Lecture 6: Recursive Preferences

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Basics

- Epstein and Zin (1989 JPE, 1991 Ecta) following work by Kreps and Porteus introduced a class of preferences which allow to break the link between risk aversion and intertemporal substitution.
- These preferences have proved very useful in applied work in asset pricing, portfolio choice, and are becoming more prevalent in macroeconomics.

Value function:

- To understand the formulation, recall the standard expected utility time-separable preferences are defined as

$$V_t = E_t \sum_{s=0}^{\infty} \beta^{s-t} u(c_{t+s}),$$

- We can also define them recursively as

$$V_t = u(c_t) + \beta E_t V_{t+1},$$

or equivalently:

$$V_t = (1-\beta) u(c_t) + \beta E_t \left(V_{t+1} \right).$$

EZ Preferences

- EZ preferences generalize this: they are defined recursively over current (known) consumption and a certainty equivalent $R_t\left(V_{t+1}\right)$ of tomorrow's utility V_{t+1} :

$$V_t = F\left(c_t, R_t\left(V_{t+1}\right)\right),$$

  where

$$R_t(V_{t+1}) = G^{-1}\left(E_t G(V_{t+1})\right),$$

  with F and G increasing and concave, and F is homogeneous of degree one.
- Note that $R_t(V_{t+1}) = V_{t+1}$ if there is no uncertainty on V_{t+1}.
- The more concave G is, and the more uncertain V_{t+1} is, the lower is $R_t(V_{t+1})$.

Functional forms

- Most of the literature considers simple functional forms for F and G:

$$\begin{aligned}
\rho &> 0 : F(c,z) = \left((1-\beta)c^{1-\rho} + \beta z^{1-\rho}\right)^{\frac{1}{1-\rho}}, \\
\alpha &> 0 : G(x) = \frac{x^{1-\alpha}}{1-\alpha}.
\end{aligned}$$

- For example:

$$V_t = \left((1-\beta)c_t^{1-\rho} + \beta\left(E_t V_{t+1}^{1-\alpha}\right)^{\frac{1-\rho}{1-\alpha}}\right)^{\frac{1}{1-\rho}}.$$

Limits

- Limits:

$$\begin{aligned} \rho &= 1 : F(c,z) = c^{1-\beta} z^{\beta}. \\ \alpha &= 1 : G(x) = \log x. \end{aligned}$$

- Hence

$$\begin{aligned} \alpha &> 0 : R_t\left(V_{t+1}\right) = E_t\left(V_{t+1}^{1-\alpha}\right)^{\frac{1}{1-\alpha}}, \\ \alpha &= 1 : R_t(V_{t+1}) = \exp\left(E_t \log\left(V_{t+1}\right)\right). \end{aligned}$$

Proof

- Define $f(x)$
$$F(c,z) = cf(x)$$
where $x = z/c$ and
$$f(x) = \left(1 - \beta + \beta x^{1-\rho}\right)^{\frac{1}{1-\rho}}$$
- So
$$\frac{f'(x)}{f(x)} = \frac{\beta x^{-\rho}}{1 - \beta + \beta x^{1-\rho}}$$
and
$$\lim_{\rho \to 1} \frac{f'(x)}{f(x)} = \frac{\beta}{X}.$$
- Since f continuous this implies
$$\lim_{\rho \to 1} f(x) = X^{\beta}$$
(note this is simply the proof that a CES function converges to a Cobb-Douglas as $\rho \to 1$).

Risk Aversion vs IES

- In general α is the relative risk aversion coefficient for static gambles and ρ is the inverse of the intertemporal elasticity of substitution for deterministic variations.
- Suppose consumption today is c today and consumption tomorrow is uncertain: $\{c_L, \bar{c}, \bar{c}, ...\}$ or $\{c_H, \bar{c}, \bar{c}, ...\}$, each has prob $\frac{1}{2}$.
- Utility today:

$$V = F\left(c, G^{-1}\left(\frac{1}{2}G(V_L) + \frac{1}{2}G(V_H)\right)\right)$$

  where $V_L = F(c_L, \bar{c})$ and $V_H = F(c_H, \bar{c})$.
- Curvature of G determines how adverse you are to the uncertainty.
  - If G is linear you only care about the expected value.
  - If not, this is the same as the definition of a certainty equivalent: $G(\widehat{V}) = \frac{1}{2}G(V_L) + \frac{1}{2}G(V_H)$.

