Pricing Strategies for Spectrum Lease in Secondary Markets

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***Abstract*—We develop analytical models to characterize pricing of spectrum rights in cellular CDMA networks. Specifically, we consider a primary license holder that aims to lease its spectrum within a certain geographic subregion of its network. Such a transaction has two contrasting economic implications: On the one hand the lessor obtains a revenue due to the exercised price of the region. On the other hand, it incurs a cost due to (i) reduced spatial coverage of its network and (ii) possible interference from the leased region into the retained portion of its network, leading to increased call blocking. We formulate this trade-off as an optimization problem, with the objective of profit maximization. We consider a range of pricing philosophies and derive near-optimal solutions that are based on a reduced load approximation for estimating blocking probabilities. The form of these prices suggests charging the lessee in proportion to the fraction of admitted calls. We also exploit the special structure of the solutions to devise an efficient iterative procedure for computing prices. We present numerical results that demonstrate superiority of the proposed strategy over several alternative strategies. The results emphasize importance of effective pricing strategies in bringing secondary markets to full realization.**

Keywords

Network economics, Traffic modeling, Cellular CDMA networks, Reduced load approximation.

I. Introduction

Legacy regulatory frameworks of cellular wireless communications grant limited property rights to license holders of spectrum: a license holder can only provide a specific service and cannot resale any part of its license. Economists have long argued against such rigid regulations [1], whose inefficiency has recently gained wider recognition and led to global regulatory effort centered around introducing reforms to encourage the development of spectrum secondary markets. In this direction, the US Federal Communications Commission (FCC) has lately adopted a set of policies and procedures to facilitate trading of spectrum [2]. Similar regulatory efforts are also underway in the EU [3].

A salient feature of spectrum markets is the possibility of spatial interactions among service providers caused by electromagnetic interference. The focus of this work is to study the problem of spectrum pricing for secondary markets in light of such interactions. In particular, the problem considered in this work involves a primary license holder, or in short a licensee, that aims to relinquish its service in a subset of its coverage area and instead lease its right to some secondary users, call them the lessees. Such a transaction has two contrasting economic implications for the licensee: On the one hand the licensee obtains a revenue due to the exercised price, or rent, of the region. On the other hand, it incurs a cost due to (i) reduced spatial coverage of its network and (ii) possible interference from the leased region into the retained portion of its network leading to increased blocking of users.

While the pricing problem can in principle be considered within the framework of monopolistic markets in classical microeconomic theory [4], complexity of network-wide consequences of interference presents a major hurdle in obtaining explicit solutions. For example, a call in progress leads to a temporal reduction in utilization in its immediate neighborhood, which may in turn help accommodate more calls in the second-tier cells around it. In view of such knock-on effects, determining the marginal cost of traffic in a given area appears involved. A seemingly appealing solution to this issue might be to eliminate interference by isolating the activity in the two subregions by way of guard bands [5]. A guard band, however, is an unutilized resource whose cost needs to be internalized either by the licensee or by the lessee involved in the transaction. The situation leads to an inevitable loss of efficiency in the transaction, which may in fact be significant enough to limit granularity and liquidity of a secondary spectrum market.

Our goal in this paper is to devise effective pricing strategies that facilitate achieving full potential of secondary spectrum markets. The main challenge within this context is to develop analytical methods that on the one hand appropriately capture the effect of interferences and blocking at a network level, and on the other hand remain tractable. For this purpose, we adopt the so-called reduced load approximation (RLA) approach, widely used in the design of circuit-switched networks to estimate blocking probabilities [6]. This approximation is known to be asymptotically exact in certain limiting regimes (e.g., the regime of many small users [7]).

The technical focus of this work is on networks that employ CDMA as the spectrum access mechanism, wherein each call uses the entire available spectrum and can be sustained as long as the level of interference originating from other calls is maintained below a certain threshold. We study such networks under an idealized model in which a call consumes communication resources both in its own cell and, typically to a lesser

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extent, in other cells in its vicinity. This situation differs from that of the alluded standard circuit-switched models [8], where a call consumes the same capacity resources from each link over which it is routed. Nevertheless, we show in this paper that insightful properties of the classical RLA model extend to the model considered herein.

Our main contribution is to exploit the properties of the RLA model to derive expressions that characterize the optimal prices. Our derivations apply to a number of pricing philosophies, e.g., flat rate or demand-based. In each case, the form of the optimal price suggests charging the lessee proportionally to the fraction of admitted calls. The charged amount is also shown to depend on the extent of generated interference, namely, it balances the corresponding loss of revenue incurred by the licensee due to the influence of an admitted call. Moreover, we show that the analytical expressions lend themselves to an efficient computational procedure based on an iterative argument.

Next, we conduct a thorough numerical study to illustrate our analytical findings. We first assess the level of accuracy of the RLA-based pricing strategy. We show that the prices computed using this strategy are very close to the optimal prices. We then use an iterative procedure for computing prices and verify the outcome by obtaining similar results via exhaustive search. Finally, we compare the performance of the RLA-based strategy with other pricing strategies, namely: (i) an oblivious strategy that ignores effects of interference; (ii) a spatial guard band strategy, that prevents usage of spectrum in the surroundings of the leased region; and (iii) a pricing strategy due to Paschalidis and Liu [9] that was developed in a related pricing context. We show that the RLA-based strategy outperforms these strategies. Perhaps more importantly, we show that the first two strategies may sometimes lead to a net loss, in which case the licensee would renounce participating in secondary markets, while RLA-based pricing yields profit, and thereby creates an incentive for licensees to enter secondary markets, to the benefit of all participants.

While pricing in communication networks is a well-studied topic (see for example [10]), the setting considered here is specific to a scenario that arises in secondary cellular wireless markets and, to the best of the authors' knowledge, it has not been considered before. In related work, [11] pursues interference based pricing in a single cell via adaptive optimization techniques, and [12, 13] adopt a performance oriented viewpoint in considering dynamic spectrum access within a cell. In [14], the authors use pricing to achieve power control in a multi-cell wireless data system. In light of the above, novel aspects of the present paper are:

- Global consideration of network: General network topologies are considered rather than a single cell. Rather than lumping any portion of the network into an approximate module, the paper accounts for sophisticated dependence between cells due to generated interference.
- Characterization of optimal price: The form of optimal prices is characterized under a general framework. Optimal prices are shown to have an interpretation that offers insight on dominant factors that determine the value of spectrum under spatial interactions.

It should be noted that this paper concerns one possible secondary market transaction, namely long-term lease of spectrum rights in a region, that is consistent with the current state of spectrum reforms in the US. Another plausible scenario involves highly dynamic spectrum markets in which a service provider admits secondary calls opportunistically to utilize temporally idle capacity. This latter scenario suggests dynamic pricing policies, which have received considerable interest in communication networks (see for example the survey [15]). Dynamic pricing and admission control for secondary spectrum access is studied in [16] for a single-cell system. The scope of the present paper does not include dynamic pricing, however the nature of optimal prices obtained here admits comparison with an important pricing policy [9] that has certain optimality properties in the dynamic context. We elaborate on this connection further in Section VIII.

The rest of this paper is organized as follows: Section II describes the CDMA teletraffic network model. In Section III, different economic aspects of the network model and potential pricing techniques are introduced. The RLA employed in the paper is developed in Section IV, and optimal pricing is formulated as a related profit maximization problem in Section V. Expressions for the form of the optimal prices, based on first order optimality conditions, are given in Section VI. Section VII specifies a suggested computational procedure for optimal prices. A numerical study in support of the provided results is given in Section VIII including some comparisons with less sophisticated pricing techniques. Conclusions with final remarks are given in Section IX.

II. Network Model

We consider a cellular CDMA network under circuit-switched operation. A *call* in the network refers to a communication session between a base station and a terminal within the associated cell. Calls are subject to interference from other calls within the same cell, as well as from calls in other cells in proportion to the strength of electromagnetic coupling between their locations. We model such interference relations with a weighted graph $G = (N, E)$ where N denotes the collection of cells, and for each edge $(i, j) \in E$, its weight $w_{ij} \geq 0$ represents a measure of electromagnetic interference between cells i and j. Since a call may generate interference on other calls in the same cell, self-loops are allowed in G. Namely, the modeled physical situation typically implies $w_{ii} > 0$ for each cell i, although this condition is not required in the analysis.

We study the network under the condition that a call can be sustained only if it experiences admissible interference, and that a new call request cannot be honored if it leads to premature termination of another call that is already in progress. To formalize this condition, let n_i denote the number of calls in progress at each cell i so that $\sum_i n_i w_{ij}$ is the total interference acting on cell j. Given a positive interference threshold κ_j for each cell j, a network load $\mathbf{n} = \{n_i : i \in N\}$ is *feasible* if

$$\sum_{i \in N} n_i w_{ij} \leq \kappa_j \quad \text{for each cell } j. \tag{1}$$

Note that a more accurate feasibility condition would enforce (1) if $n_j > 0$, that is, it would not constrain the interference acting on idle cells. Here we proceed with the more stringent feasibility condition (1) since n_j is greater than zero with reasonably high probability unless cell j is grossly underloaded; therefore the attendant loss in accuracy is arguably tolerable. Sensitivity of a key performance measure to this disparity is illustrated in Table I of Section IV. CDMA network models that are based on the constraint (1) have been previously considered in the study of cellular wireless networks. See, for example, [17] for an in-depth discussion of this model and specifications of model parameters w_{ij} and κ_j in terms of physical layer parameters. In this paper, we assume that the model parameters satisfy the following mild condition:

Assumption 2.1: All edge weights w_{ij} and thresholds κ_j are rational numbers. Hence, without loss of generality in the feasibility condition (1), these parameters are further taken as integers.

We adopt a dynamic network model in which new calls arrive at each cell i according to a Poisson process. Call arrival processes in different cells are mutually independent. An incoming call is accepted if and only if its inclusion in the network conserves the feasibility condition (1) and the call is blocked otherwise. Each accepted call has a holding time that is exponentially distributed with unit mean, independently of the history prior to its arrival. In particular, the network load is a time-homogenous Markov process with state space S where

$$S = \{\mathbf{n} \in \mathbb{Z}_+^N : \mathbf{n} \text{ is feasible}\}.$$

Given a call arrival rate $\lambda_i > 0$ for each cell $i \in N$, we refer to the collection of such rates as the *network demand* and denote it by $\boldsymbol{\lambda} = \{\lambda_i : i \in N\}$. The probability of call blocking in each cell i associated with demand $\boldsymbol{\lambda}$ is denoted by $B_i(\boldsymbol{\lambda})$. Note that due to inter-cell interference, blocking in a given cell is affected by the arrival rates in other parts of the network.

