lambda_1-C\mu-\alpha)\, V_{n-1}(x, y)\mathbf{1}\{x+y=C\} \Big\}. && (9) \end{aligned}$$

Note that if the process vanishes (which happens with probability α), then the reward is 0.

We set $V_n(-1, y) = V_n(0, y)$ and $V_n(x, -1) = V_n(x, 0)$ when required. The value of u that maximizes discounted profit is denoted $u = u_n^*(x, y)$ which is the optimal pricing decision of state (x, y) at time period n.

The first three terms in the DP formulation corresponds to the arrival of a PU and includes three distinct cases. Term (2) corresponds to the first possible case for a PU arrival. The indicator function denotes that the system is not full yet and there is no preemption. In term (3), there is a PU arrival again. However, all channels are occupied and there is at least one SU in the system. Thus, the incoming PU call preempts an SU which causes a reduction in the number of SU calls and an increase in the number of PUs. The SP is charged a cost K as a result of preemption. Term (4) corresponds to the case when all C channels are occupied but there are no SUs to preempt. In other words, all C users consist of PUs. Hence, preemption is not possible, the incoming PU is blocked and the system stays at the same state.

Next, in term (5), there is an SU arrival and there are idle channels available for its use. Terms (6) and (7) correspond to successful departures of PUs and SUs respectively. Finally, the last terms (8) and (9) are for the cases when the system remains at the same state. The former is for the case when there are idle channels and the latter is for the case when there are no idle channels.

Note that the maximization is over u in the DP equations. Hence, we can rearrange the terms such that the $max\{\cdot\}$ function only includes the terms with the variable u. Then, an alternative expression for the DP equations is the following:
For $n = 0$**:**

$$V_0(x, y) = 0 \qquad \text{for} \quad x, y \geq 0$$

For $n \geq 1$**:**

$$\begin{aligned} V_n(x, y) = \max_{u \in \mathbb{U}} \Big\{ & \lambda_2(u)(V_{n-1}(x, y+1) \\ & - V_{n-1}(x, y) + u)\mathbf{1}\{x + y < C\} \Big\} \\ & + \lambda_1 V_{n-1}(x+1, y)\mathbf{1}\{x + y < C\} \\ & + \lambda_1 (V_{n-1}(x+1, y-1) - K) \\ & \quad \cdot \mathbf{1}\{x + y = C\}\mathbf{1}\{y > 0\} \\ & + \lambda_1 V_{n-1}(x, y)\mathbf{1}\{x + y = C\}\mathbf{1}\{y = 0\} \\ & + x\mu V_{n-1}(x-1, y) \\ & + y\mu V_{n-1}(x, y-1) \\ & + (1 - \lambda_1 - x\mu - y\mu - \alpha)\, V_{n-1}(x, y). \end{aligned} \tag{10}$$

Our analysis is based on the difference between two systems where the first one has one more SU than the second one. The former starts in state $(x, y+1)$ whereas the latter starts in state (x, y) at time period n. $V_n(x, y+1) - V_n(x, y)$ corresponds to the net benefit of an additional SU when there are n periods left in the horizon which is defined as *the value of an additional SU* [31]. The following lemma demonstrates that the value of an additional SU is a function of $(x + y)$ which implies that it is a function of the total number of users in the system.

Lemma IV.1 *The value of an additional SU at time period n is a function of the total occupancy for every (x, y) such that $x + y + 1 \leq C$, i.e.*

$$V_n(x, y+1) - V_n(x, y) \ = f_n(x + y), \tag{11}$$

where $f_n(\cdot)$ is recursively defined for each n as:

$$f_n(k) = \max_{u_1 \in \mathbb{U}} \{ \min_{u_2 \in \mathbb{U}} \{ \tilde{f}_n(k, u_1, u_2) \} \} \quad \textit{and} \quad f_0(\cdot) = 0, \tag{12}$$

and

$$\tilde{f}_n(k, u_1, u_2) = \begin{cases} \lambda_2(u_1)(f_{n-1}(k+1) + u_1) \\ -\lambda_2(u_2)(f_{n-1}(k) + u_2) \\ +\lambda_1 f_{n-1}(k+1) \\ +k\mu f_{n-1}(k-1) \\ +(1 - \lambda_1 - (k+1)\mu - \alpha) f_{n-1}(k), & k < C - 1 \\ \\ -\lambda_2(u_2)(f_{n-1}(C-1) + u_2) \\ +(C-1)\mu f_{n-1}(C-2) \\ +(1 - \lambda_1 - C\mu - \alpha) f_{n-1}(C-1) - K\lambda_1, \\ & k = C - 1. \end{cases}$$

The proof of Lemma IV.1 is included in the Appendix and utilizes an induction technique. The following theorem establishes the relationship between the optimal pricing policy and the total occupancy.

Theorem IV.1 *The optimal price in state (x, y) at time period n depends only on the total number of users in the system, i.e.*

$$u_n^*(x, y) = g_n(x + y), \tag{13}$$

where $g_n(\cdot)$ is recursively defined for each $n > 0$ as:

$$g_n(k) = \operatorname*{argmax}_{u \in \mathbb{U}} \{ \lambda_2(u)(f_n(k) + u) \} \quad \textit{for} \quad 0 \leq k \leq C - 1. \tag{14}$$

Theorem IV.1 provides a drastic simplification in the determination of the optimal pricing policy. In Theorem IV.1, we have proven that the optimal pricing policy depends only on the total occupancy which illustrates an interesting result. The optimal pricing policy is a function of only the total occupancy although the profit function does not depend only on the total occupancy. The reason is that the optimal pricing policy is not determined by the profit function itself; rather it depends on the value of an additional SU. The proof of Theorem IV.1 is given in the Appendix and it directly follows from Lemma IV.1.

The following corollary brings insight to the interpretation of the optimal pricing policy. It points out that the value of an additional SU is the same for the states with identical total number of users if the pricing policy depends only on the total occupancy.

Corollary IV.1 *For all $n \geq 0$ and for all pairs of (x, y) satisfying $x + y + 2 \leq C$:*

$$V_n(x{+}1, y{+}1) - V_n(x{+}1, y) = V_n(x, y{+}2) - V_n(x, y{+}1). \tag{15}$$

Corollary IV.1 follows from Theorem IV.1 and can be obtained by direct substitution of the DP equations and Eq. (11) for $x + y \leq C - 2$.

C. Extension to Infinite Horizon

So far, we have proven our results by working on the finite horizon discounted profit to be able to use induction on n. In this section, we discuss the applicability of our results to the infinite horizon average return problems.

All conclusions apply to the infinite horizon α-discounted case by taking the limit $n \to \infty$. The limiting value exists by Proposition 3.1 in [28, p. 36] because the state-space is countable, the action space is finite, and the absolute values of rewards and costs are bounded. In the induction argument, the results are shown to hold for all values of n in the horizon. Hence, we calculate the infinite horizon α-discounted profit using the relation:

$$V(x, y) = \lim_{n \to \infty} V_n(x, y),$$

where $u^*(x, y)$ is the corresponding optimal decision at state (x, y). Furthermore, the price control space and the state space are finite and state (0,0) is accessible from every other state regardless of the pricing policy. Then, all results also apply to the average return case (see [28], pages 95-98). The average profit can be computed considering the case $\alpha \to 0$. Although the properties of the optimal pricing policy still hold, we need to formulate a new problem structure including relative rewards and an average profit.

D. Infinite Horizon Average Return DP Formulation of the Simplified System

In the previous section, we have shown that the optimal pricing policy depends only on the total occupancy. In order to derive additional properties of the optimal pricing policy, we formulate an infinite horizon average return problem by considering this simplification. We carry out our analysis starting from the continuous-time model.

Although the optimal pricing policy of our system is only determined by the total occupancy, the profit function itself does not depend only on the total occupancy which can be observed from the DP equations. Hence, we cannot completely reduce our system to a one-dimensional (1D) Markov chain. In this section, instead of using the original system, we utilize an *auxiliary system* in the derivations of the infinite horizon average return formulation of the original system. The auxiliary system is chosen such that it has the same optimal pricing policy as the original system. However, both its profit function and optimal pricing policy depend only on the total occupancy which allows to reduce it to a 1D Markov chain.