Special Case: Deterministic consumption

- If consumption is deterministic: we have the usual standard time-separable expected discounted utility with discount factor β and IES $= \frac{1}{\rho}$, risk aversion $\alpha = \rho$.
- Proof: If no uncertainty, then $R_t\left(V_{t+1}\right) = V_{t+1}$ and $V_t = F\left(c_t, V_{t+1}\right)$. With a CES functional form for F, we recover CRRA preferences:

$$V_t = \left((1-\beta)c_t^{1-\rho} + \beta V_{t+1}^{1-\rho}\right)^{\frac{1}{1-\rho}}$$

$$W_t = (1-\beta)c_t^{1-\rho} + \beta W_{t+1} = (1-\beta)\sum_{j=0}^{\infty}\beta^j c_{t+j}^{1-\rho},$$

where $W_t = V_t^{1-\rho}$.

Special Case: $\alpha = \rho$

- Similarly, if $\alpha = \rho$, then the formula

$$V_t = \left((1-\beta)c_t^{1-\rho} + \beta\left(E_t V_{t+1}^{1-\alpha}\right)^{\frac{1-\rho}{1-\alpha}}\right)^{\frac{1}{1-\rho}}$$

simplifies to

$$V_t^{1-\rho} = (1-\beta)c_t^{1-\rho} + \beta\left(E_t V_{t+1}^{1-\alpha}\right)$$

- Define $W_t = V_t^{1-\rho}$, we have

$$W_t = (1-\beta)c_t^{1-\rho} + \beta E_t\left(W_{t+1}\right),$$

i.e. expected utility.

Simple example with two lotteries:

- Lotteries:
  - lottery A pays in each period $t = 1, 2, \ldots$ c_h or c_l, the probability is $\frac{1}{2}$ and the outcome is iid across period;
  - lottery B pays starting at $t = 1$ either c_h at all future dates for sure, or c_l at all future date for sure; there is a single draw at time $t = 1$.
- With expected utility, you are indifferent between these lotteries, but with EZ lottery B is prefered iff $\alpha > \rho$.
- In general, early resolution of uncertainty is preferred if and only if $\alpha > \rho$ i.e. risk aversion $> \frac{1}{IES}$. This is another way to motivate these preferences, since early resolution seems intuitively preferable.

Resolution of uncertainty

- For lottery A, the utility once you know your consumption is either c_h, or c_l, since

$$V_h = F(c_h, V_h) = \left((1-\beta)c_h^{1-\rho} + \beta V_h^{1-\rho}\right)^{\frac{1}{1-\rho}}.$$

  The certainty equivalent before playing the lottery is

$$G^{-1}\left(\frac{1}{2}G(c_h) + \frac{1}{2}G(c_l)\right) = \left(\frac{1}{2}c_h^{1-\alpha} + \frac{1}{2}c_l^{1-\alpha}\right)^{\frac{1}{1-\alpha}}.$$

- For lottery B, the values satisfy

$$W_h^{1-\rho} = (1-\beta)c_h^{1-\rho} + \beta\left(\frac{1}{2}W_h^{1-\alpha} + \frac{1}{2}W_l^{1-\alpha}\right)^{\frac{1-\rho}{1-\alpha}},$$

$$W_l^{1-\rho} = (1-\beta)c_l^{1-\rho} + \beta\left(\frac{1}{2}W_h^{1-\alpha} + \frac{1}{2}W_l^{1-\alpha}\right)^{\frac{1-\rho}{1-\alpha}},$$