We note that the model described above does not explicitly account for user mobility. This point is elaborated further in Section IX in view of the general methodology of the paper.

III. Economic Model

Consider a service provider that aims to profit from leasing its license within a subregion comprised of cells $L \subset N$. We refer to the service provider as the *licensee* and to the other participant involved in such a transaction as the *lessee*. The subsequent formulation applies to more general cases that involve multiple lessees acting on non-overlapping regions, however we concentrate on the single lessee case for clarity of exposition. The licensee has a subscriber base that is represented by network demand $\boldsymbol{\nu} = \{\nu_i : i \in N\}$, and it generates unit revenue per admitted call. Once the lease takes place, the licensee has reduced coverage; hence its revenue from its own subscribers is then generated due to accepted calls in the retained region $N - L$ only.

A price $\mathbf{p} = \{p_i : i \in L\}$ for region L is composed of a nonnegative number p_i for each leased cell $i \in L$. We assume that, associated with each price vector $\mathbf{p}$, the licensee receives a revenue $F(\mathbf{p})$ per unit time due to the transaction. Exact specification of the transaction revenue $F(\mathbf{p})$, as well as units of the price parameters $\mathbf{p}$ depend on the pricing philosophy adopted by the licensee. This point is further illustrated via examples at the end of this section.

A given pricing philosophy does not necessarily specify how the lessee reflects the price of leased region L onto its own subscribers. However, the price affects the traffic demand in the region. Namely, a price value p_i for cell $i \in L$ leads to a call arrival rate of $\alpha_i(p_i)$ at that cell *after* the transaction. We ignore any demand substitution effects and assume that the demand in the part of the network retained by the licensee remains as $\{\nu_i : i \in N - L\}$. The overall network demand after a transaction at price $\mathbf{p}$ is denoted by $\boldsymbol{\lambda}(\mathbf{p}) = \{\lambda_i(\mathbf{p}) : i \in N\}$ where

$$\lambda_i(\mathbf{p}) = \begin{cases} \alpha_i(p_i) & \text{if } i \in L \\ \nu_i & \text{if } i \in N - L. \end{cases}$$

It is plausible to take each α_i to be non-increasing, although here we shall only assume differentiability:

Assumption 3.1: The functions $F(\cdot)$ and $\alpha_i(\cdot),\ i \in L$, are differentiable.

It is perhaps more important from a conceptual viewpoint that the transaction price has an indirect effect on the future revenue of the licensee due to the relationship between the teletraffic activity in the leased region and the exogenous interference acting on the licensee's network. We will show that taking this effect into account in pricing leads to remarkable insight and benefit in realizing the potential of spectrum markets.

We close this section with three examples of pricing philosophies that lead to different specifications for the transaction revenue F and that will be further elaborated in the paper.

Example 3.1: (Flat price) A flat price for region L refers to charging p_i units of currency per unit time in each cell $i \in L$. The expected revenue due to a flat price should arguably involve a discount factor due to the possibility of the price being rejected. Hence a possible choice of $F(\mathbf{p})$ can be

$$F(\mathbf{p}) = \sum_{i \in L} p_i Pr(\mathbf{p} \text{ is accepted})$$

for a suitable probabilistic model of price acceptance.

Example 3.2: (Price per demand) The licensee may also price the spectrum access right in region L based on a prediction of the traffic demand of the lessee. In this case p_i refers to the revenue of the licensee per Erlang of traffic in cell $i \in L$ after the transaction. The transaction revenue would then be

$$F(\mathbf{p}) = \sum_{i \in L} \alpha_i(p_i) p_i. \tag{2}$$

Example 3.3: (Price per interference) Alternatively, the licensee may price the region by taking into consideration the interference that the lessee generates on the licensee's network after the transaction. Since such interference is generated by accepted calls, this principle can be interpreted as imposing a tax p_i per accepted call in cell $i \in L$, thereby entitling the licensee to a certain share of the lessee's revenue. The rate of revenue from the lease would then be given by

$$F(\mathbf{p}) = \sum_{i \in L} (1 - B_i(\boldsymbol{\lambda}(\mathbf{p})))\, \alpha_i(p_i) p_i.$$

We emphasize that the above expression implicitly accounts for interference through blocking probabilities. As alluded earlier, these probabilities are closely related to the interference coefficients w_{ij}. An explicit form of this relationship will be given in the sequel.

IV. Blocking Probabilities

Economics of the network displays substantial dependence on blocking that arises as a consequence of the interference limitations. In turn, a tractable characterization of the relationship between blocking probabilities and network demand is essential in developing insight on pricing parts of the network. In this section, we specify an approximate characterization of this relationship, namely one that is based on "reduced load approximation," which has proved fruitful in analysis of blocking in circuit-switched telephony [6, 18].

Given a set of arrival rates $\boldsymbol{\lambda} = \{\lambda_i : i \in N\}$, the network load $\mathbf{n}$ is a Markov process obtained by truncating the state space of a reversible Markov process that corresponds to the case when interference limitations are ignored. The equilibrium distribution π_λ of the network load is therefore given by

$$\pi_\lambda(\mathbf{n}) = Z \prod_{i \in N} \frac{\lambda_i^{n_i}}{n_i!}, \qquad \mathbf{n} \in S,$$

where Z is a constant which ensures that π_λ is a probability vector [19]. In turn, the blocking probability $B_i(\boldsymbol{\lambda})$ can be expressed as

$$B_i(\boldsymbol{\lambda}) = \sum_{\mathbf{n}:\mathbf{n}+\mathbf{e}(i) \notin S} \pi_\lambda(\mathbf{n}), \tag{3}$$

where $\mathbf{e}(i) = \{e_j(i) : j \in N\}$ be such that $e_j(i) = 1$ if $j = i$ and $e_j(i) = 0$ otherwise. Despite its appealing form, computation of the equilibrium distribution π_λ is hindered by the effort required to compute the normalization constant Z. Furthermore, the expression (3) for blocking probabilities can seldom be reduced to a closed form that applies to general topologies.

Reduced load approximation refers to approximating each blocking probability $B_i(\boldsymbol{\lambda})$ by the quantity $\hat{B}_i(\boldsymbol{\lambda})$ that is defined by

$$\hat{B}_i(\boldsymbol{\lambda}) = 1 - \prod_{j \in N} (1 - b_j(\boldsymbol{\lambda}))^{w_{ij}}, \tag{4}$$

where the numbers $\{b_j(\boldsymbol{\lambda}) : j \in N\}$ satisfy the equalities

$$b_j(\boldsymbol{\lambda}) = E\left((1 - b_j(\boldsymbol{\lambda}))^{-1} \xi_j(\boldsymbol{\lambda})\ ,\ \kappa_j\right) \tag{5}$$

with

$$\xi_j(\boldsymbol{\lambda}) = \sum_{i \in N} w_{ij} \lambda_i \prod_{k \in N} (1 - b_k(\boldsymbol{\lambda}))^{w_{ki}},$$

and $E(\cdot, \cdot)$ denoting the Erlang blocking formula

$$E(x, K) = \left(\sum_{m=0}^{K} \frac{x^m}{m!} \right)^{-1} \frac{x^K}{K!}, \qquad x, K \geq 0. \tag{6}$$

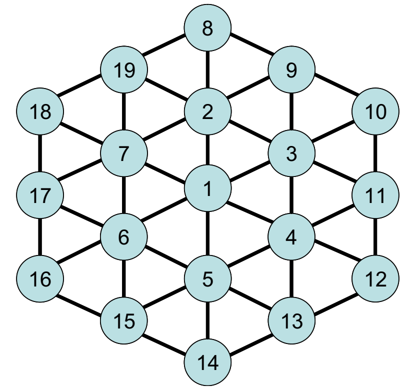


Fig. 1. The network graph of a 19-cell hexagonal lattice topology.

There exists a unique collection of numbers $\{b_j(\boldsymbol{\lambda}) : j \in N\}$ that satisfy equations (5) [18]; hence $\hat{B}_i(\boldsymbol{\lambda})$ is well-defined, and furthermore it is differentiable in the network demand $\boldsymbol{\lambda}$.

Under the feasibility condition (1), threshold κ_j can be interpreted as an "interference capacity" for cell j, and w_{ij} can be interpreted as the units of interference placed on cell j per call in progress in cell i. Consider a hypothetical model in which a call request is subject to an independent acceptance/rejection decision for each unit of interference that the call will generate at each cell. If, in this model, cell j accepts unit interference with probability $1 - b_j(\boldsymbol{\lambda})$, then $\hat{B}_j(\boldsymbol{\lambda})$ in expression (4) is the blocking probability of a call request to cell i. Under this assumption, $\xi_j(\boldsymbol{\lambda})$ is the average interference acting on cell j, and

$$(1 - b_j(\boldsymbol{\lambda}))^{-1} \xi_j(\boldsymbol{\lambda})$$

is the intensity of interference offered to cell j. The blocking probabilities in the hypothetical model would be consistent if the parameters $b_j(\boldsymbol{\lambda})$ satisfy the fixed-point relations (5).

The approximate blocking probabilities $\hat{B}_j(\boldsymbol{\lambda})$ are known to be asymptotically exact under the feasibility condition (1) along a limiting regime where the network arrival rates λ_j and thresholds κ_j increase in proportion [18]. Table I provides a numerical verification of the accuracy of the RLA in a 19-cell hexagonal lattice topology illustrated by Figure 1. The table displays RLA-based blocking estimates and simulation results with 95% confidence obtained under the original model described in Section II. Table I also quantifies the change in blocking due to imposing the condition (1) on non-idle cells only. Here $\kappa_i = 10$, $w_{ii} = 2$, and $\lambda_i = 1.0$ for all cells $i \in N$ and $w_{ij} = 1$ for all edges with $i \neq j$. In particular, the average number of calls in progress at a cell is smaller than 1. A call generates half of the interference in neighboring cells relative to its own cell; thus the *average* interference (both internally and externally generated) acting on an interior cell is roughly (but smaller than) 80% of its threshold.