The model description of the auxiliary system is the same as the original system with one exception: the system imposes a cost K when all channels are busy and a PU arrival occurs, regardless of the presence of SUs in the system. Namely, the auxiliary system is exactly the same as the original system other than the fact that a cost equal to the preemption cost occurs if a PU gets blocked because of other PUs in the system.

The average profit rate of the auxiliary system under the policy $\mathbf{u}$ is denoted $Q_{\mathbf{u}}$ and is the following:

$$\begin{aligned}
Q_{\mathbf{u}} = &\sum_{(x,y)\in S_2} \lambda_2(u(x,y))\, u(x,y)\, \pi_{\mathbf{u}}(x,y) \\
&\quad - K\lambda_1 \sum_{x+y=c} \pi_{\mathbf{u}}(x,y) \\
= &\sum_{(x,y)\in S_2} \lambda_2(u(x,y))\, u(x,y)\, \pi_{\mathbf{u}}(x,y) \\
&\quad - K\lambda_1 \sum_{(x,y)\in S_1} \pi_{\mathbf{u}}(x,y) - K\lambda_1 \pi_{\mathbf{u}}(C,0) \\
= &\sum_{(x,y)\in S_2} \lambda_2(u(x,y))\, u(x,y)\, \pi_{\mathbf{u}}(x,y) \\
&\quad - K\lambda_1 \sum_{(x,y)\in S_1} \pi_{\mathbf{u}}(x,y) - K\lambda_1 E(\lambda_1/\mu, C), \qquad (16)
\end{aligned}$$

where $E(\lambda_1/\mu, C)$ is the blocking probability of a PU, which is given by the well-known *Erlang-B* formula:

$$E(\lambda_1/\mu, C) = \frac{\frac{(\lambda_1/\mu)^C}{C!}}{\sum_{n=0}^{C} \frac{(\lambda_1/\mu)^n}{n!}}. \qquad (17)$$

Since the system is preemptive (i.e., PUs always have higher priority over SUs), a PU is blocked if and only if all the channels are occupied by PUs. Therefore, the blocking probability of PUs does not depend on the pricing policy $\mathbf{u}$ on SUs.

By comparing Eq. (1) and Eq. (16), the relationship between the profit functions $Q_{\mathbf{u}}$ and $J_{\mathbf{u}}$ is given by:

$$J_{\mathbf{u}} = Q_{\mathbf{u}} + K\lambda_1 E(\lambda_1/\mu, C). \qquad (18)$$

Thus, for any policy $\mathbf{u}$, $J_{\mathbf{u}}$ and $Q_{\mathbf{u}}$ differ by the constant $K\lambda_1 E(\lambda_1/\mu, C)$. Consequently, the policy that maximizes $Q_{\mathbf{u}}$ is the same as the policy that maximizes $J_{\mathbf{u}}$ in the average return case.

In the simplified model which considers the total occupancy to determine the optimal policy, we define new system parameters to describe the system. Let $0 \le i \le C$ denote the occupancy levels of the auxiliary system which is the sum of PUs and SUs in the system, i.e. $i = x + y$. Our system parameters are the same as before: We still have Poisson arrivals and exponentially distributed call durations with mean μ^{-1}. Thus, we still consider a continuous-time birth-death Markov Process. The only modification is that we replace the definition of state (x, y) with i.

Prices are chosen from the discrete set $\mathbb{U}$ which is defined earlier. Price advertised to SUs at the total occupancy level i is $u(i)$. The arrival rate of SUs is a function of the price denoted by $\lambda_2(u(i))$. Then, the total arrival rate to any state i is computed as follows:

$$\lambda(u(i)) = \lambda_1 + \lambda_2(u(i)).$$

At this point, we provide an average return DP formulation of the auxiliary system using *Bellman's equations* [4] in order to obtain the optimal price vector $\mathbf{u}^* \triangleq (u^*(0), u^*(1), ..., u^*(C-1))$, which provides the optimal price at each occupancy level of the original system as well.

We model the auxiliary system as a 1D MDP when the total occupancies are considered as the states. The 1D continuous-time Markov chain of the auxiliary system is illustrated in Fig. 1. $Q_{\mathbf{u}}$, the average profit rate of the auxiliary system under policy $\mathbf{u}$, is as follows:

$$Q_{\mathbf{u}} = \sum_{i=0}^{C-1} \lambda_2(u(i))\, u(i)\, \pi_{\mathbf{u}}(i) - K\lambda_1 \pi_{\mathbf{u}}(C). \qquad (19)$$

Under the same optimal pricing policy, the relationship between the optimal average profit functions Q^* and J^* is unchanged and given in Eq. (18).

Next, we formulate the average return DP problem for the auxiliary system. The use of Bellman's equations is possible in this system, since all the states in the Markov chain are recurrent [4]. However, we need to convert the continuous-time Markov chain to its discrete-time equivalent. As before, we use uniformization to achieve it. We normalize every rate coming out of each state by the maximum rate possible, which is $v' = \lambda_1 + \lambda_{2,max} + C\mu$. Without loss of generality, we set $v' = 1$. The corresponding Bellman's equations are as follows:

- $i = 0, 1, ..., C - 1$

$$\begin{aligned}
Q^* + h(i) = \max_{u\in\mathbb{U}} \, &[\lambda_2(u)u + h(i+1)\lambda(u) \qquad (20)\\
&+ h(i-1)i\mu + h(i)(1 - \lambda(u) - i\mu)]
\end{aligned}$$

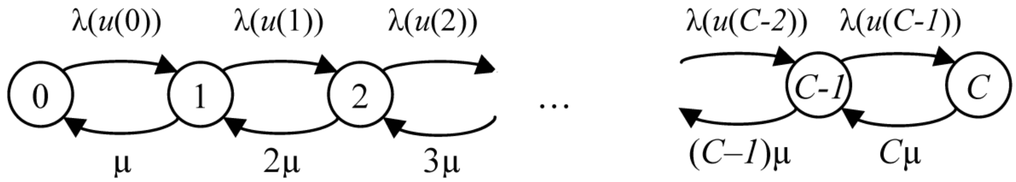

Fig. 1: 1D continuous-time Markov chain representation of the auxiliary system

• $i = C$

$$Q^* + h(C) = -\lambda_1 K + h(C-1)C\mu + h(C)(1 - C\mu). \quad (21)$$

The optimal prices and the optimal average profit are determined by solving the equations (20) and (21). We set $h(C) = 0$, so $h(i)$ is the *relative reward* earned at state i with respect to $h(C)$. Over the infinite horizon, $h(i)$ gives the difference of the total rewards of two processes where the former starts from state i and the latter starts from state C. In practice, the optimal prices can be computed by solving Bellman's equations using policy iteration [4, Vol. II, p. 227]. While the worst-case complexity of policy iteration is $O(|\mathbb{U}|^C)$, this procedure converges quickly even with a large number of channels as shown in Section IV-F.

In Eq. (20), the first term at the right-hand side gives the reward collected upon an SU arrival. The second term corresponds to an arrival from both classes and the third term gives a departure from any class. Finally, the last term is for the cases when the system stays at the same state. The Bellman's equation for the last state is different from the rest because there is no pricing at $i = C$. The first term on the right-hand side of Eq. (21) is the cost incurred when a PU arrives, finds all channels busy and gets blocked. The optimal pricing policy maximizes the right-hand side of Eq. (20).

Using the average profit function we have obtained in this section, we will discuss some monotonicity properties of the system in the next section.

E. Monotonicity Properties of the Optimal Pricing Policy

In this section, we emphasize some monotonicity properties of the simplified system. The proofs of these properties follow similar lines as the works of Paschalidis and Tsitsiklis [25] and Mutlu [20], and are given in the Appendix. The main differences are that our system is preemptive and has a different cost structure.

The following two lemmas demonstrate that the relative reward function is a decreasing and concave in the total occupancy.

