

## NONADIABATIC CORRECTIONS TO THE HANNAY–BERRY PHASE\*

SEAN B. ANDERSSON†

**Abstract.** The effect of the Coriolis force on a moving system can be described as a holonomy with respect to a particular connection known as the Cartan–Hannay–Berry connection. The resulting geometric phase is called the Hannay–Berry phase, and it provides direct information about the imposed motion on the system. This approach assumes that the imposed motion is adiabatic. In this paper we describe the use of Hamiltonian perturbation theory to develop nonadiabatic corrections to the Hannay–Berry phase for a moving system. The technique is illustrated by applying it to a rotating free-floating spring-jointed equal-sided four-bar mechanism.

**Key words.** geometric phases, perturbations, systems with slow and fast motions, averaging of perturbations

**AMS subject classifications.** 81Q70, 37J40, 70K70, 70K65

**DOI.** 10.1137/040606600

**1. Introduction.** When a system undergoes an imposed motion, the external forces can alter the natural dynamics. A simple example is the Foucault pendulum, where the rotation of the Earth causes a precession of the swing plane of the pendulum. Classically the effect of the external forces is captured by introducing fictitious forces (the centrifugal, Coriolis, and Euler forces) into the moving system (see, e.g., [26]). When the rate of the imposed motion is slow with respect to the time scale of the nominal dynamics, that is, with respect to the dynamics in the absence of any external forces, one does not expect the imposed motion to fundamentally alter the behavior of the system. (This is essentially a statement of the averaging principle. It should be kept in mind that this principle is simply one of physical intuition and not a theorem. See [6] for further comments.) For the Foucault pendulum, the rotation of the Earth slowly shifts the swing plane; on short time scales the motion of the bob is well approximated by ignoring the effects of the imposed rotation. (See [2] for a historical narrative of Foucault and the pendulum experiment.)

An effect analogous to the precession of the swing plane of the Foucault pendulum due to the Coriolis force has been observed in many other rotating dynamical systems including rotating vibrating beams [18], tuning forks [27], and shells [10]. Vibratory gyroscopes take advantage of the effect to sense the imposed rotation, and a wide variety of designs have been proposed and built [24]. Most modern analyses of slowly rotating systems are linear in nature and view the Coriolis force as providing a coupling between two vibratory modes of the system (see, e.g., [30]). However, it is desirable to have a method which, at least in principle, can be extended to a nonlinear theory and which provides a unified setting for understanding a variety of systems in which

---

\*Received by the editors April 12, 2004; accepted for publication (in revised form) March 30, 2005; published electronically October 3, 2005. This research was supported in part by Army Research Office ODDR&E MURI97 Program grant DAAG55-97-1-0114 to the Center for Dynamics and Control of Smart Structures (through Harvard University), Army Research Office ODDR&E MURI01 Program grant DAAD19-01-1-0465 to the Center for Communicating Networked Control Systems (through Boston University), and Air Force Office of Scientific Research grant F49620-01-0415, and by a fellowship from the ARCS foundation.

<http://www.siam.org/journals/siap/66-1/60660.html>

†Division of Engineering and Applied Sciences, Harvard University, Cambridge, MA 02138 (sanderss@deas.harvard.edu).

the Coriolis force plays a role. A technique developed by Marsden, Montgomery, and Ratiu, known as the moving systems approach, provides such a method [25]. The moving systems approach descends from the classical work of Cartan [11] and describes the effect of the Coriolis force as a geometric phase (holonomy) with respect to a particular Ehresmann connection known as the Cartan–Hannay–Berry connection. Research on geometric phases in physical systems was spurred by the work of Berry in the early 1980s on the geometric shift of the phase in a quantum system as a result of an adiabatic variation of a parameter-dependent Hamiltonian [7]. In previous work we have applied the moving systems approach to show that the Hannay–Berry phase in a rotating equal-sided four-bar mechanism is zero [4] and to show that the precession of the nodal points in a rotating vibrating ring can be understood as a Hannay–Berry phase [3].