V. Problem Formulation

We consider the licensee's pricing of region L under the objective of profit maximization. The licensee's revenue due to price $\mathbf{p}$ has two components: i) revenue from the leased

	Blocking Probability		
Cell No.	**Reduced load approximation**	**Simulation eq. (1)**	**Simulation eq. (1) for $n_j > 0$**
1	0.358	0.315 $\pm$0.002	0.305 $\pm$ 0.003
2-7	0.279	0.259 $\pm$0.003	0.259 $\pm$ 0.003
8-18 (even)	0.107	0.103 $\pm$0.002	0.102 $\pm$ 0.002
9-19 (odd)	0.159	0.153 $\pm$0.003	0.150 $\pm$ 0.002

TABLE I
BLOCKING ESTIMATES FOR THE NETWORK OF FIGURE 1 WITH $\kappa_i = 10$, $w_{ij} = 1, w_{ii} = 2, \lambda_i = 1$. CONFIDENCE LEVEL OF SIMULATIONS IS 95%.

region L, $F(\mathbf{p})$, and $ii)$ revenue from the retained part of the network $N - L$, which will be denoted by $Q(\mathbf{p})$. We model this latter quantity as

$$Q(\mathbf{p}) = \sum_{i \in N-L} \left(1 - \hat{B}_i\left(\boldsymbol{\lambda}\left(\mathbf{p}\right)\right)\right) \nu_i. \tag{7}$$

Note that here the RLA-based blocking estimates are adopted. Also note that, although the licensee continues to generate unit revenue per accepted call in $N - L$, the resulting revenue $Q(\mathbf{p})$ depends on the price $\mathbf{p}$ through consequences of the interference originating in the leased region L. The profit of the licensee due to price $\mathbf{p}$ is then $F(\mathbf{p}) + Q(\mathbf{p}) - R(\boldsymbol{\nu})$ where $R(\boldsymbol{\nu}) = \sum_{i \in N}(1 - \hat{B}_i(\boldsymbol{\nu}))\nu_i$ is the revenue of the licensee prior to the transaction. An optimal price for the licensee therefore solves:

$$\max_{\mathbf{p} \geq 0} \quad F(\mathbf{p}) + Q(\mathbf{p}). \tag{8}$$

In view of Assumption 3.1 and the differentiability of RLA-based blocking estimates $\hat{B}_i(\cdot)$, an inner solution $\mathbf{p}^* = (p_i^* : i \in L)$ to the licensee's problem (8) satisfies

$$\frac{\partial}{\partial p_i} F(\mathbf{p}) \bigg|_{\mathbf{p}=\mathbf{p}^*} = -\frac{\partial}{\partial p_i} Q(\mathbf{p}) \bigg|_{\mathbf{p}=\mathbf{p}^*}, \qquad i \in L.$$

In the following section, we obtain salient features of prices that satisfy this first-order optimality condition. Characteristics of such prices are closely related to the revenue function $F(\cdot)$ and the demand functions $\alpha_i(\cdot),\ i \in L$. In this respect, establishing existence and uniqueness properties of optimal prices appear difficult for general topologies and general demand functions. However, we address this issues via numerical analysis in special cases in Section VIII.

VI. Characterization of Optimal Prices

We start with notational remark that will be useful in the characterization and computation of optimal prices: For each cell $j \in N$ and any quantity H of interest in the sequel, let $\Delta_j H$ denote the amount by which H decreases when the threshold κ_j is decreased by 1. That is,

$$\Delta_j H = H|_{\kappa_j} - H|_{\kappa_j - 1}.$$

Theorem 6.1: An inner solution $\mathbf{p}^* = \{p_i^* : i \in L\}$ of the licensee's problem (8) satisfies

$$p_i^* = \left(1 - \hat{B}_i(\boldsymbol{\lambda}(\mathbf{p}^*))\right) \alpha_i(p_i^*) \gamma_i(\mathbf{p}^*), \tag{9}$$

where

$$\gamma_i(\mathbf{p}^*) = \varepsilon_i(p_i^*) \left(\frac{dF(\mathbf{p})}{dp_i} \bigg|_{\mathbf{p}=\mathbf{p}^*} \right)^{-1} \sum_{j \in N} w_{ij} \Delta_j Q(\mathbf{p}^*) \tag{10}$$

and $\varepsilon_i(p_i^*) = \frac{p_i^* \alpha_i'(p_i^*)}{\alpha_i(p_i^*)}$ is the price elasticity of demand in cell $i \in L$.

Proof: The proof of the theorem is deferred to the Appendix. ■

Theorem 6.1 can be interpreted for the three pricing philosophies alluded in Section III as follows:

a) Flat price: The form (9) suggests that optimal flat price for cell i is the same as the average revenue generated from that cell if the network demand were given by $\boldsymbol{\lambda}(\mathbf{p}^*)$ and each *admitted* call in the cell were charged an amount $\gamma_i(\mathbf{p}^*)$. In parsing the expression (10), recall that $\Delta_j Q(\mathbf{p}^*)$ is the reduction in the licensee's revenue from the retained region $N - L$ due to unit reduction in the interference threshold of cell $j \in N$, or equivalently due to imposing unit interference on that cell. Since an accepted call in cell $i \in L$ generates w_{ij} units of interference in cell j, such a call leads to a reduction of $w_{ij}\Delta_j Q(\mathbf{p}^*)$ in the licensee's revenue. The form (10) in turn indicates that the hypothetical price $\gamma_i(\mathbf{p}^*)$ balances the attendant revenue loss of the licensee, up to a multiplicative quantity that depends on the price elasticity of demand in cell i and the revenue function $F(\cdot)$.

b) Price per demand: If the licensee's revenue function is given by (2) then

$$\frac{dF(\mathbf{p})}{dp_i} = \alpha_i(p_i)(1 + \varepsilon_i(p_i))$$

and rearrangement of equalities (9) and (10) yields for all $i \in L$

$$p_i^* = \left(1 - \hat{B}_i(\boldsymbol{\lambda}(\mathbf{p}^*))\right) \left(1 + \varepsilon_i^{-1}(p_i^*)\right)^{-1} \sum_{j \in N} w_{ij} \Delta_j Q(\mathbf{p}^*). \tag{11}$$

In particular, the optimal price p_i^* per *arriving* call in cell i is proportional to the marginal cost of the licensee due to an *accepted* call in that cell, discounted by a factor that is equal to the acceptance probability.

c) Price per interference: If the licensee's revenue function is given by

$$F(\mathbf{p}) = \sum_{i \in L} \left(1 - \hat{B}_i(\boldsymbol{\lambda}(\mathbf{p}))\right) \alpha_i(p_i) p_i, \tag{12}$$

then a relatively more explicit characterization of $\mathbf{p}^*$ can be obtained by defining $U(\mathbf{p})$ as the *overall* revenue of the licensee after the transaction at price $\mathbf{p}$. That is,

$$U(\mathbf{p}) = \sum_{i \in N} \left(1 - \hat{B}_i(\boldsymbol{\lambda}(\mathbf{p}))\right) \lambda_i(\mathbf{p}) r_i(\mathbf{p}) \tag{13}$$

where

$$r_i(\mathbf{p}) = \begin{cases} p_i & \text{if } i \in L \\ 1 & \text{if } i \in N - L; \end{cases} \tag{14}$$

so that the licensee's profit due to price $\mathbf{p}$ is given by $U(\mathbf{p}) - R(\boldsymbol{\nu})$.

Theorem 6.2: (Optimal price per interference) If $F(\cdot)$ is given by (12), then an inner solution $\mathbf{p}^* = (p_i^* : i \in L)$ of the licensee's problem (8) satisfies

$$p_i^* = \left(1 + \varepsilon_i^{-1}(p_i^*)\right)^{-1} \sum_{j \in N} w_{ij} \Delta_j U(\mathbf{p}^*), \qquad i \in L. \quad (15)$$

Proof: The proof of the theorem is deferred to the Appendix. ■

VII. Computation of Optimal Prices

In this section, we establish properties of the differences $\{\Delta_j Q(\mathbf{p}) : j \in N\}$ and $\{\Delta_j U(\mathbf{p}) : j \in N\}$ that lead to computational methods for the optimal prices identified by Theorems 6.1 and 6.2. Here, we first give an informal derivation of these properties and refer the reader to the Appendix for a formal proof.

Consider the hypothetical network model that motivates the RLA when the network load is $\boldsymbol{\lambda}$ and each accepted call at cell $i \in N$ generates a generic revenue r_i. Let $S(\boldsymbol{\lambda})$ denote the rate of revenue generation in this network. Supposing that the blocking rates of all other cells are kept fixed, unit decrease in the threshold κ_j of cell j increases the blocking parameter $b_j(\boldsymbol{\lambda})$ of this cell by

$$\begin{aligned} \eta_j(\boldsymbol{\lambda}) &= E\left((1 - b_j(\boldsymbol{\lambda}))^{-1} \xi_j(\boldsymbol{\lambda}), \kappa_j - 1\right) \\ &\quad - E\left((1 - b_j(\boldsymbol{\lambda}))^{-1} \xi_j(\boldsymbol{\lambda}), \kappa_j\right). \end{aligned}$$

This leads to an increase in the rate that unit-capacity demands from other cells are rejected, specifically, from cell i, by an amount

$$w_{ij} \eta_j(\boldsymbol{\lambda})(1 - b_j(\boldsymbol{\lambda}))^{-1} \rho_i(\boldsymbol{\lambda}),$$

where

$$\rho_i(\boldsymbol{\lambda}) = \lambda_i \prod_{l \in N} (1 - b_l(\boldsymbol{\lambda}))^{w_{il}}.$$

Note here that $(1 - b_j(\boldsymbol{\lambda}))^{-1} \rho_i(\boldsymbol{\lambda})$ is the rate of unit-capacity demands at cell i, evaluated after thinning at all cells, including i, except cell j. Rejecting such a demand in cell j results in dropping an additional $w_{ij} - 1$ unit-demands in cell j and w_{ik} unit-demands at cells $k \neq j$. On the one hand, this event leads to a revenue loss of r_i; on the other hand it frees up some capacity which would not be available for future calls had the demand been granted. This latter effect can be interpreted as increasing the threshold of cell j by an amount $w_{ij} - 1$ (thereby the revenue by $(w_{ij} - 1)\Delta_j S(\boldsymbol{\lambda})$), and the threshold of each cell $k \neq j$ by an amount w_{ik} (thereby the revenue by $w_{ik} \Delta_k S(\boldsymbol{\lambda})$). Considering the consequences at all cells i, it may be argued that unit decrease in the threshold κ_j decreases the network revenue by

$$\Delta_j S(\boldsymbol{\lambda}) = \eta_j(\boldsymbol{\lambda})(1 - b_j(\boldsymbol{\lambda}))^{-1} \sum_{i \in N} w_{ij} \rho_i(\boldsymbol{\lambda}) \times \left(r_i - (w_{ij} - 1)\Delta_j S(\boldsymbol{\lambda}) - \sum_{k \in N - j} w_{ik} \Delta_k S(\boldsymbol{\lambda}) \right).$$

This intuition is formalized by the following theorem.