Lemma IV.2 *As total occupancy increases, the value of relative reward decreases i.e.,*

$$h(i+1) - h(i) \leq 0 \quad \text{for} \quad 0 \leq i < C.$$

Lemma IV.3 *The relative rewards are concave in occupancy i.e.,*

$$h(i) - h(i-1) \geq h(i+1) - h(i) \quad \text{for} \quad 0 < i < C.$$

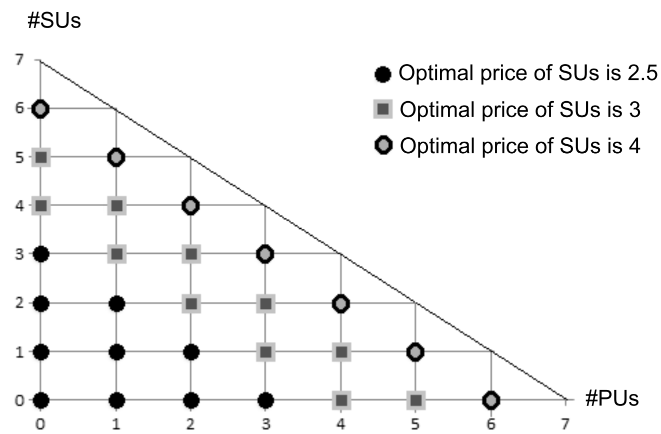

Fig. 2: The optimal pricing policy of SUs in the system of Example IV.1, with inelastic PUs. The advertised price is constant on diagonal states on which the total occupancy is the same.

Theorem IV.2 *As total occupancy increases, the optimal prices increase as well i.e.,*

$$u^*(i+1) \geq u^*(i) \quad \text{for} \quad 0 \leq i < C-1.$$

Now that we have specified the properties of the optimal pricing policy, we present an example to illustrate the results of Theorem IV.1 and Theorem IV.2.

Example IV.1 *In this example, we set $C = 7$, $\mu = 1$, $\lambda_1 = 3$, $K = 10$, $\Delta u = 0.5$, $\mathbb{U} = [0, 4]$ and the demand function is $\lambda_2(u) = (4-u)_+$ where $(\cdot)_+ = max(\cdot, 0)$. The resulting optimal pricing policy is illustrated in Fig. 2. We observe that the states with the same occupancy have the same optimal pricing decision. Furthermore, the optimal prices increase with the total occupancy.*

F. Numerical Results

We present numerical results to illustrate the performance of the optimal pricing policy, in terms of average profit and running times, in systems with large number of channels and large sets of possible prices. We provide a comparison with the optimal static pricing policy (i.e., the same price is advertised regardless of the occupancy state of the system), which is known to perform well in systems where all classes

C	t_{run}^{OP} (sec)	t_{run}^{SP} (sec)	J_{OP}	J_{SP}
100	0.302	0.001	9.671	3.518
150	0.545	0.001	18.209	14.632
200	0.603	0.001	22.583	21.365
250	0.753	0.001	24.243	23.921

TABLE I: Running times of the optimal pricing policy t_{run}^{OP} and of the optimal static pricing policy t_{run}^{SP}, and average profits of the optimal pricing policy J_{OP} and of the optimal static pricing policy J_{SP}. System parameters: $\lambda_1 = 0.78C$, $K = 100$, $\lambda_2(u) = (10 - u)_+$ for $u \geq 0$ and $\Delta u = 10^{-4}$.

Δu	t_{run}^{OP} (sec)	t_{run}^{SP} (sec)	J_{OP}	J_{SP}
0.0001	0.5508	0.0010	18.2090	14.6327
0.002	0.3777	0.0009	18.2090	14.6327
0.001	0.2478	0.0006	18.2090	14.6327
0.02	0.2274	0.0006	18.2090	14.6327
0.01	0.1534	0.0006	17.8833	14.6327

TABLE II: Running times and average profits of the optimal pricing policy and the optimal static pricing policy for different price granularity Δu and $C = 150$ channels. Other system parameters are as in Table I.

of users are price-sensitive [25]. The code for both policies is implemented in MATLAB and run on an Intel Core i7 PC, with a processor speed of 2.30 GHz, 8 GB of RAM, and 64-bit Windows 10 OS.

Table I shows results for a secondary demand that is linearly decreasing with price. The set of possible prices $\mathbb{U}$ ranges from 0 to 10 with a granularity of $\Delta u = 10^{-4}$. Hence, the cardinality of the set of prices is $|\mathbb{U}| = 10^5+1$. The table shows that the optimal pricing policy can be efficiently computed (i.e., within less than one second even for $C = 250$ channels). While the optimal static pricing policy can be computed faster, the average profit of the optimal pricing policy is significantly higher in certain cases. For instance, with $C = 100$ channels, the average profit of the optimal pricing policy is about three times higher than that of the optimal static pricing policy.

Table II shows the effect of changing the price granularity, for $C = 150$ channels. We observe that using lower granularity yields the same profit performance (but faster execution time) up to $\Delta u = 10^{-2}$, at which point the profit of the optimal pricing policy starts to degrade.

V. Special Cases

In this section, we examine some special cases of the optimal pricing policy we have characterized. First, we extend our results to the optimal admission control of SUs in a two-class network. Next, we consider optimal pricing with QoS-sensitive SUs and generalize our results to such settings. Last, we show that if both PUs and SUs are price-sensitive, then the structure of the optimal pricing policy changes.

A. Optimal Admission Control Policy of SUs

In admission control, the SP either accepts a user upon arrival or rejects it by following an admission control policy $\mathbf{a}$. If an SU call is accepted, a constant reward of $R > 0$ is collected per call and the corresponding arrival rate is $\lambda_2(R)$. If a call is rejected, the SP sets the price to u_{max}, hence the arrival rate of SUs $\lambda_2(u_{max})$ drops to 0. We consider admission control as a special case of pricing which includes two possible prices in the control set. There is a mapping between the pricing decision and admission control decision at state (x, y). We define a new control parameter $a(x, y)$ which is a restricted binary version of $u(x, y)$ (i.e. $a(x, y) \in \{0, 1\}$). $a(x, y)$ is the admission control decision of an arriving SU when the current state is (x, y). The admission control set is the following:

$$a(x,y) = \begin{cases} 1 & \text{for} \quad \text{accept with reward } R \\ 0 & \text{for} \quad \text{reject with reward } 0. \end{cases} \tag{22}$$

The corresponding demand function $\lambda_2(a(x, y))$ is defined as follows:

$$\lambda_2(a(x,y)) = \begin{cases} \lambda_2(R) & \text{for} \quad a = 1 \\ 0 & \text{for} \quad a = 0. \end{cases} \tag{23}$$

The following corollary applies theorems IV.1 and IV.2 to the optimal admission control problem.

Corollary V.1 *The optimal admission control policy* $\mathbf{a}^*$ *of SUs is of threshold type and it depends only on the total number of users in the system. Thus, there exists an optimal occupancy threshold* T^* *for a system at state* (x, y) *such that* $0 \leq x + y < T^*$*. If the total occupancy is less than* T^**, an incoming SU call is accepted. Otherwise, it is rejected.*

Note that the same result is obtained in the work of Turhan et al. [30] and we have corroborated its results by using a different technique.

B. Optimal Pricing with QoS-sensitive SUs

The secondary demand may not only affected by the advertised price, but also by the perceived quality of service (QoS). In particular, if SUs experience a high level of forced termination, then their demand may decrease. We extend here our results to handle such cases.

Denote by P_T the probability that an SU call terminates prematurely due to preemption, and assume the secondary demand takes the form $\lambda_2(u, P_T)$. We next establish an iterative prcoedure to compute P_T and the optimal pricing policy $\mathbf{u}^*$. We use the same notation as that introduced in Section IV-D.

The forced termination probability P_T corresponds to the ratio of the average rate at which SU calls are preempted to the average rate at which SU calls enter the system. The average rate of SU preemptions is given by the product of the arrival rate of PUs λ_1 and the probability that the system is full and contains at least one SU, which is $\pi_{\mathbf{u}}(C) - E(\lambda_1/\mu, C)$ (recall that $\pi_{\mathbf{u}}(C)$ is the probability that all C channels are occupied while $E(\lambda_1/\mu, C)$ is the probability that all C channels are occupied by PUs only).