Resolution of uncertainty

- We want to compare $G^{-1}\left(\frac{1}{2}G\left(W_h\right)+\frac{1}{2}G\left(W_l\right)\right)$ to $G^{-1}\left(\frac{1}{2}G\left(c_h\right)+\frac{1}{2}G\left(c_l\right)\right)$.
- Note that the function $x \to x^{\frac{1-\rho}{1-\alpha}}$ is concave if $1-\rho < 1-\alpha$, i.e. $\rho > \alpha$, and convex otherwise. As a result, if $\rho > \alpha$,

$$
\begin{aligned}
\left(\frac{1}{2}W_h^{1-\alpha}+\frac{1}{2}W_l^{1-\alpha}\right)^{\frac{1-\rho}{1-\alpha}} &\geq \frac{1}{2}\left(W_h^{1-\alpha}\right)^{\frac{1-\rho}{1-\alpha}}+\frac{1}{2}\left(W_l^{1-\alpha}\right)^{\frac{1-\rho}{1-\alpha}} \\
&= \frac{1}{2}W_h^{1-\rho}+\frac{1}{2}W_l^{1-\rho}
\end{aligned}
$$

- Also

$$
\begin{aligned}
W_h^{1-\rho} &\geq (1-\beta)c_h^{1-\rho}+\beta\left(\frac{1}{2}W_h^{1-\rho}+\frac{1}{2}W_l^{1-\rho}\right) \\
W_l^{1-\rho} &\geq (1-\beta)c_l^{1-\rho}+\beta\left(\frac{1}{2}W_h^{1-\rho}+\frac{1}{2}W_l^{1-\rho}\right)
\end{aligned}
$$

Continued

- These results imply that if $\rho > \alpha$ then

$$\frac{W_h^{1-\rho} + W_l^{1-\rho}}{2} \geq \frac{c_h^{1-\rho} + c_l^{1-\rho}}{2}.$$

  in which case the certainty equivalent of lottery A is higher than the certainty equivalent of lottery B and agents prefer late to early resolution of uncertainty.

- Technically, EZ is an extension of EU which relaxes the independence axiom. Recall the independence axion is: if $x \succeq y$, then for any z,α : $\alpha x + (1-\alpha)z \succeq \alpha y + (1-\alpha)z$. With EZ, "Intertemporal composition of risk matters": we cannot reduce compound lotteries.

Euler's Thereom:

- We have

$$V_t = \left((1-\beta) C_t^{1-\rho} + \beta R_t (V_{t+1})^{1-\rho} \right)^{\frac{1}{1-\rho}}$$

where

$$R_t (V_{t+1}) = \left(E_t \left(V_{t+1}^{1-\alpha} \right) \right)^{\frac{1}{1-\alpha}}$$

- Since V_t is homogenous of degree one, Euler's thereom implies

$$V_t = MC_t C_t + E_t MV_{t+1} V_{t+1}$$

Euler equation:

- Taking derivatives:

$$MC_t = \frac{\partial V_t}{\partial C_t} = (1-\beta) V_t^{\rho} C_t^{-\rho}$$

and

$$MV_{t+1} = \frac{\partial V_t}{\partial R_t (V_{t+1})} \frac{\partial R_t (V_{t+1})}{\partial V_{t+1}}$$

where

$$\frac{\partial V_t}{\partial R_t (V_{t+1})} = V_t^{\rho} \beta R_t (V_{t+1})^{-\rho}$$

and

$$\frac{\partial R_t (V_{t+1})}{\partial V_{t+1}} = R_t (V_{t+1})^{\alpha} V_{t+1}^{-\alpha}$$

- This implies

$$MV_{t+1} = \beta V_t^{\rho} R_t (V_{t+1})^{\alpha-\rho} V_{t+1}^{-\alpha}$$

IES

- Define the intertemporal marginal rate of substitution as

$$
\begin{aligned}
S_{t,t+1} &= \frac{MV_{t+1}MC_{t+1}}{MC_t} \\
&= \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left(\frac{V_{t+1}}{R_t\left(V_{t+1}\right)}\right)^{\rho-\alpha}
\end{aligned}
$$

- The first term is familiar. The second term is next period's value relative to its certainty equivalent.
- If $\rho = \alpha$ or there is no uncertainty so that $V_{t+1} = R_t\left(V_{t+1}\right)$ this term equals unity.