Given $\mathbf{p}$, define the vectors $\Delta U(\mathbf{p}) = \{\Delta_j U(\mathbf{p}) : j \in N\}$ and $\Delta Q(\mathbf{p}) = \{\Delta_j Q(\mathbf{p}) : j \in N\}$.

Theorem 7.1: Given price vector $\mathbf{p}$

$$\Delta U(\mathbf{p}) = f(\mathbf{p}, \Delta U(\mathbf{p})), \quad (16)$$

where for each $j \in N$

$$f_j(\mathbf{p}, \Delta U(\mathbf{p})) = \eta_j(\boldsymbol{\lambda}(\mathbf{p}))(1 - b_j(\boldsymbol{\lambda}(\mathbf{p})))^{-1} \sum_{i \in N} w_{ij} \rho_i(\boldsymbol{\lambda}(\mathbf{p})) \times \left(r_i(p_i) + \Delta_j U(\mathbf{p}) - \sum_{k \in N} w_{ik} \Delta_k U(\mathbf{p}) \right).$$

The same relation is satisfied by $\Delta Q(\mathbf{p})$ by replacing $r_i(\mathbf{p})$ by 0 for $i \in L$ and by 1 for $i \in N - L$.

Proof: The proof of the theorem is deferred to the Appendix. ■

Note that for each value of the price vector $\mathbf{p}$, the mapping $f(\mathbf{p}, \cdot) : \mathbb{R}^N \mapsto \mathbb{R}^N$ is linear, and techniques of [8] extend to the present setting to establish that the relation (16) has a unique solution in the vector $\Delta U(\mathbf{p})$. Furthermore, the sequence of vectors $\mathbf{c}^k : k = 1, 2, \cdots$ obtained via the recursion

$$\mathbf{c}^{k+1} = (1 - a)\mathbf{c}^k + a f(\mathbf{p}, \mathbf{c}^k), \qquad k \geq 1, \quad (17)$$

converges to that solution provided that $a \in (0, 1]$ is chosen small enough.

A similar iterative approach can be adopted in computation of optimal prices as well, although establishing convergence properties of such approach appears difficult in the generality of the model considered in the present paper. To describe the method, given a vector $\Delta U \in \mathbb{R}^N$ we define the mapping $g(\Delta U, \cdot) : \mathbb{R}^L \mapsto \mathbb{R}^L$ as

$$g_i(\Delta U, \mathbf{p}) = \left(1 + \varepsilon_i^{-1}(p_i)\right)^{-1} \sum_{j \in N} w_{ij} \Delta_j U, \quad i \in L, \ \mathbf{p} \in \mathbb{R}^L.$$

In particular, an optimal price $\mathbf{p}^*$, when the revenue function is (12), satisfies $\mathbf{p}^* = g(\Delta U(\mathbf{p}^*), \mathbf{p}^*)$. One may thus consider possible limits of the iteration

$$\mathbf{p}^{k+1} = (1 - a)\mathbf{p}^k + a g(\Delta U(\mathbf{p}^k), \mathbf{p}^k) \quad (18)$$

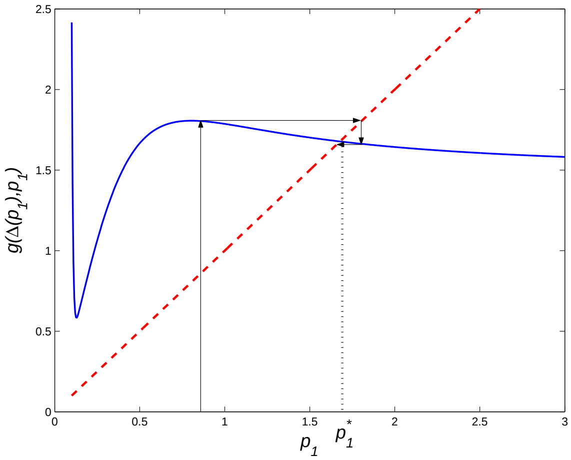
to compute an optimal price.

Figure 2 illustrates the mapping $g(\Delta U(\mathbf{p}), \mathbf{p})$ for the topology of Figure 1 in which the leased region involves only cell 1 (i.e. $L = \{1\}$), under two separate demand functions. Figure 2(a) depicts the case when $\alpha_1(p_1) = p_1^{-2}$; and hence the price elasticity $\varepsilon_1(\cdot)$ is constant. The undamped version of iteration (18) with $a = 1$ converges to the unique fixed point shown in the figure irrespective of the initial value. Figure 2(b) concerns the case where $\alpha_1(p_1) = e^{-p_1}$, and thus $g(\Delta U(p_1), p_1) < 0$ for $p_1 < 1$. Iteration (18) has two fixed points in this case and it does not converge if $a = 1$.

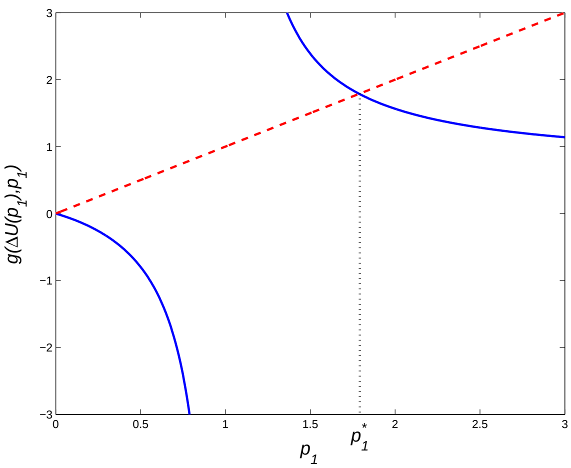
The procedure described by (18) entails nested iterations and its computational burden can be reduced by carrying out the two iterations (17)–(18) simultaneously via

$$\begin{aligned} \mathbf{c}^{k+1} &= (1 - a)\mathbf{c}^k + a f(\mathbf{p}^k, \mathbf{c}^k) \quad (19) \\ \mathbf{p}^{k+1} &= (1 - a)\mathbf{p}^k + a g(\mathbf{c}^k, \mathbf{p}^k). \quad (20) \end{aligned}$$

Note that the function $f(\cdot, \cdot)$ entails computing probabilities $\{b_j(\boldsymbol{\lambda}(\cdot)) : j \in N\}$, which is typically carried out via



(a) $\alpha_1(p_1) = p_1^{-2}$.



(b) $\alpha_1(p_1) = e^{-p_1}$.

Fig. 2. Computation of optimal price of cell 1 in the network of Figure 1 using the iteration (18).

iterative techniques itself. The complexity of (19)–(20) is then $C+|N|d^2+|L|d$ operations per iteration, where C represents complexity of the blocking-probability computation and d is the degree of the network graph.

The iterations (19)–(20) are generally nonlinear, but a linearized analysis sheds light on their convergence properties. Namely, let $\mathbf{c}^*, \mathbf{p}^*$ be a fixed point and define the matrix

$$A = (1-a)I + a \left[\begin{array}{cc} \partial_{\mathbf{c}} f(\mathbf{p}^*, \mathbf{c}^*) & \partial_{\mathbf{p}} f(\mathbf{p}^*, \mathbf{c}^*) \\ \partial_{\mathbf{c}} g(\mathbf{c}^*, \mathbf{p}^*) & \partial_{\mathbf{p}} g(\mathbf{c}^*, \mathbf{p}^*) \end{array} \right],$$

where $\partial_{\mathbf{c}}$ and $\partial_{\mathbf{p}}$ denote partial derivatives with respect to vectors $\mathbf{c}$ and $\mathbf{p}$ respectively. Let $\delta\mathbf{c}^k = \mathbf{c}^* - \mathbf{c}^k$, $\delta\mathbf{p}^k = \mathbf{p}^* - \mathbf{p}^k$. Taylor expansion of $f(\mathbf{p},\mathbf{c})$ and $g(\mathbf{p},\mathbf{c})$ around $\mathbf{c}^*, \mathbf{p}^*$ yields

$$\left[\begin{array}{c} \delta\mathbf{c}^{k+1} \\ \delta\mathbf{p}^{k+1} \end{array} \right] = A \left[\begin{array}{c} \delta\mathbf{c}^{k} \\ \delta\mathbf{p}^{k} \end{array} \right] + \text{h.o.t.}$$ [1]

Note that since $f(\mathbf{p},\mathbf{c})$ is linear in $\mathbf{c}$, the alluded convergence of iteration (17) implies that eigenvalues of the first diagonal block $(1-a)I + a\partial_{\mathbf{c}} f(\mathbf{p}^*, \mathbf{c}^*)$ in A are smaller than 1 for small values of a. In this case, the eigenvalues are also bounded away from 0. The remaining diagonal block $(1-a)I + a\partial_{\mathbf{p}} g(\mathbf{c}^*, \mathbf{p}^*)$ has the same property if

$$\frac{d}{dp_i} \left(1 + \varepsilon_i^{-1}(p_i)\right)^{-1} \Bigg|_{p_i = p_i^*} \sum_{j \in N} w_{ij} c_j^* < 1, \quad \text{for } i \in L. \quad (21)$$

Hence, if condition (21) holds then the spectral radius of A is smaller than 1 provided that a is chosen small enough [20]. A plausible technique to compute an optimal price is then to employ iterations (19)–(20) starting from a reasonably fine grid of the solution space and to choose the limit point that leads to the largest profit. In the example associated with Figure 2(b), iterations (19)–(20) converge to one of the two fixed points when $a = 1$.

VIII. Numerical Study

Throughout this section we adopt the following system parameters for every topology studied: Equal interference thresholds $\kappa_i = 10$ at each cell i, $w_{ij} = 1$ for each edge such that $i \neq j$ and self-loop weights are $w_{ii} = 2$ for all i.

A. Accuracy of the Reduced Load Approximation

We start by examining sensitivity of optimal prices to modeling errors in the blocking probabilities due to inaccuracy of the RLA. Our investigation here involves computing optimal price per interference of a single cell using the RLA and also using the exact equilibrium distribution of the network load. We adopt the 7-cell topology whose graph representation is shown in Figure 3, where cell 1 is for lease. In this study, a smaller topology is selected in order to avoid otherwise lengthy numerical procedures to compute the exact profit for every value of the price. For this small topology the equilibrium distribution can be exactly computed in a relatively short time.