Inherent in the moving systems approach is an assumption of adiabaticity; that is, it is assumed that the imposed motion is infinitely slow. In practice, of course, while the imposed motion may be extremely slow with respect to the nominal dynamics of the system, it is not infinitely slow, and neglecting this fact introduces some error. By accounting for the nonadiabatic nature of the imposed motion, more accurate models of the resulting response of the system can be developed.

Since Berry’s original work on geometric phases in quantum systems, various techniques have been proposed to account for the finite rate of change of the parameters in the Hamiltonian. Berry developed an iterative scheme in which the geometric phase at each step is incorporated into the nominal dynamics [8], while other authors have showed that the Berry phase can be viewed as the first-order term in a perturbation expansion of the system [15, 33]. This work has been applied to systems such as nuclear quadrupole resonance [34], hysteresis loops in manganese acetate crystals [16], and magnetic resonance [19]. A few authors have also considered the effect of the finite rate of change of the parameters on the Hannay angles, the geometric phase in classical integrable systems. Bhattacharjee and Sen used a perturbative method [9], which was later compared by Gjaja and Bhattacharjee to a classical analogue of the iterative scheme of Berry [20].

In this work we consider a perturbative approach to account for the nonadiabatic nature of the imposed motion. However, classical perturbation techniques ignore the underlying geometric structure of the Hamiltonian system to which the moving systems method applies. We will show that the averaged Hamiltonian which gives rise to the Hannay–Berry phase Poisson commutes with the nominal Hamiltonian, using the canonical Poisson bracket on the phase space. This leads us to Hamiltonian perturbation theory and Hamiltonian normal forms. We note that when not considering the imposed motion as a small perturbation from the nominal system, the effect of the external forces is often incorporated into an amended potential (see, e.g., [26]). Here we wish to develop a technique which takes advantage of the fact that the imposed motion is slow when describing the resulting system dynamics. If the original system is integrable, then the techniques developed here are similar to a perturbation expansion in the action-angle variables as in [23].

In the next section we briefly recall the moving system approach, and in section 3 give a short introduction to Hamiltonian normal form theory. In section 4 the theory of normal forms is applied to the moving systems approach to develop higher-order corrections. The technique is illustrated in section 5 by applying it to a spring-jointed equal-sided four-bar mechanism, where we show that the effect of an imposed rotation is zero to second order.

**2. The moving systems approach.** Inspired by classical examples such as the Foucault pendulum and the ball in a hoop, one is naturally led to consider the effect in the phase space of a mechanical system as a parameter is slowly varied along a closed loop in parameter space. Well-known examples in quantum physics, optics, and other settings [7, 31, 35] reveal that the essential calculation is a geometric one and is in fact captured by the holonomy of a connection on a fiber bundle. (For background material on fiber bundles, connections, and holonomy see [17, 28].)

Marsden, Montgomery, and Ratiu have developed an approach to moving systems using the tools of Ehresmann connections on fiber bundles [25]. Here we provide a brief review of their approach. Let  $S$  be a Riemannian manifold, and let  $M$  be the space of embeddings of a manifold  $Q$  into  $S$ . We think of  $S$  as the ambient space in which  $Q$  is being moved and of  $Q$  as the configuration space for a system of interest. A tangent vector to  $M$  at  $m$  is a map  $u_m : Q \rightarrow TS$  such that  $u_m(q) \in T_{m(q)}S$ . Given a tangent vector  $u_m(q)$ , one can construct a tangent vector to  $T_qQ$  as follows. Relative to the metric on  $S$ , orthogonally project  $u_m(q)$  to  $T_{m(q)}m(Q) \subset (T_qm)(T_qQ)$ , denote this vector  $u_m^T(q)$ , and then pull-back  $u_m^T(q)$  by  $Tm^{-1}$  to  $T_qQ$ . This natural construction defines an Ehresmann connection on the product bundle  $\pi : Q \times M \rightarrow M$  as follows.