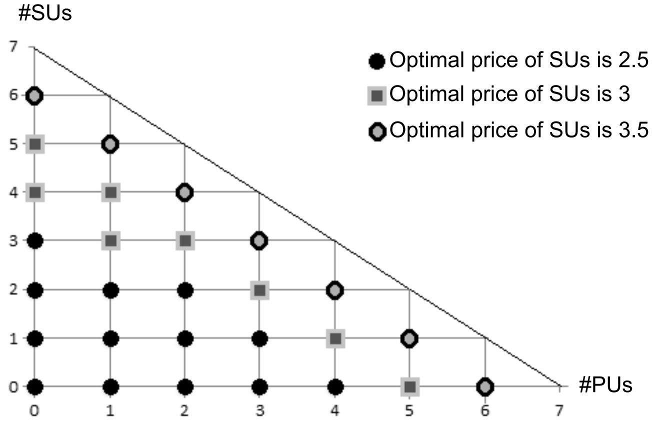

Fig. 3: The optimal pricing policy of SUs in the system of Example V.2, with price-sensitive PUs. The advertised price is not constant on the diagonal states.

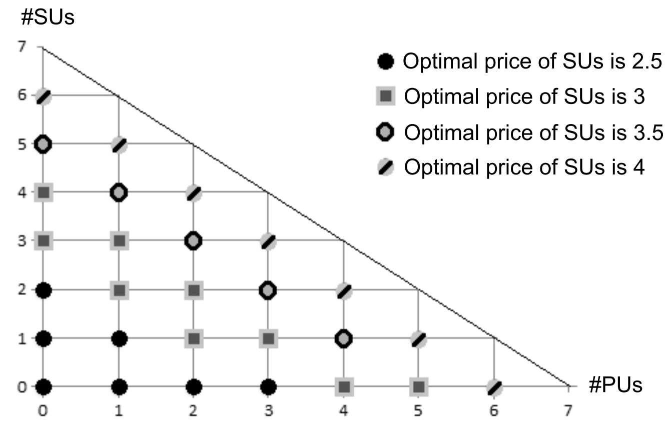

Fig. 4: The optimal pricing policy of SUs in the system of Example V.3

The average rate at which SU calls enter the system is $\sum_{i=0}^{C-1} \lambda_2(u(i), P_T)\pi_{\mathbf{u}}(i)$. Hence,

$$P_T = \frac{\lambda_1(\pi_{\mathbf{u}}(C) - E(\lambda_1/\mu, C))}{\sum_{i=0}^{C-1} \lambda_2(u(i), P_T)\pi_{\mathbf{u}}(i)}. \tag{24}$$

Equation (24) can be solved using successive substitutions:

1) Start with an initial estimate of P_T, say $P_T = 0$.
2) Fix the secondary demand $\lambda_2(u, P_T)$.
3) Compute the optimal pricing policy $\mathbf{u}^*$ by solving Bellman's equations (20) and (21).
4) Determine the stationary probabilities $\pi_{\mathbf{u}^*}(i)$ of the 1D Markov chain (see Fig. 1).
5) Use the right-hand side of Equation (24) to obtain a new estimate of P_T.
6) Return to 2) and repeat the procedure until the difference between the last two estimates of P_T is smaller than some accuracy parameter ϵ.

Example V.1 *Let $C = 7$, $\mu = 1$, $\lambda_1 = 3$, $K = 10$, $\Delta u = 0.5$, $\mathbb{U} = [0, 4]$ and the demand function be $\lambda_2(u) = (4-u)_+ \times (1-\alpha P_T)_+$ where $\alpha = 10$ and $(\cdot)_+ = max(\cdot, 0)$. Using the iterative procedure with $\epsilon = 10^{-6}$, we get $\mathbf{u}^* = (2, 2, 2.5, 2.5, 2.5, 3, 4)$ and $P_T = 0.0731$. The procedure converges after 10 iterations.*

C. Discussion of the Optimal Pricing with Price-sensitive PUs

Assume that every definition and parameter in the original system model stays the same except the PU arrival rates. In the original model, PUs have inelastic Poisson arrival rates with mean λ_1. We consider a variant of the original system where the PUs are sensitive to price and a price $\hat{u}(x, y)$ is advertised to PUs at state (x, y). Furthermore, the PUs have a demand function $\lambda_1(\hat{u})$, which follows Assumptions 1 and 2. We aim to determine the optimal pricing policies of both PUs and SUs that maximize the overall profit by solving a joint maximization problem. That is, at every state (x, y), there exists an optimal price for SUs and another optimal price for PUs. The next example demonstrates that Theorem IV.1 does not anymore apply to the optimal pricing policy of SUs.

Example V.2 *Set $C = 7$, $\mu = 1$, $K = 7$, $\Delta u = 0.5$, $\Delta \hat{u} = 1$ and the demand functions for PUs and SUs are $\lambda_1(\hat{u}) = (10-\hat{u})_+$ and $\lambda_2(u) = (4-u)_+$. The resulting optimal pricing policy is demonstrated in Fig. 3. For $x + y = 4$, we have different optimal pricing decisions for different states.*

Next, consider a system where the price optimization is conducted separately for PUs and SUs. That is, we first optimize the prices of PUs ignoring the existence of SUs. We formulate a 1D profit maximization DP problem for a system that includes only PUs similar to the one class profit maximization problem introduced by Paschalidis and Tsitsiklis [25]. We denote the optimal pricing policy of PUs in this system by $\tilde{\mathbf{u}}^* \triangleq (\tilde{u}^*(0), \tilde{u}^*(1), ..., \tilde{u}^*(C-1))$, which maximizes the long-run average profit of the system obtained solely from the PUs. Similar to the original problem, we substitute the optimal PU arrival rates corresponding to the optimal prices of PUs to the 2D Markov chain and determine the optimal pricing policy of SUs. Note that, in the original problem, the arrival rate of PUs at every total occupancy level is constant whereas for this case, the demand varies with the total occupancy level. The aim is to observe whether the optimal pricing policy of SUs obeys Theorem IV.1 and IV.2. Example V.3 demonstrates that even in that case, the optimal pricing policy of SUs is not a function of total occupancy and Theorem IV.1 does not apply.

Example V.3 *We have $C = 7$, $\mu = 1$, $K = 7$, $\Delta u = 0.5$, $\Delta \tilde{u} = 1$ and the demand functions for PUs and SUs are $\lambda_1(\tilde{u}) = (10 - \tilde{u})_+$ and $\lambda_2(u) = (4 - u)_+$. The optimal pricing policy of PUs is $\tilde{\mathbf{u}}^* = (5, 5, 5, 5, 4, 4, 3)$. Each entry in $\tilde{\mathbf{u}}^*$ corresponds to the optimal price of PUs for a given* *total occupancy level. The resulting optimal pricing policy of SUs after substituting $\lambda_1(\tilde{u}^*)$ values is demonstrated in Fig. 4. For $x+y=3$ and $x+y=5$, we have different optimal pricing decisions for different states.*

VI. Optimal Pricing in Non-preemptive Systems

In this section, by applying the techniques developed for the preemptive loss systems, we formally prove that the optimal admission control and pricing policies in non-preemptive loss systems are the same as the ones in a preemptive loss system. In fact, Ramjee et al. [27] and Mutlu et al. [21] assume that the total occupancy is sufficient to describe the state of the system instead of keeping track of each class separately. Specifically, Ramjee et al. [27] includes a footnote stating "One could possibly enhance the state description by keeping track of new calls and handoff calls separately, rather than the total occupancy alone. However, this new state descriptor is not expected to change any of the conclusions of the paper given the memoryless nature of the arrival process."