The traffic demand of the licensee prior to the transaction in call per unit time is taken as

$$\nu_i = \begin{cases} 0 & \text{if } i = 1 \\ 1 & \text{if } i = 2, \ldots 7, \end{cases} \quad (22)$$

and the demand function of the lessee α_1 in cell 1 is taken as

$$\alpha_1(p_1) = p_1^{-2}. \quad (23)$$

Figure 4 shows exact and approximated profit of the licensee for different prices for cell 1. The figure suggests that the profit maximization problem admits a unique solution for this particular setup. The disparity in the profit $F(\mathbf{p}) - C(\mathbf{p})$ appears small. More importantly, the approximate optimal price is very close to the exact price, both values are about 1.3. Hence, the profit achieved with the approximate optimal price under the exact model is very close to the optimal profit.

B. Computation of Prices

We next study an example where optimal prices are computed using the iterations (19)–(20). We consider the 19-cell hexagonal lattice topology with the corresponding network graph shown in Figure 1. The leased region is $L = \{1, \cdots 7\}$

[1] the acronym represents "higher order terms" that are significantly smaller than entries of $\delta\mathbf{c}^k$ and $\delta\mathbf{p}^k$ whenever the latter quantities are small themselves.

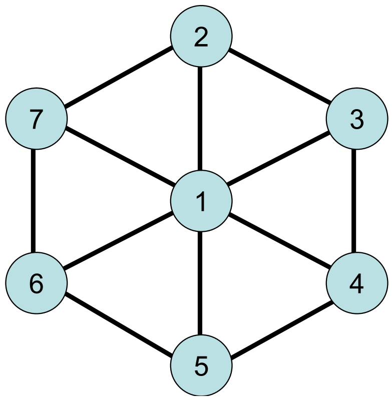


Fig. 3. Graph representation of a 7-cell network topology.

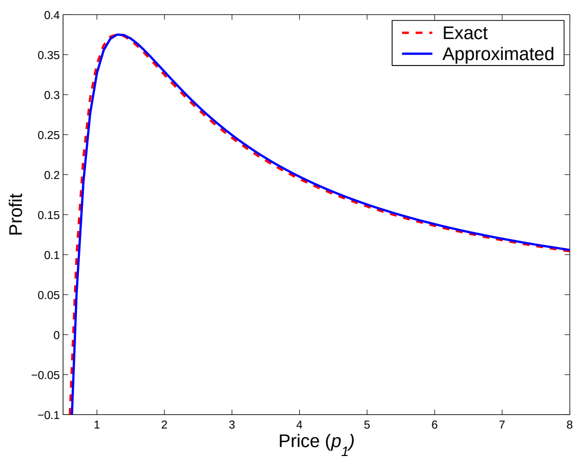


Fig. 4. Approximated and exact profit from leasing cell 1 in the network shown in Figure 3 under the demand function (23) and price per interference.

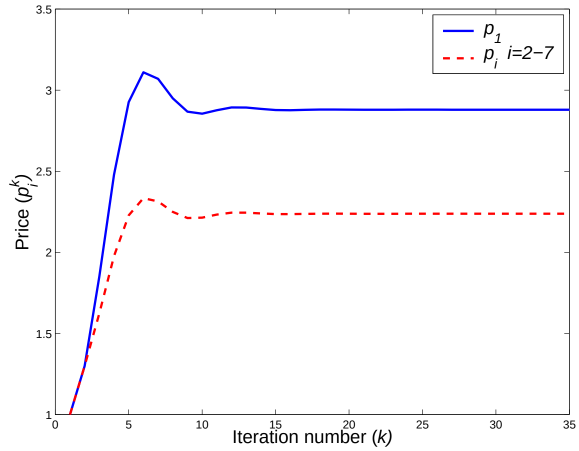


Fig. 5. Computation of (per-interference) prices for cells $1-7$ in the network shown in Figure 1 under demand function (25). Values are obtained via iterations (19)–(20).

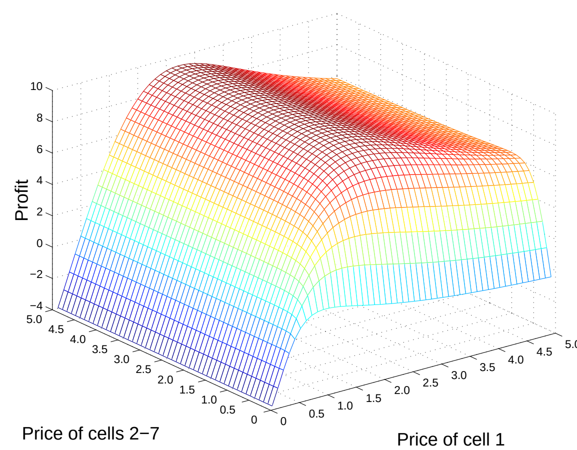


Fig. 6. Profit of the licensee from leasing cells $1-7$ in the network shown in Figure 1 using pricing per interference. Call demand on the leased cells is taken to be as in (25). Profit-maximizing prices match those obtained by iterations (19)–(20) as illustrated in Figure (5).

and the traffic demand of the licensee prior to the transaction in call per unit time is taken as

$$\nu_i = \begin{cases} 0 & \text{if } i = 1, \dots 7 \\ 1 & \text{if } i = 8, \dots 19. \end{cases} \tag{24}$$

The traffic of the lessee is assumed to be following the demand function

$$\alpha_i(p_i) = \beta_i p_i^{-2} \qquad i = 1, \dots 7, . \tag{25}$$

where

$$\beta_i = \begin{cases} 1 & \text{if } i = 1 \\ 5 & \text{if } i = 2, \dots 7. \end{cases} \tag{26}$$

Figure 5 illustrates effectiveness of iterations (19)–(20) in computing prices for this example when price per interference is implemented. Iterations are performed using a moderate damping factor of value 0.5. They converge relatively quickly, in less than 25 iterations, to the values $p_1 = 2.88$ and $p_i = 2.24$ for $i = 2, \dots 7$.

Verifying optimality of the previous prices requires an exhaustive search over all possible prices $\mathbf{p} \geq 0$. A task that is computationally infeasible. However, given the results from the iterations, we perform a search on a subdomain of possible prices which have at most two distinct price values, one for cell 1 and one for cells $2-7$. This setup helps visualize the licensee's profit in a 3-dimensional setting as in Figure 6. A price step of 0.1 is used in producing profit values comprising the figure. The numerical results suggest that the licensee's profit maximization problem for the range of prices used admits a unique solution where, roughly, $p_1^* = 2.9$ and $p_i^* = 2.2$ for $i = 2, \dots 7$, which match the values observed from Figure 5.

C. *Comparison with other Pricing Strategies*

In this section, we numerically compare performance of the proposed RLA-based pricing strategy with those of three other arguable strategies. The adopted measure of performance is profit as defined in Section V. The following pricing strategies are considered:

Interference-oblivious pricing: Consider the strategy when the licensee does not account for the cost resulting from the interference caused by the traffic in the leased region. The

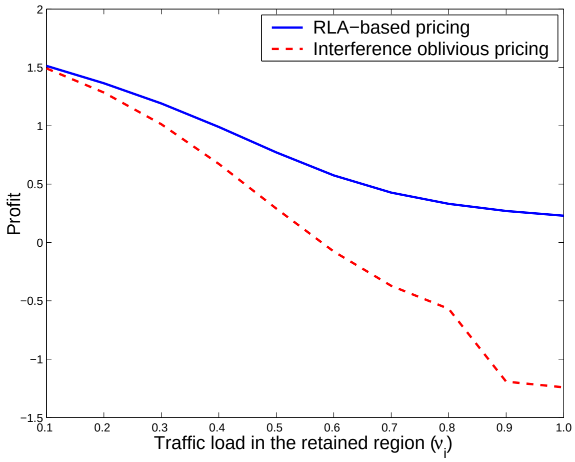


Fig. 7. Profit comparison with interference-oblivious pricing when cell 1 in Figure 1 is leased under the demand function (25).

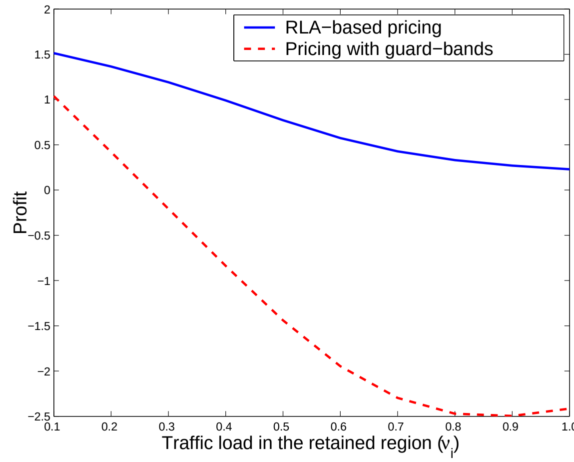


Fig. 8. Profit comparison with pricing subject to spatial guard-bands when cell 1 in Figure 1 is leased under the demand function (25).

exercised price solves

$$\max_{\mathbf{p}} \quad F(\mathbf{p}). \tag{27}$$

To understand the consequences of this strategy, we consider the 19-cell network in Figure 1, where we are interested in price per interference for cell 1 under call demand function given by (25) with $\beta_1 = 1.0$.

In Figure 7, we show the optimal profit for different traffic rates ν_i in the retained region of the licensee, i.e. cells $\{2, \cdots 19\}$. We also show the profit when the price solves (27) for cell 1. Values in the figure are computed via RLA, and it is assumed that there is no traffic demand on cell 1 prior to the transaction, i.e. $\nu_1 = 0$. Observe that the profit gap between the optimal and the interference oblivious techniques widens as ν_i increases. This can be intuitively explained by the fact that spectral resources in the retained network become scarce as the traffic intensity ν_i in the network increases: While calls in cell 1 typically complete service without having noticeable effect in the retained region with low traffic intensity, such calls are likely to lead to blocked calls as the intensity increases. Hence the burden of a call in cell 1 is not the same under the two traffic scenarios. The reason is the interference generated by cell 1, which strategy (27) does not account for.

Spatial guard-bands: Another simplistic pricing strategy involves the technique of spatial guard bands. It subscribes to a philosophy that can be considered as an opposite extreme of interference oblivion: it eliminates interference by isolating the activities between the leased and the retained subregions [5]. For example in the network in Figure 1, traffic in cell 1 can be isolated from the rest of the network by prohibiting traffic to cells $(2-7)$. This implies losing some potential revenue from those cells, and, even though the licensee does not incur blocking due to exogenous interference, its profit is suboptimal. To get an exposure on the extent of suboptimality, we consider the 19-cell network in Figure 1 where cells $(2-7)$ are taken as guard bands for cell 1. As in the previous example, we are interested in the price per interference for cell 1 facing a demand function given by (25) where $\beta_1 = 1.0$. Figure 8 illustrates the profit gained by solving (27) for cell 1 for different traffic intensities on the cells $8-19$. The licensee may commit significant losses out of this pricing technique, although the percentage loss in profit should be expected to decrease if the leased region L is a connected component of N that is significantly larger than its boundary. Note that this strategy and the interference-oblivious strategy can cause a net loss (negative profit). In this case, the strategies drop the profitability incentive for the licensee and can lead to renounce participating in secondary markets.