DEFINITION 2.1 (see [25]). *The Cartan connection on  $\pi : Q \times M \rightarrow M$  is given by the vertical-valued one-form  $\gamma_c$  defined by*

$$(2.1) \quad \gamma_c(q, m)(v_q, u_m) = (v_q + (T^{-1}m \circ u_m^T)q, 0).$$

The Cartan connection induces a connection on  $\rho : T^*Q \times M \rightarrow M$  as follows.

DEFINITION 2.2 (see [25]). *The induced Cartan connection on  $\rho : T^*Q \times M \rightarrow M$  is given by the vertical-valued one-form  $\gamma_o$  defined by*

$$(2.2) \quad \gamma_o(\alpha_q, m)(U_{\alpha_q}, u_m) = (U_{\alpha_q} + X_{\mathcal{P}(u_m)}(\alpha_q), 0),$$

where  $\mathcal{P}(u_m)$  is the function defined by

$$(2.3) \quad (\mathcal{P}(u_m))\alpha_q = \alpha_q \cdot (T^{-1}m \circ u_m^T)(q)$$

and  $X_{\mathcal{P}(u_m)}$  is the Hamiltonian vector field of  $\mathcal{P}(u_m)$ .

To separate the effects of the imposed motion on the system (as defined by the embeddings  $m_t$ ) from the nominal dynamics (when the imposed motion is zero) we assume we are given a left action of a compact Lie group  $G$  on  $T^*Q$  with respect to which we can define an average and then make the following definition.

DEFINITION 2.3 (see [25]). *The Cartan–Hannay–Berry connection on  $\rho : T^*Q \times M \rightarrow M$  is given by the vertical-valued one-form  $\gamma$  defined by*

$$(2.4) \quad \gamma(\alpha_q, m)(U_{\alpha_q}, u_m) = (U_{\alpha_q} + X_{\langle \mathcal{P}(u_m) \rangle}(\alpha_q), 0),$$

where  $\langle \cdot \rangle$  denotes the average with respect to the action of the Lie group  $G$ .

In [25] Marsden, Montgomery, and Ratiu show that this is an Ehresmann connection. The horizontal lift of a vector field  $Z$  on  $M$  relative to  $\gamma$  is

$$(2.5) \quad (\text{hor } Z)(\alpha_q, m) = (-X_{\langle \mathcal{P}(Z(m)) \rangle}(\alpha_q), Z(m)).$$

DEFINITION 2.4 (see [25]). *The holonomy of the Cartan–Hannay–Berry connection is called the Hannay–Berry phase for a moving system.*

**2.1. The adiabatic assumption.** The Hannay–Berry phase captures the effects of the imposed motion on a system under the assumption that this imposed motion is slow with respect to the nominal dynamics. To better understand this adiabatic assumption, we consider the following system (as in [26]). If a particle in  $Q$  is following a curve  $q(t)$  and if  $Q$  is in turn being moved in the ambient space  $S$  by superposing the motion  $m_t$ , then the path of the particle in  $S$  is given by  $m_t(q(t))$ . The velocity in  $S$  is then

$$(2.6) \quad T_{q(t)}m_t\dot{q}(t) + \mathcal{Z}_t(m_t(q(t))),$$

where  $\mathcal{Z}_t(m_t(q)) = \frac{d}{dt}m_t(q)$  (with  $q$  viewed as fixed). The standard Lagrangian is given by the kinetic energy minus the potential energy,

$$(2.7) \quad L(q, v) = \frac{1}{2}\|T_{q(t)}m_tv + \mathcal{Z}_t(m_t(q))\|^2 - V(q) - U(m_t(q)).$$