We next explain how to obtain such structural results for optimal pricing in a non-preemptive loss system, using the model introduced in Mutlu et al. [21]. This model considers a loss network with the same arrivals and capacity constraint as described in this paper, however the nature of the loss system is *non-preemptive*. Under a non-preemptive system, a PU arrival no longer causes the provider to drop a SU from the system. Rather, when a PU arrives and finds the system full, it is blocked. The model of Mutlu et al. differs from our paper in that it has no associated preemption cost but instead imposes a cost K on the provider per each blocked PU. Under the described model, the DP optimality equations for the finite horizon discounted problem become:

For $n=0$:

$$V_0(x,y)=0 \qquad \text{for} \quad x,y\geq 0.$$

For $n\geq 1$:

$$\begin{aligned} V_n(x,y) = \max_{u\in\mathbb{U}} \Big\{ & \lambda_1 V_{n-1}(x+1,y)\mathbf{1}\{x+y<C\} \\ & + \lambda_1(V_{n-1}(x,y)-K)\mathbf{1}\{x+y=C\} \\ & + \lambda_2(u)(V_{n-1}(x,y+1)+u)\mathbf{1}\{x+y<C\} \\ & + x\mu V_{n-1}(x-1,y) \\ & + y\mu V_{n-1}(x,y-1) \\ & + (1-\lambda_1-\lambda_2(u)-x\mu-y\mu-\alpha)\, V_{n-1}(x,y) \\ & \quad \cdot \mathbf{1}\{x+y<C\} \\ & + (1-\lambda_1-C\mu-\alpha)\, V_{n-1}(x,y)\mathbf{1}\{x+y=C\} \Big\}. \end{aligned} \tag{25}$$

Compared to the formulation for the preemptive case, the only difference is the replacement of Eq. (3) and Eq. (4) by Eq. (25). As one can readily observe from above, $V_n(x,y)$ depends only on the total occupancy $x+y$, a result which we formalize with the following lemma:

Lemma VI.1 *In a non-preemptive loss system, the value of $V_n(x,y)$ only depends on the total occupancy such that:*

$$V_n(x,y)=q_n(x+y), \tag{26}$$

where $q_n(\cdot)$ is recursively defined for each n as:

$$q_n(k)=\max_{u\in\mathbb{U}}\{\tilde{q}_n(k,u)\} \quad \textit{and} \quad q_0(\cdot)=0,$$

and

$$\tilde{q}_n(k,u)=\left\{ \begin{array}{l} \lambda_1 q_{n-1}(k+1)\mathbf{1}\{k<C\} \\ +\lambda_1(q_{n-1}(k)-K)\mathbf{1}\{k=C\} \\ +\lambda_2(u)(q_{n-1}(k+1)+u)\mathbf{1}\{k<C\} \\ +k\mu q_{n-1}(k-1) \\ +\left(1-\lambda_1-\lambda_2(u)-k\mu-\alpha\right) q_{n-1}(k)\mathbf{1}\{k<C\} \\ +\left(1-\lambda_1-C\mu-\alpha\right) q_{n-1}(k)\mathbf{1}\{k=C\}, \end{array}\right.$$

The proof of Lemma VI.1 is included in the Appendix.

In the next theorem, we utilize the result of Lemma VI.1 to demonstrate that the optimal pricing policy that maximizes the average profit rate of the non-preemptive system is the same as the pricing policy that maximizes profit rate of the 1D auxillary system described in Section IV-D.

Theorem VI.1 *The optimal policy $\mathbf{u}^*$ that maximizes the average profit for a preemptive loss system is also optimal for a non-preemptive loss system.*

The proof of Theorem VI.1 is provided in the Appendix. Having proven that the optimal pricing policy $\mathbf{u}^*$ is the same for preemptive and non-preemptive loss systems, one can readily extend the results obtained in Mutlu et al. [21] and Mutlu et al. [22] for occupancy-based pricing policies in non-preemptive loss systems to preemptive systems.

Specifically, in the work by Mutlu et al. [21], it was demonstrated that a *threshold*-type pricing policy performs very close to the optimal pricing policy u^*. The main advantage of the threshold pricing policy is that it is sufficiently described by only two parameters: a threshold value and a single price. Building on this threshold type policy, Mutlu et al. [22] provides an online dynamic measurement algorithm that helps to determine the optimal threshold and price values. Since Theorem VI.1 states that the optimal pricing policy is the same for preemptive and non-preemptive systems, the threshold pricing policy and the applicable online measurement techniques described in Mutlu et al. [21] and Mutlu et al. [22] can also be applied to preemptive loss models.

VII. Conclusions

In a system of inelastic PUs and price-sensitive SUs, we analytically proved that the optimal pricing policy of SUs depends only on the total number of users in the system. The technique to prove this result, stated in Lemma IV.1, is novel and of independent interest in solving other types of dynamic programming problems.

We discussed how our results generalize previous ones from Turhan et al. [30] on optimal admission control in preemptive systems and carry over to non-preemptive loss systems. Yet, we also showed that our structural results do not carry over for systems with price-sensitive PUs. Our results have a wide range of applications in both revenue management and admission control, not necessarily restricted to that of bandwidth allocation in wireless networks.

As future work, it should be possible to extend our results beyond Poisson arrivals. For example, the work of Ali et al. [2] demonstrates that state-dependent threshold control is also optimal in loss networks with arrivals generated by a finite number of sources (also known as the Engset Model) to which we expect our pricing policies can be applied. Other interesting open problems include analyzing systems where SUs are queued rather than evicted, considering call length distributions of PUs and SUs that are not necessarily exponential and identical, and extending the results to multi-hop configurations with spatial reuse.

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References

[1] I.F. Akyildiz, W.L. Lee, M.C. Vuran, and S. Mohanty. Next generation/dynamic spectrum access/cognitive radio wireless networks: A survey. *Elsevier Computer Networks Journal*, 50:2127–2159, 2006.

[2] Ahsan-Abbas Ali, Shuangqing Wei, and Lijun Qian. Optimal admission and preemption control in finite-source loss systems. *Operations Research Letters*, 43(3):241–246, 2015.

[3] Eitan Altman, Tania Jiménez, and Ger Koole. On optimal call admission control in resource-sharing system. *Communications, IEEE Transactions on*, 49(9):1659–1668, 2001.

[4] D.P. Bertsekas. *Dynamic Programming and Optimal Control*. Athena Scientific, third edition, 2007.

[5] A.J.F.G. Brouns. *Queueing models with admission and termination control - Monotonicity and threshold results*. PhD thesis, Eindhoven University of Technology, 2003.

[6] A.J.F.G. Brouns and J. van der Wal. Optimal threshold policies in a two-class preemptive priority queue with admission and termination control. *Queueing Systems*, 54:21–33, 2006.

[7] Hong Chen and Murray Z Frank. State dependent pricing with a queue. *IIE Transactions*, 33(10):847–860, 2001.

[8] Ericsson. Ericsson predicts mobile data traffic to grow 10-fold by 2016. http://www.ericsson.com/news/1561267, 2011.

[9] FCC. Spectrum leasing. http://wireless.fcc.gov/licensing/index.htm?job=spectrum_leasing, 2010.

[10] Eugene A Feinberg and Fenghsu Yang. Optimality of trunk reservation for an m/m/k/n queue with several customer types and holding costs. *Probability in the Engineering and Informational Sciences*, 25(04):537–560, 2011.

[11] N. Gans and S.V. Savin. Pricing and capacity rationing for rentals with uncertain durations. *Management Science*, 53:390–407, 2007.

[12] J.A. Garay and I.S. Gopal. Call preemption in communication networks. In *IEEE International Conference on Computer Communications*, volume 3 of *INFOCOM 1992*, pages 1043–1050, 1992.

[13] S. Geirhofer, L. Tong, and B. M. Sadler. Cognitive radios for dynamic spectrum access - dynamic spectrum access in the time domain: Modeling and exploiting white space. *IEEE Communications Magazine*, 45(5):66–72, May 2007.

[14] Avi Giloni, Yaşar Levent Koçağa, and Phil Troy. State dependent pricing policies: Differentiating customers through valuations and waiting costs. *Journal of Revenue & Pricing Management*, 12(2):139–161, 2013.

[15] Andrea Goldsmith, Syed Ali Jafar, Ivana Marić, and Sudhir Srinivasa. Breaking spectrum gridlock with cognitive radios: An information theoretic perspective. *Proceedings of the IEEE*, 97(5):894–914, 2009.

[16] S. Haykin. Cognitive radio: Brain-empowered wireless communications. *IEEE Journal on Selected Areas in Communications*, 23(2):201–220, 2005.

[17] W. Helly. Two doctrines for the handling of two-priority traffic by a group of n servers. *Operations Research*, 10(2):268–269, 1962.

[18] S.A. Lippman. Applying a new device in the optimization of exponential queueing systems. *Operations Research*, 23:687–710, 1975.

[19] David W Low. Optimal dynamic pricing policies for an m/m/s queue. *Operations Research*, 22(3):545–561, 1974.

[20] H. Mutlu. *Spot Pricing of Secondary Access to Wireless Spectrum*. PhD thesis, Boston University, 2010.

[21] H. Mutlu, M. Alanyali, and D. Starobinski. Spot pricing of secondary spectrum access in wireless cellular networks. *IEEE/ACM Transactions on Networking*, 17:1794–1804, 2009.