Paschalidis-Liu Pricing: The last pricing strategy for comparison is the one given in [9] where prices solve a certain optimization problem. The authors establish that charging a static price per call, that does not depend on system state, is asymptotically optimal along a limiting regime in which traffic load and system capacity tend to infinity. The reader is referred to [9] for details of this technique, but note that the problem setup there does not involve the notion of a leased region. Yet the technique of [9] yields price values that are applied to incoming calls, and thereby it admits comparison with the technique studied in this paper.

An adaptation of the methodology of [9] suggests a solution of the following optimization problem as an alternative pricing policy in the present context:

$$\begin{aligned} &\max_{\mathbf{p}} \quad \sum_{i\in L} p_i \alpha_i(p_i) \\ s.t. \quad &\sum_{i\in L} \alpha_i(p_i) w_{ij} + \sum_{i\in N-L} \nu_i w_{ij} \le \kappa_j \quad \forall j \in N. \end{aligned} \tag{28}$$

The objective here is to maximize the revenue from the leased region subject to the capacity constraints based on mean network demand. Hence, interactions between the leased and retained regions due to statistical fluctuations of instantaneous network load are neglected. The approach may therefore be interpreted to assume that revenue from the retained region is not affected by the secondary load as long as the constraints

	Interference threshold (κ_j)				
	7.0	8.0	9.0	10.0	11.0
Paschalidis-Liu	0.1275	0.0957	0.0830	0.1099	0.1774
RLA-based	0.1556	0.2121	0.2839	0.3751	0.4862

(a) Under demand function (23).

	Interference threshold (κ_j)				
	7.0	8.0	9.0	10.0	11.0
Paschalidis-Liu	0.9814	1.8583	2.5570	3.0326	3.2430
RLA-based	1.6997	2.1280	2.5868	3.0666	3.5490

(b) Under demand function (29).

TABLE II
PROFIT COMPARISON WITH PASCHALIDIS AND LIU STRATEGY [9] WHEN CELL 1 IN FIGURE 3 IS LEASED.

are satisfied.

In Table II we compare the two pricing strategies on the 7-cell topology illustrated in Figure 3. Here $L = \{1\}$ and traffic prior to the transaction is as given in (22). All interference thresholds are taken to be identical. Tables II.(a) and II.(b) report profits respectively under the demand functions (23) and

$$\alpha_1(p_1) = \begin{cases} 5 - p_1 & \text{if } 0 \leq p_1 \leq 5 \\ 0 & \text{if } p_1 \geq 5. \end{cases} \tag{29}$$

We point out that the demand function (29) complies with a key assumption that is needed for optimality in [9], whereas (23) does not.

While both pricing techniques are based on approximations that are asymptotically exact in the same limiting regime, RLA-based pricing appears to yield substantially higher profit, possibly due to its more detailed accounting of the effects of interference.

IX. CONCLUSION AND DISCUSSION

In this paper, an optimization perspective is studied for pricing spectrum in CDMA-based cellular wireless networks. Reduced load approximation is employed to obtain tractable expressions for blocking probabilities in the network. A computational technique is outlined and the resulting prices are shown to yield higher profit than three alternative pricing strategies.

It is possible to imagine cases in which leasing is not profitable even under optimal pricing - for example when the demand function is confined to low prices that do not justify the loss of primary revenue due to the interference generated by a secondary call. The breakpoint, however, appears difficult to identify in closed-form. The technique of the paper may also be used to verify whether leasing a given region would yield positive profit. However, it does not suggest an efficient method to search for the most profitable region, and finding one would possibly require exhaustive search. In this connection, it may be useful to see the proposed technique as a toolbox that can be applied in making a variety of transaction decisions pertaining to secondary markets.

The model adopted in this paper does not explicitly account for user mobility. An implicit method to involve mobility may be to lump its overall effect in the demand function by allowing the call arrival rates $\alpha_i(\cdot)$ to depend on the entire price vector $\mathbf{p}$. This approach models handoffs exactly as new call requests and confines the additional complexity in the demand function. If such demand function is known, then the development of Section VI can be readily adapted to identify first order optimality conditions. If demand function is known for only new call requests but an explicit pattern of mobility is available, then the same effect may be obtained through the rate calculations of [21]. Markovian network models, and in particular blocking therein, were studied in [22] under a certain reversible routing condition, and in [23] under more relaxed assumptions. The attendant modeling accuracy comes at the expense of increased complexity in representations of critical performance measures, and striking a favorable balance between these two aspects appears challenging.

Narrowband networks, whose operational constraints require that any two calls in the same or neighboring cells are assigned different channels, generally appear harder to analyze due to combinatorial consequences of interference. In such systems, the channel assignment of each call in progress, rather than solely cell occupancies, dictate whether a new call can be admitted. While the presented techniques can be shown to apply to certain topologies and channel assignment policies, a general treatment of narrowband networks remains an area open for future work. Another direction for future work is to consider an oligopoly market form with multiple service providers. In this case, game theoretical techniques seem appealing for analyzing and stabilizing interactive market strategies.

ACKNOWLEDGMENT

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X. APPENDIX

In this section, we provide proofs of Theorems 6.1, 6.2, and 7.1. The proofs require several auxiliary lemmas that identify sensitivity of the revenue function to various model parameters. We establish these lemmas by generalizing a technique that is used by Kelly [8] for the special case $w_{ij} \in \{0, 1\}$. This special case does not readily yield the conclusions required here; we therefore provide a self-contained analysis that can be read in isolation.

For convenience of analysis we consider an extended model in which, in addition to regular calls, each cell receives a Poisson stream of *local* calls that generate unit interference at their cell of residence but do not generate interference at other cells. The quantities associated with such calls in cell $i \in N$ will be indexed by i'; hence all primed indices have a specific meaning in this section. It will be convenient to represent this model in terms of the original model of Section II by augmenting the network graph G by cell i' for each (original) cell $i \in N$, and by setting

$$w_{i'j} = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{else.} \end{cases}$$

Note that the specification on the right hand side involves cell $i \in N$ rather than i'. We denote the set of cells in this extended model by

$$D = \{i, i' : i \in N\}.$$

The extended model thus has twice as many cells as the original model but we shall always assume that both the arrival rates $\lambda_{i'} = 0$ and call revenues $r_{i'} = 0$; hence the local calls do not alter the network dynamics or revenue, but merely serve as an analytical instrument.

Let r_i denote a generic revenue per accepted call at cell i. For a given demand vector $\boldsymbol{\lambda}$, denote the associated rate of revenue from the network by

$$S(\boldsymbol{\lambda}) = \sum_{i \in N} r_i \rho_i(\boldsymbol{\lambda}), \tag{30}$$

where

$$\rho_i(\boldsymbol{\lambda}) = \lambda_i \prod_{j \in N} (1 - b_j(\boldsymbol{\lambda}))^{w_{ij}} \tag{31}$$

is the rate of accepted calls at cell i.

Lemma 10.1: For $j \in N$

$$\frac{d}{d\lambda_{j'}} S(\boldsymbol{\lambda}) = -\sum_{k \in N} (1 - b_k(\boldsymbol{\lambda}))^{-1} \times \left(\sum_{i \in D} w_{ik} r_i \rho_i(\boldsymbol{\lambda}) \right) \frac{d}{d\lambda_{j'}} b_k(\boldsymbol{\lambda}).$$

Proof: Given a vector $\mathbf{b} = (b_k : k \in N) \in [0,1]^N$, define

$$S(\boldsymbol{\lambda}; \mathbf{b}) = \sum_{i \in D} r_i \lambda_i \prod_{l \in N} (1 - b_l)^{w_{il}} \tag{32}$$

so that

$$\frac{d}{d\lambda_{j'}} S(\boldsymbol{\lambda}) = \left[\frac{d}{d\lambda_{j'}} + \sum_{k \in N} \frac{d}{d\lambda_{j'}} b_k(\boldsymbol{\lambda}) \frac{\partial}{\partial b_k} \right] S(\boldsymbol{\lambda}; \mathbf{b}) \Bigg|_{\mathbf{b} = \mathbf{b}(\boldsymbol{\lambda})}$$

where $\mathbf{b}(\boldsymbol{\lambda}) = (b_k(\boldsymbol{\lambda}) : k \in N)$. The claim of the lemma is verified by observing that $\frac{d}{d\lambda_{j'}} S(\lambda; \mathbf{b}) = 0$ and

$$\begin{aligned} \frac{\partial}{\partial b_k} S(\boldsymbol{\lambda}; \mathbf{b}) &= -\sum_{i \in D} r_i \lambda_i w_{ik} (1 - b_k)^{w_{ik} - 1} \prod_{l \in N - k} (1 - b_l)^{w_{il}} \\ &= -(1 - b_k)^{-1} \sum_{i \in D} w_{ik} r_i \lambda_i \prod_{l \in N} (1 - b_l)^{w_{il}}. \end{aligned}$$

■

For each vector $\mathbf{b} = (b_k : k \in N)$ let

$$\xi_k(\mathbf{b}) = (1 - b_k)^{-1} \sum_{i \in D} w_{ik} \lambda_i \prod_{j \in N} (1 - b_j)^{w_{ij}}$$

and

$$\eta_k(\mathbf{b}) = E(\xi_k(\mathbf{b}), \kappa_k - 1) - E(\xi_k(\mathbf{b}), \kappa_k), \quad k \in N.$$

Here, we point to an intentional abuse of notation to reduce the notational burden: Namely, the symbols ξ_k and η_k have analogous definitions respectively in Sections IV and VII, although with different arguments. For the correctness of proofs, it is enough to notice that the present definitions match with the earlier ones when $\mathbf{b}$ is replaced by $\mathbf{b}(\boldsymbol{\lambda})$.

Let δ_{jk} be 1 if $j = k$ and 0 otherwise.