Here  $V$  is a given potential on  $Q$ , and  $U$  is a given potential on  $S$ . To compute the associated Hamiltonian we take the Legendre transform. Taking the derivative of  $L$  with respect to  $v$  in the direction  $w$  yields

$$\frac{\partial L}{\partial v} \cdot w = p \cdot w = \langle T_{q(t)}m_tv + \mathcal{Z}_t(m_t(q(t)))^T, T_{q(t)}m_tw \rangle_{m_t(q(t))},$$

where  $p \cdot w$  is the natural pairing between the covector  $p \in T_{q(t)}^*Q$  and the vector  $w \in T_{q(t)}Q$ ,  $\langle \cdot, \cdot \rangle_{m_t(q(t))}$  denotes the metric inner product on  $S$  at the point  $m_t(q(t))$ , and  $T$  denotes the orthogonal projection to  $Tm_t(Q)$  using the metric of  $S$  at  $m_t(q(t))$ .  $Q$  inherits a metric from  $S$  such that  $m_t$  is an isometry for each  $t$ . Thus

$$(2.8) \quad \begin{aligned} p \cdot w &= \left\langle v + (T_{q(t)}m_t)^{-1} \mathcal{Z}_t^T(m_t(q(t)))^T, w \right\rangle_{q(t)} \\ &\Rightarrow p = \left( v + (T_{q(t)}m_t)^{-1} \mathcal{Z}_t^T(m_t(q(t)))^T \right)^{\flat} \triangleq (v + Z_t)^{\flat}, \end{aligned}$$

where we have defined the tangent vector  $Z_t$ . Here  $\flat : T_qQ \rightarrow T_q^*Q$  is the map defined by

$$(2.9) \quad z^{\flat} \cdot w = \langle z, w \rangle_q \quad \forall w \in T_qQ.$$

The Hamiltonian for the moving system is given by

$$(2.10) \quad H(q, p) = \frac{1}{2}\|p\|^2 - \mathcal{P}(Z_t) - \frac{1}{2}\|\mathcal{Z}_t^{\perp}\|^2 + V(q) + U(m_t(q)),$$

where  $\mathcal{Z}_t^{\perp} = \mathcal{Z}_t - \mathcal{Z}_t^T$  is the orthogonal complement of  $\mathcal{Z}_t$  and  $\mathcal{P}(Z_t)$  is the function on  $T^*Q$  (defined in (2.3)) given by

$$\mathcal{P}(Z_t)(q, p) = p \cdot Z_t(q).$$

The nominal Hamiltonian  $H_0$  is defined by setting  $\mathcal{Z}_t = 0$  and  $U = 0$ . The term  $\mathcal{P}(Z_t)$  captures what are classically referred to as the Coriolis terms, and  $\|\mathcal{Z}_t^{\perp}\|^2$  captures the centrifugal terms.

Recall now that we have a compact Lie group  $G$  acting on  $T^*Q$  on the left. We assume that the group action leaves the nominal Hamiltonian invariant. Applying the corresponding average, we obtain

$$(2.11) \quad \langle H \rangle(q, p) = \frac{1}{2}\|p\|^2 - \langle \mathcal{P}(Z_t) \rangle - \frac{1}{2}\langle \|\mathcal{Z}_t^{\perp}\|^2 \rangle + V(q) + \langle U(m_t(q)) \rangle.$$

Invoking the adiabatic assumption, we discard  $\langle \|\mathcal{Z}_t^\perp\|^2 \rangle$  since it is small with respect to the other terms in the averaged Hamiltonian. After discarding the centrifugal terms, the dynamics of the Hamiltonian system are governed by the Hamiltonian vector field

$$(2.12) \quad X_{\langle H \rangle} = X_{H_0} - X_{\langle \mathcal{P}(Z_t) \rangle} + X_{\langle U_{\text{om}t} \rangle}.$$

The second term captures the effect of the imposed motion in the adiabatic limit and is precisely the term given by the horizontal lift of the vector field  $\mathcal{Z}_t$  with respect to the Cartan–Hannay–Berry connection as defined in (2.5).