[22] H. Mutlu, M. Alanyali, D. Starobinski, and Turhan A. On-line pricing of secondary spectrum access with unknown demand function. *IEEE Journal on Selected Areas in Communications*, 30:2285–2294, 2012.

[23] Pinhas Naor. The regulation of queue size by levying tolls. *Econometrica: journal of the Econometric Society*, pages 15–24, 1969.

[24] D. Pacheco-Paramo, V. Pla, and J. Martinez-Bauset. Optimal admission control in cognitive radio networks. In *Cognitive Radio Oriented Wireless Networks and Communications, 2009. CROWNCOM '09. 4th International Conference on*, pages 1–7, June 2009.

[25] I.C. Paschalidis and J.N. Tsitsiklis. Congestion-dependent pricing of network services. *IEEE/ACM Transactions on Networking*, 8(2):171–184, 2000.

[26] J.M. Peha. Approaches to spectrum sharing. *IEEE Communications Magazine*, 43(2):10–12, 2005.

[27] R. Ramjee, D.F. Towsley, and R. Nagarajan. On optimal call admission control in cellular networks. *Wireless Networks*, 3:29–41, 1996.

[28] S.M. Ross. *Introduction to Stochastic Dynamic Programming*. Academic Press, 1983.

[29] P. Tortelier, S. Grimoud, P. Cordier, S. B. Jemaa, L. Iacobelli, C. Le Martret, J. van de Beek, S. Subramani, W. H. Chin, M. Sooriyabandara, L. Gavrilovska, V. Atanasovski, P. Mahonen, and J. Riihijarvi. Flexible and spectrum aware radio access through measurements and modelling in cognitive radio systems. Faramir Tech. Rep. 248351, Faramir Project, 2010.

[30] A. Turhan, M. Alanyali, and D. Starobinski. Optimal admission control in two-class preemptive loss systems. *Operations Research Letters*, 40:510–515, 2012.

[31] M.Y. Ulukus, R. Gullu, and L. Ormeci. Admission and termination control of a two class loss system. *Stochastic Models*, 27(1):2–25, 2011.

[32] J. Walrand. *An Introduction to Queueing Networks*. Prentice Hall, 1988.

[33] WSJ. Verizon in spectrum spat. http://online.wsj.com/article/SB10001424052970203960804577239083364357176.html, 2012.

[34] S.H. Xu and J.G. Shanthikumar. Optimal expulsion control - a dual approach to admission control of an ordered-entry system. *Operations Research*, 41:1137–1152, 1993.

[35] Utku Yildirim and John J Hasenbein. Admission control and pricing in a queue with batch arrivals. *Operations Research Letters*, 38(5):427–431, 2010.

[36] Z. Zhao, S. Weber, and J.C. Oliveira. Preemption rates for a parallel link loss network. *Performance Evaluation*, 66(1):21–46, 2006.

[37] X. Zhu, L. Shen, and T. s. P. Yum. Analysis of cognitive radio spectrum access with optimal channel reservation. *IEEE Communications Letters*, 11(4):304–306, April 2007.

APPENDIX

Proof of Lemma IV.1. We prove this result by induction on n. Although the DP equations of $V_n(x, y)$ depend on the value of y due to the indicator functions when $x+y=C$, we prove that this is not the case for the value of an additional SU. We start the induction from the end of the horizon $n=0$, i.e. $V_0(x, y+1) - V_0(x, y) = 0$. Thus, Lemma IV.1 holds for $n=0$ by definition.

Induction step: Assume that for any $n > 0$ the following holds:

$$\begin{aligned} V_n(x, y+1) - V_n(x, y) &= f_n(x+y) \\ &= \max_{u_1 \in \mathbb{U}} \{ \min_{u_2 \in \mathbb{U}} \{ \tilde{f}_n(x+y, u_1, u_2) \} \}. \end{aligned} \tag{27}$$

We show that the value of an additional SU at time period $n+1$ is a function of $(x+y)$ only as well, i.e.

$$\begin{aligned} &V_{n+1}(x, y+1) - V_{n+1}(x, y) \\ &= f_{n+1}(x+y) = \max_{u_1 \in \mathbb{U}} \{ \min_{u_2 \in \mathbb{U}} \{ \tilde{f}_{n+1}(x+y, u_1, u_2) \} \}. \end{aligned} \tag{28}$$

We need to consider two distinct cases. In the first case, $x+y < C-1$ hence, both $(x, y+1)$ and (x, y) are non-preemptive states as there are idle channels. In the second case, we consider $x+y = C-1$ where $(x, y+1)$ is a preemptive state since all channels are busy and there is at least one SU in the system. However, (x, y) is non-preemptive again as the system has an idle channel. We analyze these cases separately because the corresponding DP equations are different.

<u>Case 1.</u> $x+y < C-1$

$$\begin{aligned} &V_{n+1}(x, y+1) - V_{n+1}(x, y) \\ &= \max_{u_1 \in \mathbb{U}} \left\{ \lambda_2(u_1)(V_n(x, y+2) - V_n(x, y+1) + u_1) \right\} \\ &+ \lambda_1 V_n(x+1, y+1) \\ &+ x\mu V_n(x-1, y+1) \\ &+ (y+1)\mu V_n(x, y) \\ &+ (1 - \lambda_1 - x\mu - (y+1)\mu - \alpha) V_n(x, y+1) \\ &- \max_{u_2 \in \mathbb{U}} \left\{ \lambda_2(u_2)(V_n(x, y+1) - V_n(x, y) + u_2) \right\} \\ &- \lambda_1 V_n(x+1, y) \\ &- x\mu V_n(x-1, y) \\ &- y\mu V_n(x, y-1) \\ &- (1 - \lambda_1 - x\mu - y\mu - \alpha) V_n(x, y). \end{aligned}$$

We rearrange the terms such that the terms with the same factor are grouped together which yields the following expression for $V_{n+1}(x, y+1) - V_{n+1}(x, y)$:

$$\begin{aligned} &V_{n+1}(x, y+1) - V_{n+1}(x, y) \\ &= \max_{u_1 \in \mathbb{U}} \Big\{ \min_{u_2 \in \mathbb{U}} \big\{ \lambda_2(u_1)(V_n(x, y+2) - V_n(x, y+1) + u_1) \\ &- \lambda_2(u_2)(V_n(x, y+1) - V_n(x, y) + u_2) \\ &+ \lambda_1(V_n(x+1, y+1) - V_n(x+1, y)) \\ &+ x\mu(V_n(x-1, y+1) - V_n(x-1, y)) \\ &- (y+1)\mu(V_n(x, y+1) - V_n(x, y)) \\ &+ y\mu(V_n(x, y) - V_n(x, y-1)) \\ &+ (1 - \lambda_1 - x\mu - \alpha)(V_n(x, y+1) - V_n(x, y)) \big\} \Big\}. \end{aligned}$$

We substitute Eq. (27) to the above expression and we obtain the following:

$$\begin{aligned} &V_{n+1}(x, y+1) - V_{n+1}(x, y) \\ &= \max_{u_1 \in \mathbb{U}} \Big\{ \min_{u_2 \in \mathbb{U}} \big\{ \lambda_2(u_1)(f_n(x+y+1) + u_1) \\ &- \lambda_2(u_2)(f_n(x+y) + u_2) \\ &+ \lambda_1 f_n(x+y+1) \\ &+ x\mu f_n(x+y-1) && (29) \\ &- (y+1)\mu f_n(x+y) && (30) \\ &+ y\mu f_n(x+y-1) && (31) \\ &+ (1 - \lambda_1 - x\mu - \alpha) f_n(x+y) \big\} \Big\}. && (32) \end{aligned}$$

First, we merge terms (29) and (31) into a single term. Then, we repeat the same procedure for the terms (30) and (32).

$$\begin{aligned} &V_{n+1}(x, y+1) - V_{n+1}(x, y) \\ &= \max_{u_1 \in \mathbb{U}} \Big\{ \min_{u_2 \in \mathbb{U}} \big\{ \lambda_2(u_1)(f_n(x+y+1) + u_1) \\ &- \lambda_2(u_2)(f_n(x+y) + u_2) \\ &+ \lambda_1 f_n(x+y+1) \\ &+ (x+y)\mu f_n(x+y-1) \\ &+ (1 - \lambda_1 - (x+y+1)\mu - \alpha) f_n(x+y) \big\} \Big\} \\ &= \max_{u_1 \in \mathbb{U}} \Big\{ \min_{u_2 \in \mathbb{U}} \big\{ \tilde{f}_{n+1}(x+y, u_1, u_2) \big\} \Big\} \\ &= f_{n+1}(x+y), \end{aligned}$$

which proves the induction for this case.