Household wealth:

- Start with the value function:

$$V_t = MC_t C_t + E_t MV_{t+1} V_{t+1}$$

- Divide by MC_t :

$$\frac{V_t}{MC_t} = C_t + E_t \left(\frac{MV_{t+1} MC_{t+1}}{MC_t} \right) \frac{V_{t+1}}{MC_{t+1}}$$

- Define

$$W_t = \frac{V_t}{MC_t}$$

then

$$W_t = C_t + E_t S_{t,t+1} W_{t+1}$$

is the present-discounted value of wealth.

The return on wealth

- Define the cum-dividend return on wealth:

$$R_{m,t+1} = \frac{W_{t+1}}{W_t - C_t}$$

- Note that

$$W_{t+1} = \frac{V_{t+1}}{MC_{t+1}} = \frac{V_{t+1}^{1-\rho} C_{t+1}^{\rho}}{1-\beta}$$

Hence

$$R_{m,t+1} = \frac{V_{t+1}^{1-\rho} C_{t+1}^{\rho}}{V_t^{1-\rho} C_t^{\rho} - C_t} = \left(\frac{C_{t+1}}{C_t}\right)^{\rho} \left(\frac{V_{t+1}^{1-\rho}}{V_t^{1-\rho} - (1-\beta) C_t^{1-\rho}}\right)$$

- Now use fact that

$$V_t^{1-\rho} = (1-\beta)\, C_t^{1-\rho} + \beta R_t (V_{t+1})^{1-\rho}$$

to obtain

$$R_{m,t+1} = \left[\beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left(\frac{R_t\,(V_{t+1})}{V_{t+1}}\right)^{1-\rho}\right]^{-1}$$

Certainty Equivalent

- Use this equation to solve for the value function relative to the certainty equivalent:

$$R_{m,t+1}^{-1} = \left[\beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left(\frac{R_t\left(V_{t+1}\right)}{V_{t+1}}\right)^{1-\rho}\right]$$

$$\frac{V_{t+1}}{R_t\left(V_{t+1}\right)} = \left(\beta R_{m,t+1} \left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right)^{1/(1-\rho)}$$

- Comment: we can use this to directly evaluate the cost of uncertain returns and consumption.

SDF:

- From above:

$$S_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left(\frac{V_{t+1}}{R_t\left(V_{t+1}\right)}\right)^{\rho-\alpha}$$

$$= \beta^{\theta} R_{m,t+1}^{\theta-1} \frac{C_{t+1}}{C_t}^{-\frac{\theta}{\psi}}$$

where

$$\theta = \frac{1-\alpha}{1-\rho} \; and \;\; \psi = 1/\rho$$

- Note if $\rho = \alpha$ we have

$$S_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho}$$

- Now take logs

$$\log S_{t,t+1} = \theta \log \beta - \frac{\theta}{\psi} \Delta c_{t+1} - (1-\theta)\, r_{m,t+1}$$

Expected returns

- The return on the ith asset satisfies:

$$E_t S_{t,t+1} R^i_{t,t+1} = 1$$

- Taking logs:

$$\begin{aligned}
\log\left(\frac{E_t R^i_{t,t+1}}{R^f_{t+1}}\right) &= -cov(\log S_{t,t+1}, \log R^i_{t,t+1}) \\
&= \frac{\theta}{\psi}\left(cov\left(\Delta c_{t+1}, r_{i,t+1}\right)\right) \\
&+ (1-\theta)\, cov(r_{m,t+1}, r_{i,t+1})
\end{aligned}$$

- Epstein-Zin is a linear combination of the CAPM and the CCAPM model.

Market return:

- For the market return we have

$$\log\left(\frac{ER_m}{R^f}\right) = \frac{\theta}{\psi} cov\left(\Delta c, r_m\right) + \left(1-\theta\right)\sigma_m^2$$

- We can write as:

$$r_m + \frac{\sigma_m}{2} = r_f + \frac{\theta}{\psi} cov\left(\Delta c, r_m\right) + \left(1-\theta\right)\sigma_m^2$$

Special case: $\rho = 1$

- In this case
$$R_{m,t+1} = \left(\beta\left(\frac{C_{t+1}}{C_t}\right)^{-1}\right)^{-1}$$
and
$$\sigma_{\Delta_c} = \sigma_m$$
- This implies that:
$$\log\left(\frac{ER_m}{R^f}\right) = \sigma_m^2$$