**2.2. Geometric character of the Hannay–Berry phase.** The effect of the vector field  $X_{\langle \mathcal{P}(Z_t) \rangle}$  is geometric in nature. By this we mean that the resulting change in the system is independent of the parametrization of the curve followed in the base space  $M$ ; i.e., the effect depends only on the loop itself and not on how it is traversed. To see this explicitly, recall that the vector field  $-X_{\langle \mathcal{P}(Z_t) \rangle}$  is the horizontal lift of a vector field  $\mathcal{Z}_t$  on the base space  $M$  to the fiber  $T^*Q$  with respect to the Cartan–Hannay–Berry connection and is thus a linear map of  $\mathcal{Z}_t$ . Denoting points in  $T^*Q$  by  $z$ , the ordinary differential equation defining the Hannay–Berry phase may be expressed as

$$(2.13) \quad \frac{dz}{dt} = -X_{\langle \mathcal{P}(Z_t) \rangle} = D(z)\mathcal{Z}_t.$$

In coordinates,  $D(z)$  is a matrix taking tangent vectors on  $M$  to tangent vectors on  $T^*Q$ . We now change the time parametrization by taking  $t \mapsto \tau(t)$  with  $\frac{d\tau}{dt}$  strictly positive. Under this new parametrization, the vector field  $\mathcal{Z}_t$  is scaled by  $\frac{d\tau}{dt}$ , and thus

$$(2.14) \quad \frac{dz}{dt} = \frac{dz}{d\tau} \frac{d\tau}{dt} = D(z) \left( \frac{d\tau}{dt} \right) \mathcal{Z}_\tau.$$

We then have

$$(2.15) \quad \frac{dz}{d\tau} = D(z)\mathcal{Z}_\tau,$$

which has the same form as the differential equation in the original parametrization.

**3. Hamiltonian normal forms.** The theory of Hamiltonian normal forms (or Hamiltonian perturbation theory) is a generalization of Lie perturbation techniques (see, e.g., [12]), which in turn is built upon the perturbation methods developed by Poincaré and von Ziepel (see [5] for historical comments). In this section we provide a brief description of the theory and refer the reader to [13, 14] for more details and further references.

Recall the definition of a Poisson manifold.

**DEFINITION 3.1.** A Poisson manifold is a smooth manifold  $M$  together with a  $\mathbb{R}$ -bilinear map on  $C^\infty(M)$ ,

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M),$$

which for all  $f, g, h \in C^\infty(M)$  satisfies

- (i) *skew symmetry:*  $\{f, g\} = -\{g, f\}$ ,
- (ii) *Leibniz identity:*  $\{f, gh\} = \{f, g\}h + g\{f, h\}$ ,
- (iii) *Jacobi identity:*  $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ .

Consider a Poisson manifold  $(M, \{\cdot, \cdot\})$ . Let  $\mathcal{F}(M)$  be the vector space of formal power series in  $\epsilon$  with coefficients in  $C^\infty(M)$ . That is,

$$(3.1) \quad \mathcal{F}(M) = \left\{ f_\epsilon \in C^\infty(M) \left| f_\epsilon = \sum_{i=0}^{\infty} \epsilon^i f_i, f_i \in C^\infty(M) \right. \right\}.$$

Let  $\text{ad}_f g = \{g, f\}$  and define  $\text{ad}_f^0 g = g$ . Recursively define  $\text{ad}_f^i$  by

$$(3.2) \quad \text{ad}_f^i g = \{\text{ad}_f^{i-1} g, f\}.$$

DEFINITION 3.2. *The Lie series of  $f$  is the formal power series*

$$(3.3) \quad \phi_\epsilon^f = \exp(\epsilon \text{ad}_f) = \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \text{ad}_f^i.$$

Here  $\phi_\epsilon^f$  is the formal flow of the Hamiltonian vector field  $X_f$  with  $\epsilon$  as the time parameter.