<u>Case 2.</u> $x+y = C-1$

$$\begin{aligned} &V_{n+1}(x, y+1) - V_{n+1}(x, y) \\ &= \lambda_1(V_n(x+1, y) - K) \\ &+ x\mu V_n(x-1, y+1) \\ &+ (y+1)\mu V_n(x, y) \\ &+ (1 - \lambda_1 - x\mu - (y+1)\mu - \alpha) V_n(x, y+1) \\ &- \max_{u_2 \in \mathbb{U}} \left\{ \lambda_2(u_2)(V_n(x, y+1) - V_n(x, y) + u_2) \right\} \\ &- \lambda_1 V_n(x+1, y) \\ &- x\mu V_n(x-1, y) \\ &- y\mu V_n(x, y-1) \\ &- (1 - \lambda_1 - x\mu - y\mu - \alpha) V_n(x, y). \end{aligned}$$

Similar to Case 1, we rearrange the terms which results in the following expression for $V_{n+1}(x, y+1) - V_{n+1}(x, y)$:

$$\begin{aligned}
&V_{n+1}(x, y+1) - V_{n+1}(x, y) \\
&= \min_{u_2 \in \mathbb{U}} \big\{ -\lambda_2(u_2)(V_n(x, y+1) - V_n(x, y) + u_2) - K\lambda_1 \\
&+ \lambda_1(V_n(x+1, y) - V_n(x+1, y)) \qquad (33)\\
&+ x\mu(V_n(x-1, y+1) - V_n(x-1, y)) \\
&- (y+1)\mu(V_n(x, y+1) - V_n(x, y)) \\
&+ y\mu(V_n(x, y) - V_n(x, y-1)) \\
&+ (1 - \lambda_1 - x\mu - \alpha)\,(V_n(x, y+1) - V_n(x, y))\big\}.
\end{aligned}$$

Note that term (33) is zero and vanishes. We substitute Eq. (27) to the above expression.

$$\begin{aligned}
&V_{n+1}(x, y+1) - V_{n+1}(x, y) \\
&= \min_{u_2 \in \mathbb{U}} \big\{ -\lambda_2(u_2)(f_n(x+y) + u_2) - K\lambda_1 \\
&+ x\mu f_n(x+y-1) \qquad (34)\\
&- (y+1)\mu f_n(x+y) \qquad (35)\\
&+ y\mu f_n(x+y-1) \qquad (36)\\
&+ (1 - \lambda_1 - x\mu - \alpha)\, f_n(x+y)\big\}. \qquad (37)
\end{aligned}$$

We first merge terms (34) and (36) into a single term and repeat it for the pair (35) and (37). Finally, we substitute $x + y = C - 1$ which is already given for Case 2. We obtain the following expression for $V_{n+1}(x, y+1) - V_{n+1}(x, y)$:

$$\begin{aligned}
&V_{n+1}(x, y+1) - V_{n+1}(x, y) \\
&= \min_{u_2 \in \mathbb{U}} \big\{ -\lambda_2(u_2)(f_n(C-1) + u_2) - K\lambda_1 \\
&+ (C-1)\mu f_n(C-2) \\
&+ (1 - \lambda_1 - C\mu - \alpha)\, f_n(C-1)\big\} \\
&= \min_{u_2 \in \mathbb{U}} \big\{\tilde{f}_{n+1}(C-1, u_1, u_2)\big\} \\
&= f_{n+1}(C-1),
\end{aligned}$$

which concludes the proof of Case 2.

Combining the two cases we have examined, the induction hypothesis given in Eq. (28) is correct and we have proven that $V_n(x, y+1) - V_n(x, y)$ depends only on $(x + y)$ for all values of n. □

Proof of Theorem IV.1. The optimal price $u_n^*(x, y)$ maximizes the right-hand side of the DP equations. If we discard the terms that do not include the price variable u, we deduce that $u_n^*(x, y)$ maximizes only the term given in Eq. (10), i.e. for $x + y \leq C - 1$ we have the following:

$$u_n^*(x, y) = \operatorname*{argmax}_{u \in \mathbb{U}} \left\{ \lambda_2(u)(V_{n-1}(x, y+1) - V_{n-1}(x, y) + u) \right\}. \qquad (38)$$

From Lemma IV.1, we know that $V_{n-1}(x, y+1) - V_{n-1}(x, y) = f_n(x + y)$. When we substitute it to Eq. (38), we obtain the following:

$$\begin{aligned}
u_n^*(x, y) &= \operatorname*{argmax}_{u \in \mathbb{U}} \left\{ \lambda_2(u)(f_n(x+y) + u) \right\} \\
&= g_n(x+y),
\end{aligned}$$

which gives the desired result. □

Proof of Lemma IV.2. Let System A and System B be two coupled systems except for an additional user in System A. System A starts from state $i + 1$ and System B starts from state i. Statistically, all inter-arrival times and arrival processes are the same for those two systems. Assume that all actual arrivals are identical and common users other than the extra user in System A depart at the same time instants. At each time instant, System B follows the optimal pricing policy and System A imitates all actions of System B. Under this setting, we consider two cases: First, the additional user in System A successfully leaves the system before the total number of users in System A reaches C. Then, System A and System B have the same reward. Second, System A has C users and System B has $C - 1$ users. Then, either a PU or an SU arrival occurs at both systems. If an SU arrival occurs, System A blocks the incoming user whereas System B accepts the user by earning a reward. System A has less reward than System B for this case. If a PU arrival occurs, System A blocks the incoming user with a cost or preempts an SU if there exist any. On the other hand, System B accepts the user. System B does not earn a reward but System A is penalized either because of preemption or blocking. In all possible cases, the total reward of System B is always greater than or equal to the total reward of System A. □

Proof of Lemma IV.3. We consider two copies of a system. System A starts at state $i - 1$ and System B starts at state i and they have the same users except for the extra user in System B. The common users depart from the systems at the same time instants. System A always follows the optimal pricing policy whereas System B employs different pricing policies before and after a time juncture n. n is the time period at which System B moves to state $i + 1$ or the additional user in System B departs from the system depending on which event happens first. Before n, System B imitates whatever System A does, i.e. advertises the same prices. After n, System B follows the optimal pricing policy. We investigate these two cases and the corresponding rewards. In the first case, before System B reaches state $i + 1$, the additional user in System B leaves. In this case, System A and System B will be in the same state. They will advertise the same prices and will end up with the same average rewards over the infinite horizon. In the second case, with probability $0 \leq p \leq 1$, System B reaches state $i + 1$ before the additional user departs. Then, System A will be in state i as both systems experience the same arrivals. After this point, both systems enforce the optimal pricing policy. System B has an additional average reward of $p(h(i+1) - h(i))$ more than System A.

Due to the nature of optimal pricing policies, the difference between the total average rewards of two systems would be no smaller than this value if System B had followed the optimal pricing policy from the beginning. As a result:

$$h(i) - h(i-1) \geq p(h(i+1) - h(i)).$$

From Lemma IV.2 and $p \leq 1$, we can write:

$$h(i) - h(i-1) \geq h(i+1) - h(i),$$

which concludes the proof. □

Lemma IV.2 and Lemma IV.3 lead to the following theorem on the relationship between the total occupancy and the optimal pricing policy.