Lemma 10.2: For $k, j \in N$

$$\begin{aligned} \eta_k(\boldsymbol{\lambda})^{-1} \frac{d}{d\lambda_{j'}} b_k(\boldsymbol{\lambda}) &= \delta_{jk}(1 - b_k(\boldsymbol{\lambda})) \\ &\quad -(1 - b_k(\boldsymbol{\lambda}))^{-1} \left(\sum_{i \in D} w_{ik}(w_{ik} - 1) \rho_i(\boldsymbol{\lambda}) \right) \frac{d}{d\lambda_{j'}} b_k(\boldsymbol{\lambda}) \\ &\quad - \sum_{l \in N - k} (1 - b_l(\boldsymbol{\lambda}))^{-1} \left(\sum_{i \in D} w_{il} w_{ik} \rho_i(\boldsymbol{\lambda}) \right) \frac{d}{d\lambda_{j'}} b_l(\boldsymbol{\lambda}). \end{aligned}$$

Proof: Given $\mathbf{b} = (b_k : k \in N)$, we define

$$\mathcal{E}_k(\boldsymbol{\lambda}; \mathbf{b}) = E\left((1 - b_k)^{-1} \sum_{i \in D} w_{ik} \lambda_i \prod_{l \in N} (1 - b_l)^{w_{il}}, \kappa_k \right) \tag{33}$$

so that

$$\frac{d}{d\lambda_{j'}} b_k(\boldsymbol{\lambda}) = \left[\frac{d}{d\lambda_{j'}} + \sum_{l \in N} \frac{d}{d\lambda_{j'}} b_l(\boldsymbol{\lambda}) \frac{\partial}{\partial b_l} \right] \mathcal{E}_k(\boldsymbol{\lambda}; \mathbf{b}) \Bigg|_{\mathbf{b} = \mathbf{b}(\boldsymbol{\lambda})}. \tag{34}$$

Since for each integer $\kappa > 0$

$$\frac{d}{dx} E(x, \kappa) = [1 - E(x, \kappa)] \left[E(x, \kappa - 1) - E(x, \kappa) \right] \tag{35}$$

(see [8, Lemma 2.1]), it follows that

$$\frac{d}{d\lambda_{j'}} \mathcal{E}_k(\boldsymbol{\lambda}; \mathbf{b}) = \delta_{jk} \eta_k(\boldsymbol{\lambda})(1 - b_k). \tag{36}$$

By the same token

$$\begin{aligned} &\frac{\partial}{\partial b_l} \mathcal{E}_k(\boldsymbol{\lambda}; \mathbf{b}) \\ &= -\eta_k(\boldsymbol{\lambda}) \sum_{i \in D} w_{ik} \lambda_i w_{il} (1 - b_l)^{w_{il} - 1} \prod_{j \in N - l} (1 - b_j)^{w_{ij}} \\ &= -\eta_k(\boldsymbol{\lambda})(1 - b_l)^{-1} \sum_{i \in D} w_{ik} w_{il} \lambda_i \prod_{j \in N} (1 - b_j)^{w_{ij}} \end{aligned} \tag{37}$$

for $l \neq k$, and

$$\begin{aligned} &\frac{\partial}{\partial b_k} \mathcal{E}_k(\boldsymbol{\lambda}; \mathbf{b}) \\ &= -\eta_k(\boldsymbol{\lambda})(1 - b_k) \sum_{i \in D} w_{ik} \lambda_i (w_{ik} - 1) \times \\ &\quad (1 - b_k)^{w_{ik} - 2} \prod_{j \in N - k} (1 - b_j)^{w_{ij}} \\ &= -\eta_k(\boldsymbol{\lambda})(1 - b_k)^{-1} \sum_{i \in D} w_{ik}(w_{ik} - 1) \lambda_i \prod_{j \in N} (1 - b_j)^{w_{ij}}. \end{aligned} \tag{38}$$

The desired result is obtained by substituting (36)-(38) in (34).

■

We next express Lemma 10.2 in a matrix form that will be useful in the proofs. Towards this end, define the matrices

$$\begin{aligned} w &= [w_{ij}]_{D\times N} \\ \frac{db}{d\lambda} &= [\sigma_{jk}]_{N\times N}, \quad \text{where} \quad \sigma_{jk} = \frac{d}{d\lambda_{k'}} b_j(\boldsymbol{\lambda}) \\ \Lambda &= [\Lambda_{jk}]_{N\times N}, \quad \text{where} \\ & \Lambda_{jk} = \begin{cases} \sum_{i\in D} w_{ik}(w_{ik}-1)\rho_i(\boldsymbol{\lambda}) & \text{if } j=k \\ \sum_{i\in D} w_{ij} w_{ik} \rho_i(\boldsymbol{\lambda}) & \text{otherwise,} \end{cases} \end{aligned}$$

the diagonal matrices

$$\begin{aligned} \beta &= diag[1-b_i(\boldsymbol{\lambda})]_N, \\ \eta &= diag[\eta_i(\mathbf{b}(\boldsymbol{\lambda}))]_N, \\ \rho &= diag[\rho_i(\boldsymbol{\lambda})]_D, \end{aligned}$$

and the row vectors

$$\begin{aligned} r &= [r_i]_D, \\ \frac{dS}{d\lambda} &= \left[\frac{d}{d\lambda_{j'}} S(\boldsymbol{\lambda})\right]_N. \end{aligned}$$

Lemmas 10.1 and 10.2 can be expressed respectively in terms of these matrices as

$$\begin{aligned} \frac{dS}{d\lambda} &= -r\rho w \beta^{-1} \frac{db}{d\lambda} \quad (39) \\ \frac{db}{d\lambda} &= \eta(\beta - \Lambda\beta^{-1}\frac{db}{d\lambda}). \end{aligned}$$

In particular, the last equality can be written as

$$\frac{db}{d\lambda} = (I + \eta\Lambda\beta^{-1})^{-1}\eta\beta. \quad (40)$$

Lemma 10.3: For $k, j \in N$

$$\begin{aligned} \frac{d}{d\lambda_j} b_k(\boldsymbol{\lambda}) &= \sum_{i\in N} w_{ji}(1-b_i(\boldsymbol{\lambda}))^{-1} \times \\ & \left(\prod_{l\in N}(1-b_l(\boldsymbol{\lambda}))^{w_{jl}}\right) \frac{d}{d\lambda_{i'}} b_k(\boldsymbol{\lambda}). \end{aligned}$$

Proof: Consult the definition (33) of $\mathcal{E}_k(\cdot;\cdot)$ to observe that

$$\frac{d}{d\lambda_j} b_k(\boldsymbol{\lambda}) = \left[\frac{d}{d\lambda_j} + \sum_{l\in N} \frac{d}{d\lambda_j} b_l(\boldsymbol{\lambda}) \frac{\partial}{\partial b_l}\right] \mathcal{E}_k(\boldsymbol{\lambda};\mathbf{b})\bigg|_{\mathbf{b}=\mathbf{b}(\boldsymbol{\lambda})}, \quad (41)$$

and that by (35)

$$\frac{d}{d\lambda_j}\mathcal{E}_k(\boldsymbol{\lambda};\mathbf{b}) = \eta_k(\boldsymbol{\lambda}) w_{jk} \prod_{i\in N}(1-b_i)^{w_{ji}}.$$

Note that the above equalities are valid for j' as well. We define the matrices

$$\begin{aligned} \frac{db}{d\bar{\lambda}} &= [\bar{\sigma}_{ki}]_{N\times D}, \quad \text{where} \quad \bar{\sigma}_{ki} = \frac{d}{d\lambda_i} b_k(\boldsymbol{\lambda}) \\ \bar{\beta} &= [\bar{\beta}_{ki}]_{N\times D}, \quad \text{where} \quad \bar{\beta}_{ki} = w_{ik} \prod_{j\in N}(1-b_j)^{w_{ij}}, \end{aligned}$$

and appeal to (37)-(38) to express equalities (41) in the matrix form

$$\frac{db}{d\bar{\lambda}} = \eta(\bar{\beta} - \Lambda\beta^{-1}\frac{db}{d\lambda}).$$

Manipulation of this equality yields

$$\frac{db}{d\bar{\lambda}} = (I + \eta\Lambda\beta^{-1})^{-1}\eta\bar{\beta} = \frac{db}{d\lambda}\beta^{-1}\bar{\beta},$$

where the last equality follows from (40). The assertion of the lemma follows by componentwise consideration of this matrix equality. ∎

Lemma 10.4: For $j \in D$

$$\begin{aligned} \frac{d}{d\lambda_j} S(\boldsymbol{\lambda}) &= \left(\prod_{i\in N}(1-b_i(\boldsymbol{\lambda}))^{w_{ji}}\right) \times \\ & \left(r_j + \sum_{i\in N} w_{ji}(1-b_i(\boldsymbol{\lambda}))^{-1} \frac{d}{d\lambda_{i'}} S(\boldsymbol{\lambda})\right). \end{aligned}$$

Proof: Notice that for the form (32)

$$\frac{\partial}{\partial\lambda_j} S(\boldsymbol{\lambda};\mathbf{b}) = r_j \prod_{i\in N}(1-b_i)^{w_{ji}}.$$

Now using Lemma 10.3

$$\begin{aligned} & \frac{d}{d\lambda_j} S(\boldsymbol{\lambda}) \\ &= \left[\frac{\partial}{\partial\lambda_j} + \sum_{k\in N} \frac{d}{d\lambda_j} b_k(\boldsymbol{\lambda}) \frac{\partial}{\partial b_k}\right] S(\boldsymbol{\lambda};\mathbf{b}) \\ &= \left(\prod_{i\in N}(1-b_i(\boldsymbol{\lambda}))^{w_{ji}}\right) \times \\ & \left(r_j + \sum_{k\in N}\sum_{i\in N} w_{ji}(1-b_i(\boldsymbol{\lambda}))^{-1} \frac{d}{d\lambda_{i'}} b_k(\boldsymbol{\lambda}) \frac{\partial}{\partial b_k} S(\boldsymbol{\lambda};\mathbf{b})\right) \\ &= \left(\prod_{i\in N}(1-b_i(\boldsymbol{\lambda}))^{w_{ji}}\right) \times \\ & \left(r_j + \sum_{i\in N} w_{ji}(1-b_i(\boldsymbol{\lambda}))^{-1} \sum_{k\in N} \frac{d}{d\lambda_{i'}} b_k(\boldsymbol{\lambda}) \frac{\partial}{\partial b_k} S(\boldsymbol{\lambda};\mathbf{b})\right) \\ &= \left(\prod_{i\in N}(1-b_i(\boldsymbol{\lambda}))^{w_{ji}}\right) \times \\ & \left(r_j + \sum_{i\in N} w_{ji}(1-b_i(\boldsymbol{\lambda}))^{-1} \frac{d}{d\lambda_{i'}} S(\boldsymbol{\lambda})\right). \end{aligned}$$

∎

Lemma 10.5: For $k, j \in N$

$$\Delta_j b_k(\boldsymbol{\lambda}) = -(1-b_j(\boldsymbol{\lambda}))^{-1} \frac{d}{d\lambda_{j'}} b_k(\boldsymbol{\lambda}).$$