Risk free rate:

- We have:

$$\log R^f_{t+1} \quad = \quad \log E_t \exp(-\log S_{t,t+1})$$

- In logs:

$$\begin{aligned} r^f_{t+1} &= -\theta \log \beta + \frac{\theta}{\psi} E_t \Delta c_{t+1} + (1-\theta)\, r_m \\ &- \left(\frac{\theta}{\psi}\right)^2 \frac{\sigma^2_{\Delta c}}{2} - (1-\theta)^2 \frac{\sigma^2_m}{2} - \frac{\theta\,(1-\theta)}{\psi} cov(\Delta c, r_m) \end{aligned}$$

- Substitute in for the market return to obtain:

$$\begin{aligned} (1-\theta)\, r_m &= (1-\theta)\, r_f - \frac{(1-\theta)\,\sigma_m}{2} + \\ &+ (1-\theta)^2\, \sigma_m + \frac{(1-\theta)\,\theta}{\psi} cov\,(\Delta c, r_m) \end{aligned}$$

Risk free rate:

- Simplify

$$r_t^f = -\log\beta + \frac{1}{\psi}E_t\Delta c_{t+1} - \frac{\theta}{\psi^2}\frac{\sigma_{\Delta c}^2}{2} - (1-\theta)\frac{\sigma_m^2}{2}$$

- Again if $\rho = \alpha$ so $\theta = 1$ we have the standard risk-free rate equation.
- If $\alpha > \rho$ then $\theta < 1$ and the volatility from the market return reduces the real interest rate.

Iid Consumption

- Let

$$\Delta C_{t+1} = g + \sigma_c \varepsilon_{t+1}$$

- Let $v_t = \frac{V_t}{C_t}$ and write value function as

$$v_t = \left(1 - \beta + \beta E_t \left(v_{t+1}^{1-\alpha} \left(\frac{C_{t+1}}{C_t} \right)^{1-\alpha} \right)^{\frac{1-\rho}{1-\alpha}} \right)^{\frac{1}{1-\rho}}$$

- Since consumption is iid v is constant.

SDF with iid consumption

- With $v_t = v$

$$
\begin{aligned}
S_{t,t+1} &= \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left(\frac{V_{t+1}}{R_t\left(V_{t+1}\right)}\right)^{\rho-\alpha} \\
&= \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\alpha} \left(\frac{1}{E_t\left(\left(\frac{C_{t+1}}{C_t}\right)^{1-\alpha}\right)}\right)^{-(1-\theta)}
\end{aligned}
$$

- Take logs:

$$
\begin{aligned}
\log S_{t,t+1} &= \log\beta - \alpha\Delta c_{t+1} + (1-\theta)\log E_t \exp((1-\alpha)\Delta c_{t+1}) \\
&= \log\beta - \alpha\Delta c_{t+1} + (\alpha-\rho)\, g + (1-\theta)\,(1-\alpha)^2 \,\frac{\sigma_c}{2}
\end{aligned}
$$

Risk free rate with iid consumption

- Risk free rate:

$$\begin{aligned} r_f &= -\log E_t S_{t,t+1} = -\left(E_t \log S_{t,t+1} + \frac{\sigma_s^2}{2}\right) \\ &= -\log\beta + \rho g - \left[(1-\theta)(1-\alpha) + \alpha^2\right]\frac{\sigma_c^2}{2} \end{aligned}$$

- If $\rho = \alpha$ this is the standard expression

$$r_f = -\log\beta + \rho g - \rho^2 \frac{\sigma_c^2}{2}$$

Price-Dividend ratio

- Log dividend price ratio: Conjecture a constant q

$$q = E_t S_{t,t+1} \frac{C_{t+1}}{C_t} (1+q)$$

where

$$\begin{aligned}
\log S_{t,t+1} + \Delta c_{t+1} &= \log \beta + (1-\alpha)\, \Delta c_{t+1} \\
&+ (1-\theta) \log E_t \exp((1-\alpha)\Delta c_{t+1}) \\
&= \log \beta + (1-\alpha)\, \Delta c_{t+1} \\
&+ (\alpha - \rho)\, g + (1-\theta)\, (1-\alpha)^2 \, \frac{\sigma_c}{2}
\end{aligned}$$

Gordon-Growth Formula

- So the price-dividend ratio satsifies:

$$\log \frac{q}{1+q} = \log \beta + (1-\rho)g - (1-\alpha)^2 \theta \frac{\sigma_c^2}{2}$$
$$= -r_f + \left(g + \frac{\sigma_c^2}{2}\right) - \alpha\sigma_c^2$$

  where the term in brackets is expected consumption growth $\log(E_t C_{t+1}/C_t)$.
- Hence this is a risk-adjusted Gordon growth formula.