Proof of Theorem IV.2. Throughout this proof, we consider the price as a function of the demand. By Assumption 1 and Assumption 2, we are guaranteed that the inverse of the demand function exists. We denote the price at state i as $u(\lambda_2(i))$. Revisiting the DP equation Eq. (20) for $i < C$, we find the optimal demand by maximizing the right-hand side of Eq. (20). We discard the terms that are irrelevant to maximization and maximize the following sum:

$$\lambda_2(i)u(\lambda_2(i)) + \underbrace{(h(i+1) - h(i))}_{q(i)}\lambda_2(i).$$

We set $q(i) = h(i+1) - h(i)$. Now, we elaborate on the relation between $\lambda_2^*(i)$ and $\lambda_2^*(i-1)$ which are the optimal demand decisions of two neighbor states i and $i-1$. Due to Lemma IV.3, we need to consider two possible cases. First, if $q(i) = q(i-1)$, we have $\lambda_2^*(i) = \lambda_2^*(i-1)$. Second, if $q(i) < q(i-1)$, from the optimality of $\lambda_2^*(i)$ for state i, it must outperform $\lambda_2^*(i-1)$. Then, we have the following:

$$\begin{aligned}\lambda_2^*(i)u(\lambda_2^*(i)) + q(i)\lambda_2^*(i) \geq& \lambda_2^*(i-1)u(\lambda_2^*(i-1)) \\ &+ q(i)\lambda_2^*(i-1).\end{aligned} \tag{39}$$

In addition, as $\lambda_2^*(i-1)$ is optimal for state $i-1$, we have:

$$\begin{aligned}\lambda_2^*(i-1)u(\lambda_2^*(i-1)) + q(i-1)\lambda_2^*(i-1) \geq& \lambda_2^*(i)u(\lambda_2^*(i)) \\ &+ q(i-1)\lambda_2^*(i).\end{aligned} \tag{40}$$

Combining Eq. (39) and Eq. (40) yields:

$$(\lambda_2^*(i) - \lambda_2^*(i-1))(h(i+1) + h(i-1) - 2h(i)) \geq 0. \tag{41}$$

The term $(h(i+1) + h(i-1) - 2h(i)) \leq 0$ due to Lemma IV.3. Hence, $(\lambda_2^*(i) - \lambda_2^*(i-1))$ must be negative or zero as well. Then, we have $\lambda_2^*(i) \leq \lambda_2^*(i-1)$. Since $\lambda_2(u(i))$ is a decreasing function of $u(i)$, we have $u^*(i) \geq u^*(i-1)$. □

Proof of Lemma VI.1. We prove this result by induction on n. We start the induction from the end of the horizon $n = 0$, i.e. $V_0(x, y) = 0$. Thus, Lemma VI.1 holds for $n = 0$ by definition.

Induction step: Assume that for any $n > 0$ the following holds:

$$V_n(x, y) = q_n(x + y) = \max_{u \in \mathbb{U}} \ \{\tilde{q}_n(x + y, u)\}. \tag{42}$$

We need to show that $V_{n+1}(x, y) = q_{n+1}(x + y) = \max_{u \in \mathbb{U}} \ \{\tilde{q}_{n+1}(x + y, u)\}$. Using the DP equations provided for the non-preemptive systems we write:

$$\begin{aligned}V_{n+1}(x, y) = \max_{u \in \mathbb{U}} \Big\{ & \lambda_1 V_n(x+1, y)\mathbf{1}\{x + y < C\} \\ &+ \lambda_1(V_n(x, y) - K)\mathbf{1}\{x + y = C\} \\ &+ \lambda_2(u)(V_n(x, y+1) + u)\mathbf{1}\{x + y < C\} \\ &+ x\mu V_n(x-1, y) \\ &+ y\mu V_n(x, y-1) \\ &+ (1 - \lambda_1 - \lambda_2(u) - x\mu - y\mu - \alpha)\, V_n(x, y) \\ &\quad \cdot \mathbf{1}\{x + y < C\} \\ &+ (1 - \lambda_1 - C\mu - \alpha)\, V_n(x, y)\mathbf{1}\{x + y = C\} \Big\}.\end{aligned}$$

We substitute Eq. (42) to the above expression and we obtain the following:

$$\begin{aligned}V_{n+1}(x, y) = \max_{u \in \mathbb{U}} \Big\{ & \lambda_1 q_n(x + y + 1)\mathbf{1}\{x + y < C\} \\ &+ \lambda_1(q_n(x + y) - K)\mathbf{1}\{x + y = C\} \\ &+ \lambda_2(u)(q_n(x + y + 1) + u)\mathbf{1}\{x + y < C\} \\ &+ x\mu q_n(x + y - 1) && (43) \\ &+ y\mu q_n(x + y - 1) && (44) \\ &+ (1 - \lambda_1 - \lambda_2(u) - x\mu - y\mu - \alpha)\, q_n(x + y) \\ &\quad \cdot \mathbf{1}\{x + y < C\} \\ &+ (1 - \lambda_1 - C\mu - \alpha)\, q_n(x + y)\mathbf{1}\{x + y = C\} \Big\}.\end{aligned}$$

We collect terms (43) and (44) into a single term.

$$\begin{aligned}V_{n+1}(x, y) = \max_{u \in \mathbb{U}} \Big\{ & \lambda_1 q_n(x + y + 1)\mathbf{1}\{x + y < C\} \\ &+ \lambda_1(q_n(x + y) - K)\mathbf{1}\{x + y = C\} \\ &+ \lambda_2(u)(q_n(x + y + 1) + u)\mathbf{1}\{x + y < C\} \\ &+ (x + y)\mu q_n(x + y - 1) \\ &+ (1 - \lambda_1 - \lambda_2(u) - x\mu - y\mu - \alpha)\, q_n(x + y) \\ &\quad \cdot \mathbf{1}\{x + y < C\} \\ &+ (1 - \lambda_1 - C\mu - \alpha)\, q_n(x + y)\mathbf{1}\{x + y = C\} \Big\} \\ = & \max_{u \in \mathbb{U}} \left\{\tilde{q}_{n+1}(x + y, u)\right\} && (45) \\ = & \ q_{n+1}(x + y). && (46)\end{aligned}$$

which proves the induction. □

Proof of Theorem VI.1 We will prove the theorem in two parts. First, we will show that for the purpose of finding the optimal pricing policy of the non-preemptive system, we can use a 1D Markov Chain. Next, we will show that the average profit rate of the non-preemptive system differs from the average profit rate of a preemptive loss system by a constant, which leads to our theorem statement.

We will follow the same argument as provided in the proof of Theorem IV.1 to show that the optimal price only depends on the total system occupancy. Observe that the part of the DP optimality equations that depends on the optimal price

$u^*(x, y)$ for a non-preemptive system is the following:

$$u_n^*(x, y) = \underset{u \in \mathbb{U}}{\operatorname{argmax}} \left\{ \lambda_2(u)(V_{n-1}(x, y+1) - V_{n-1}(x, y) + u) \right\}. \tag{47}$$

From Lemma VI.1, we know that $V_n(x, y) = q_n(x+y)$. When we substitute it to Eq. (38), we obtain the following:

$$u_n^*(x, y) = \underset{u \in \mathbb{U}}{\operatorname{argmax}} \left\{ \lambda_2(u)(q_n(x+y+1) - q_n(x+y) + u) \right\}.$$

Therefore, the optimal pricing policy of a non-preemptive system only depends on the total system occupancy $x + y$. This result means that to find the optimal policy for a non-preemptive loss systems we can use a 1D Markov chain, where the state i is given by the total occupancy $x+y$. Further, define the arrival rates and call durations as given in section IV-D. Let $Q'_{\mathbf{u}}$ denote the average profit rate of non-preemptive system under policy $\mathbf{u}$.

$$Q'_{\mathbf{u}} = \sum_{i=0}^{C-1} \lambda_2(u(i))\, u(i)\, \pi_{\mathbf{u}}(i) - K\lambda_1 \pi_{\mathbf{u}}(C). \tag{48}$$

The relationship between the average profit of a preemptive loss system $J_{\mathbf{u}}$ and the auxillary systems described in Subsection IV-D was given by:

$$J_{\mathbf{u}} = Q_{\mathbf{u}} + K\lambda_1 E(\lambda_1/\mu, C). \tag{49}$$

By comparing Eq. (19) and Eq. (48), we conclude that $Q_{\mathbf{u}} = Q'_{\mathbf{u}}$, which leads to:

$$J_{\mathbf{u}} = Q'_{\mathbf{u}} + K\lambda_1 E(\lambda_1/\mu, C). \tag{50}$$

Hence, any policy that maximizes the average profit $J_{\mathbf{u}}$ of a preemptive system also maximizes the average profit $Q'_{\mathbf{u}}$ of a non-preemptive system. $\square$