DEFINITION 3.3. *For  $f \in \mathcal{F}(M)$ ,  $X_f$  is said to have periodic flow if there exists a positive, smooth function  $T$  on  $M$  such that for all  $m \in M$  and for all  $g \in \mathcal{F}(M)$ ,  $((\phi_T^f)^*(g))(m) = g(m)$ .*

DEFINITION 3.4. *Consider  $H \in \mathcal{F}(M)$  and suppose  $X_{H_0}$  has periodic flow.  $H$  is said to be in normal form with respect to  $H_0$  if  $\{H_0, H_i\} = 0$  for  $i = 1, 2, \dots$  and in normal form up to order  $n$  with respect to  $H_0$  if  $\{H_0, H_i\} = 0$ ,  $i = 1, 2, \dots, n$ .*

To bring a Hamiltonian into normal form we will use a formal change of coordinates of the form  $\phi_\epsilon^f$  for some appropriate  $f \in \mathcal{F}(M)$ . The following lemma from [14] shows how the Hamiltonian is modified under such a change of coordinates.

LEMMA 3.5 (see [14]). *Let  $H, f \in \mathcal{F}(M)$ . If  $\phi_\epsilon^f$  is the flow of  $X_f$ , then*

$$(3.4) \quad (\phi_\epsilon^f)^* H = \exp(\epsilon \text{ad}_f) H.$$

The use of (3.3) in (3.4) yields

$$(3.5) \quad \begin{aligned} (\phi_\epsilon^f)^* H &= \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \text{ad}_f^i \left( \sum_{j=0}^{\infty} \epsilon^j H_j \right) \\ &= H_0 + \epsilon(H_1 + \text{ad}_f H_0) + \epsilon^2 \left( H_2 + \text{ad}_f H_1 + \frac{1}{2} \text{ad}_f^2 H_0 \right) + O(\epsilon^3). \end{aligned}$$

To bring  $H$  into first-order normal form, we seek a function  $f \in \mathcal{F}(M)$  such that

$$(3.6) \quad \{H_0, H_1 + \text{ad}_f H_0\} = 0.$$

To find this function we use the following lemma from [13].

LEMMA 3.6 (see [13]). *If  $X_{H_0}$  has periodic flow on  $M$ , then*

$$(3.7) \quad C^\infty(M) = \ker(\text{ad}_{H_0}) \oplus \text{im}(\text{ad}_{H_0}).$$

Let  $\langle \cdot \rangle$  denote the average over the orbits of  $H_0$ ; i.e., for  $g \in C^\infty(M)$

$$(3.8) \quad \langle g \rangle = \frac{1}{T} \int_0^T (\phi_t^{H_0})^* g dt.$$

To use Lemma 3.6 we first show the following.

LEMMA 3.7. *Let  $g \in C^\infty(M)$ . Then  $\langle g \rangle \in \ker(\text{ad}_{H_0})$ .*

*Proof.* The equation  $\text{ad}_{H_0}\langle g \rangle = \rho$  is equivalent to the dynamical system

$$\frac{d}{dt} \left( \phi_t^{H_0} \right)^* \langle g \rangle = \left( \phi_t^{H_0} \right)^* \rho$$

(see, e.g., Proposition 10.2.3 of [26]). We show that  $\rho = 0$ . The use of the definition of the average of  $g$  yields

$$\begin{aligned} \frac{d}{dt} \left( \phi_t^{H_0} \right)^* \langle g \rangle &= \frac{d}{dt} \left( \phi_t^{H_0} \right)^* \frac{1}{T} \int_0^T \left( \phi_\tau^{H_0} \right)^* g d\tau \\ &= \frac{1}{T} \frac{d}{dt} \int_0^T \left( \phi_{t+\tau}^{H_0} \right)^* g d\tau \\ &= \frac{1}{T} \frac{d}{dt} \int_t^{T+t} \left( \phi_\sigma^{H_0} \right)^* g d\sigma \\ &= \frac{1}{T} \left( \left( \phi_{T+t}^{H_0} \right)^* g - \left( \phi_t^{H_0} \right)^* g \right) \\ &= \frac{1}{T} \left( \left( \phi_t^{H_0} \right)^* g - \left( \phi_t^{H_0} \right)^* g \right) \\ &= 0. \end{aligned}$$