Risk Premium

- The risk premium on a consumption claim is then

$$\log E_t R_{t+1} = \log E_t \frac{q+1}{q} \frac{C_{t+1}}{C_t}$$

so that

$$r_m + \frac{\sigma_m}{2} - r_f = \alpha \sigma_c^2$$

Consumption-Wealth Ratio:

- Start with the identity:

$$W_{t+1} = R_{m,t+1}\left(W_t - C_t\right)$$

to obtain the log-linear equation:

$$\Delta w_{t+1} = r_{m,t+1} + k + \left(1 - \frac{1}{\rho}\right)\left(c_t - w_t\right)$$

where $\rho = 1 - \exp(c - w)$.

- Rearrange to obtain

$$\begin{aligned}(1-\rho)\left(c_t - w_t\right) &= \rho r_{m,t+1} - \rho \Delta w_{t+1} + \rho k \\ &= \rho r_{m,t+1} + \rho\left[\Delta\left(c_{t+1} - w_{t+1}\right) - \Delta c_{t+1}\right] + \rho k\end{aligned}$$

- Present value relationship:

$$\begin{aligned}c_t - w_t &= \rho\left(r_{m,t+1} - \Delta c_{t+1}\right) + \rho\left(c_{t+1} - w_{t+1}\right) + \rho k \\ &= \sum_{s=1}^{\infty} \rho^s \left[r_{m,t+s} - \Delta c_{t+s}\right] + \frac{\rho}{1-\rho} k\end{aligned}$$

Present value expression

- Now combine the risk free and market rate Euler equations to obtain:

$$r_{m,t+s} - \Delta c_{t+s} = (1 - \psi) \, r_{m,t+s} - \mu_m$$

where μ_m is a constant that depends on conditional covariances etc..

$$c_t - w_t = (1 - \psi) \, E_t \sum_{s=1}^{\infty} \rho^s r_{m,t+s} + \frac{\rho \, (\kappa - \mu_m)}{1 - \rho}$$

- The consumption-wealth ratio is an increasing function of expected future returns if the IES < 1.
- Note, we started with an identity and combined it with the Euler equation for safe vs risky returns for a given IES. Thus these expressions are general and do not depend specifically on EZ preferences.

Unexpected changes in consumption

- Now use

$$\begin{aligned} c_{t+1} - E_t c_{t+1} &= W_{t+1} - E_t W_{t+1} \\ &+ (1-\psi)\left(E_{t+1} - E_t\right) \sum_{s=1}^{\infty} \rho^s r_{m,t+s+1} \\ &= r_{m,t+1} - E_t r_{m,t+1} \\ &+ (1-\psi)\left(E_{t+1} - E_t\right) \sum_{s=1}^{\infty} \rho^s r_{m,t+s+1} \end{aligned}$$

- Unexpected returns increase consumption growth.
- Unexpected future returns increase current consumption growth if the IES < 1.

Some comments

- If returns are not forecastable, the consumption-wealth ratio is a constant.
- In this case, consumption volatility equals the volatility of wealth, or equivalently the market return.
- In the data this is obviously not true – hence returns must be predictable.