Proof: Let $\tilde{b}_k(\boldsymbol{\lambda};\cdot) : R_+^N \mapsto [0,1]$ be the piecewise linear function obtained by linearly interpolating $b_k(\boldsymbol{\lambda})$ at integer values of the threshold vector $\boldsymbol{\kappa} = (\kappa_i : i \in N)$. In the scope of the proof, we shall interpret $\frac{d}{d\kappa_j}\tilde{b}_k(\boldsymbol{\lambda};\boldsymbol{\kappa})$ as the left derivative; so that in particular

$$\Delta_j b_k(\boldsymbol{\lambda}) = \frac{d}{d\kappa_j}\tilde{b}_k(\boldsymbol{\lambda};\boldsymbol{\kappa}).$$

The proof of Lemma 10.2, applied to derivatives of $\tilde{b}_k$ with respect to κ_j, leads to

$$\eta_k(\mathbf{b}(\boldsymbol{\lambda}))^{-1}\Delta_j b_k(\boldsymbol{\lambda}) = -\delta_{jk} \\ -(1-b_k(\boldsymbol{\lambda}))^{-1}\left(\sum_{i\in N} w_{ik}(w_{ik}-1)\rho_i(\boldsymbol{\lambda})\right)\Delta_j b_k(\boldsymbol{\lambda}) \\ -\sum_{l\in N-k}(1-b_l(\boldsymbol{\lambda}))^{-1}\left(\sum_{i\in N} w_{il}w_{ik}\rho_i(\boldsymbol{\lambda})\right)\Delta_j b_l(\boldsymbol{\lambda}). \quad (42)$$

Define the matrix

$$\frac{db}{d\kappa} = [\tilde{\sigma}_{jk}]_{N\times N}, \quad \text{where} \quad \tilde{\sigma}_{jk} = \Delta_k b_j(\boldsymbol{\lambda}),$$

which, by (42), can be expressed as

$$\frac{db}{d\kappa} = -\eta\left(I + \Lambda\beta^{-1}\frac{db}{d\kappa}\right).$$

The lemma follows by rearranging this equality as

$$\frac{db}{d\kappa} = -\left(I+\eta\Lambda\beta^{-1}\right)^{-1}\eta = -\frac{db}{d\lambda}\beta^{-1},$$

where the last equality follows from equation (40). ■

Lemma 10.6: For $j \in N$

$$\Delta_j S(\boldsymbol{\lambda}) = -(1-b_j(\boldsymbol{\lambda}))^{-1}\frac{d}{d\lambda_{j'}}S(\boldsymbol{\lambda}).$$

Proof: We continue to interpret $\Delta_j S(\boldsymbol{\lambda})$ and $\Delta_j b_k(\boldsymbol{\lambda})$ as left derivatives of associated piecewise linear functions; so that

$$\Delta_j S(\boldsymbol{\lambda}) = \sum_{k\in N}\Delta_j b_k(\boldsymbol{\lambda})\frac{\partial}{\partial b_k}S(\boldsymbol{\lambda},\mathbf{b})\bigg|_{\mathbf{b}=\mathbf{b}(\boldsymbol{\lambda})}$$

where $S(\boldsymbol{\lambda},\mathbf{b})$ is given by (32). By substituting the value of $\Delta_j b_k(\boldsymbol{\lambda})$ given by Lemma 10.5, we obtain

$$\begin{aligned}\Delta_j S(\boldsymbol{\lambda}) &= -(1-b_j(\boldsymbol{\lambda}))^{-1}\sum_{k\in N}\frac{d}{d\lambda_{j'}}b_k(\boldsymbol{\lambda})\frac{\partial}{\partial b_k}S(\boldsymbol{\lambda};\mathbf{b}) \\ &= -(1-b_j(\boldsymbol{\lambda}))^{-1}\frac{d}{d\lambda_{j'}}S(\boldsymbol{\lambda}).\end{aligned}$$

■

Proof of Theorem 6.1 Let $S(\cdot)$ be the expected revenue function (30) with

$$r_i = \begin{cases} 0 & \text{if } i \in L \\ 1 & \text{else.}\end{cases}$$

The first-order optimality condition of the licensee's problem (8) dictates

$$\begin{aligned}\frac{d}{dp_i}F(\mathbf{p})\bigg|_{\mathbf{p}=\mathbf{p}^*} &= -\frac{d}{dp_i}Q(\boldsymbol{\lambda}(\mathbf{p}))\bigg|_{\mathbf{p}=\mathbf{p}^*} \\ &= -\alpha_i'(\mathbf{p}^*)\frac{d}{d\lambda_i}S(\boldsymbol{\lambda})\bigg|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}(\mathbf{p}^*)} \quad (43)\end{aligned}$$

for each leased cell $i \in L$. Lemmas 10.4 and 10.6 imply that for such i

$$\frac{d}{d\lambda_i}S(\boldsymbol{\lambda}) = -\left(1-\hat{B}_i(\boldsymbol{\lambda})\right)\sum_{j\in N} w_{ij}\Delta_j S(\boldsymbol{\lambda}). \quad (44)$$

The theorem now follows by substituting (44) in (43) and rearranging the terms. □

Proof of Theorem 6.2 Given $\mathbf{p}$, let $S_{\mathbf{r}(\mathbf{p})}(\cdot)$ be the expected revenue function (30) with $r_i = r_i(\mathbf{p})$ which is defined by equality (14). Note that $U(\mathbf{p}) = S_{\mathbf{r}(\mathbf{p})}(\boldsymbol{\lambda}(\mathbf{p}))$. For $i \in L$

$$\begin{aligned}\frac{d}{dp_i}U(\mathbf{p}) =& \sum_{j\in N}\left(\lambda_j(\mathbf{p})\left(1-\hat{B}_j(\boldsymbol{\lambda}(\mathbf{p}))\right)\frac{d}{dp_i}r_j(\mathbf{p}) + r_j(\mathbf{p})\frac{d}{dp_i}\left(\lambda_j(\mathbf{p})\left(1-\hat{B}_j(\boldsymbol{\lambda}(\mathbf{p}))\right)\right)\right) \\ =& \alpha_i(p_i)\left(1-\hat{B}_i(\boldsymbol{\lambda}(\mathbf{p}))\right) + \sum_{j\in N} r_j(\mathbf{p})\alpha_i(p_i)'\frac{d}{d\lambda_i}\left(\lambda_j\left(1-\hat{B}_j(\boldsymbol{\lambda})\right)\right)\bigg|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}(\mathbf{p})} \\ =& \alpha_i(p_i)\left(1-\hat{B}_i(\boldsymbol{\lambda}(\mathbf{p}))\right) + \alpha_i(p_i)'\frac{d}{d\lambda_i}\left(\sum_{j\in N} r_j(\mathbf{p})\lambda_j\left(1-\hat{B}_j(\boldsymbol{\lambda})\right)\right)\bigg|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}(\mathbf{p})} \\ =& \alpha_i(p_i)\left(1-\hat{B}_i(\boldsymbol{\lambda}(\mathbf{p}))\right) + \alpha_i(p_i)'\frac{d}{d\lambda_i}S_{\mathbf{r}(\mathbf{p})}(\boldsymbol{\lambda})\bigg|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}(\mathbf{p})}.\end{aligned}$$

Now notice by Lemmas 10.4 and 10.6 that for $i \in L$

$$\frac{d}{d\lambda_i}S_{\mathbf{r}(\mathbf{p})}(\boldsymbol{\lambda}) = \prod_{j\in N}(1-b_j(\boldsymbol{\lambda}))^{w_{ij}}\left(p_i - \sum_{j\in N} w_{ij}\Delta_j S_{\mathbf{r}(\mathbf{p})}(\boldsymbol{\lambda})\right).$$

Therefore,

$$\frac{d}{dp_i}U(\mathbf{p}) = \left(1-\hat{B}_i(\boldsymbol{\lambda}(\mathbf{p}))\right)\times \left(\alpha_i(p_i)+\alpha_i(p_i)'\left(p_i-\sum_{j\in N} w_{ij}\Delta_j S_{\mathbf{r}(\mathbf{p})}(\boldsymbol{\lambda}(\mathbf{p}))\right)\right).$$

Hence, the first-order optimality condition of the licensee's problem (8) is given by

$$\alpha_i(p_i^*) = -\alpha_i(p_i^*)'\left(p_i^* - \sum_{j\in N} w_{ij}\Delta_j U(\mathbf{p}^*)\right), \qquad i \in L,$$

and the theorem follows by a rearrangement of terms. □

Proof of Theorem 7.1 Appeal to equalities (39)-(40) to obtain the matrix equation

$$\begin{aligned}\frac{dS}{d\lambda} &= -r\rho w\beta^{-1}(I+\eta\Lambda\beta^{-1})\eta\beta \\ &= -r\rho w\beta^{-1}(I-(I+\eta\Lambda\beta^{-1})^{-1}\eta\Lambda\beta^{-1})\eta\beta,\end{aligned}$$

where the second equality uses the Matrix Inversion Lemma and can be verified directly. The matrices η and β have full rank; therefore they commute, yielding

$$\begin{aligned}\frac{dS}{d\lambda}\eta^{-1} &= -r\rho w + r\rho w\beta^{-1}(I+\eta\Lambda\beta^{-1})^{-1}\eta\Lambda \\ &= -r\rho w - \frac{dS}{d\lambda}\beta^{-1}\Lambda.\end{aligned}$$

The jth component of this vector equality is:

$$\begin{aligned}&\eta_j(\mathbf{b}(\boldsymbol{\lambda}))^{-1}\frac{d}{d\lambda_{j'}}S(\boldsymbol{\lambda}) = -\sum_{i\in D} w_{ij}\rho_i(\boldsymbol{\lambda})r_i \\ &\quad -\left(\sum_{i\in D} w_{ij}(w_{ij}-1)\rho_i(\boldsymbol{\lambda})\right)(1-b_j(\boldsymbol{\lambda}))^{-1}\frac{d}{d\lambda_{j'}}S(\boldsymbol{\lambda}) \\ &\quad -\sum_{k\in N-j}\left(\sum_{i\in D} w_{ij}w_{ik}\rho_i(\boldsymbol{\lambda})\right)(1-b_k(\boldsymbol{\lambda}))^{-1}\frac{d}{d\lambda_{k'}}S(\boldsymbol{\lambda}).\end{aligned}$$

The proof is completed by consulting Lemma 10.6 to express both sides of this equality in terms of the differences $\Delta_j S(\boldsymbol{\lambda})$ and $\Delta_k S(\boldsymbol{\lambda})$. □

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