Asset pricing implications:

- We can now compute

$$cov_t\left(r_{i,t+1}, \Delta c_{t+1}\right) = \sigma_{ic} = \sigma_{im} + (1-\psi)\,\sigma_{ih}$$

where σ_{ih} is the covariance of $r_{i,t+1}$ with the surprise in future market returns:

$$\sigma_{ih} = cov(r_{i,t+1}, (E_{t+1} - E_t)\sum_{s=1}^{\infty} \rho^s r_{m,t+s+1})$$

Epstein-Zin preferences and Risk Premiums:

- Using EZ preferences, the risk premium is:

$$E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} = \theta \frac{\sigma_{ic}}{\psi} + (1-\theta)\,\sigma_{im}$$

- The risk premium for asset i depends on its covariance between current returns and its covariance with news about future market returns:

$$E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} = \alpha\sigma_{im} + (\alpha - 1)\,\sigma_{ih}$$

  where α is the coefficient of relative risk aversion.

- Note we don't need to know the IES or consumption growth to price risk in this framework.

EZ preferences and the Equity premium puzzle:

- For EZ preferences we can now write the risk premium on the market return as: .

$$E_t r_{m,t+1} - r_{f,t+1} + \frac{\sigma_m^2}{2} = \alpha \sigma_m^2 + (\alpha - 1)\, \sigma_{mh}$$

- If returns are unforecastable, $\sigma_{ih} = 0$. Given $\sigma_{im} = 0.17$ we need $\alpha = 2$ to obtain a risk premium of 6% So we succeed in matching the risk premium with low relative risk aversion but fail on the fact that the consumption-wealth ratio will be a constant, and consumption volatility should equal wealth volatility.
- If there is mean reversion and future returns are negatively correlated with current returns then $\sigma_{mh} < 0$ we would need a higher α. Since mean-reversion is difficult to determine, the estimate could be substantially higher.

Predictable consumption growth:

- Consumption growth:

$$\begin{aligned} \Delta c_{t+1} &= g + x_t + u_t \\ x_t &= \phi x_{t-1} + v_t \end{aligned}$$

- Again use Campbell-Shiller decomposition:

$$r_{t+1} = \rho q_{t+1} - q_t + \Delta c_{t+1}$$

where $q_t = P_t / C_t$ the price of a consumption claim and

$$\rho = \frac{q}{1+q} < 1$$

- Solving forward

$$q_t = E_t \sum_{s=1}^{\infty} \rho^s \left[\Delta c_{t+s+1} - r_{t+s+1} \right]$$

Price-dividend ratio

- Euler equation

$$\Delta c_{t+s} = \psi r_{t+s}$$

where ψ is the IES so that

$$[\Delta c_{t+s+1} - r_{t+s+1}] = \left[1 - \frac{1}{\psi}\right] \Delta c_{t+s+1}$$

- The price-dividend ratio satisifies

$$q_t = \left(\frac{\rho\phi}{1 - \rho\phi}\right) \left[1 - \frac{1}{\psi}\right] x_t$$

Implications

- If the $IES > 1$ then an increase in current consumption growth causes an increase in the price-dividend ratio.
- Intuition: A persistent increase in consumption growth provides news about future cash flows and discount rates that go in opposite directions.
- If the IES is high, interest rates don't need to move very much in response to the change in consumption growth. The cash flow effect dominates.

Comments

- We need a high ϕ to get large volatility in the price-dividend ratio.
- But this comes from predictable dividend growth not from time-varying returns (the risk free rate is moving but the risk premium is not).

Time-varying volatility:

- Now add time-varying volatility:

$$\begin{aligned} x_t &= \phi x_{t-1} + \sigma_t u_t \\ \sigma_t &= (1-\gamma)\sigma + \gamma \sigma_{t-1} + v_t \end{aligned}$$

  We then have a solution of the form

$$q_t = \left(\frac{\rho\phi}{1-\rho\phi}\right)\left[1 - \frac{1}{\psi}\right] x_t + a\sigma_t^2$$

  where $a > 0$ if $IES > 0$ and risk-aversion > 0.

- We will also get time-varying risk premia – "discount rate news" that offsets the "cash flow" news of the consumption growth shock.
- In other words, we need time-varying volatility to match the equity premium combined with persistent movements in consumption growth to match the volatility of the price dividend ratio.

Calibration and empirical implementation

- Calibration:
  - IES = 1.5, $\alpha = 10$
  - Very high persistence and volatility for shocks to volatility and peristent consumption-growth process.
- Issues to think about:
  - $IES > 1$ is controversial.
  - Difficult to estimate long-run risk.