Therefore  $\text{ad}_{H_0}\langle g \rangle = 0$ .  $\square$

To put the Hamiltonian into normal form to first order we write

$$(3.9) \quad H_1 = \langle H_1 \rangle + (H_1 - \langle H_1 \rangle)$$

and then substitute (3.9) into (3.6). Thus

$$(3.10) \quad \begin{aligned} 0 &= \{H_0, \langle H_1 \rangle + (H_1 - \langle H_1 \rangle) + \text{ad}_f H_0\} \\ &= \{H_0, \langle H_1 \rangle\} + \{H_0, (H_1 - \langle H_1 \rangle) + \text{ad}_f H_0\} \\ &= \{H_0, (H_1 - \langle H_1 \rangle) + \text{ad}_f H_0\}, \end{aligned}$$

where the last step follows from the fact that, from 3.7,  $\langle H_1 \rangle \in \ker(\text{ad}_{H_0})$ . We then seek a solution to the *homological equation*

$$(3.11) \quad \text{ad}_f H_0 = -(H_1 - \langle H_1 \rangle),$$

where  $f$  is the unknown function.

PROPOSITION 3.8 (see [13]). *The solution to (3.11) is given by*

$$(3.12) \quad f = \frac{1}{T} \int_0^T t \left( \phi_t^{H_0} \right)^* (H_1 - \langle H_1 \rangle) dt.$$

*Proof.* Let  $g = -\text{ad}_f H_0 = \text{ad}_{H_0} f$ . This is equivalent to the dynamical system

$$\frac{d}{dt} \left( \phi_t^{H_0} \right)^* f = \left( \phi_t^{H_0} \right)^* g.$$

We show that  $g = H_1 - \langle H_1 \rangle$  by direct substitution. Thus

$$\begin{aligned}
\frac{d}{dt} \left( \phi_t^{H_0} \right)^* f &= \frac{d}{dt} \left( \frac{1}{T} \int_0^T \tau \left( \phi_\tau^{H_0} \right)^* (H_1 - \langle H_1 \rangle) d\tau \right) \\
&= \frac{1}{T} \frac{d}{dt} \int_0^T \tau \left( \phi_{t+\tau}^{H_0} \right)^* (H_1 - \langle H_1 \rangle) d\tau \\
&= \frac{1}{T} \frac{d}{dt} \int_t^{t+T} (\sigma - t) \left( \phi_\sigma^{H_0} \right)^* (H_1 - \langle H_1 \rangle) d\sigma \\
&= \frac{1}{T} \left( T \left( \phi_{t+T}^{H_0} \right)^* (H_1 - \langle H_1 \rangle) - \int_t^{t+T} \left( \phi_\sigma^{H_0} \right)^* (H_1 - \langle H_1 \rangle) d\sigma \right) \\
&= \left( \phi_{t+T}^{H_0} \right)^* (H_1 - \langle H_1 \rangle) - \frac{1}{T} \int_t^{t+T} \left( \phi_\sigma^{H_0} \right)^* H_1 d\sigma + \langle H_1 \rangle \\
&= \left( \phi_{t+T}^{H_0} \right)^* (H_1 - \langle H_1 \rangle) - \langle H_1 \rangle + \langle H_1 \rangle \\
&= \left( \phi_t^{H_0} \right)^* (H_1 - \langle H_1 \rangle).
\end{aligned}$$

Therefore  $g = (H_1 - \langle H_1 \rangle)$ . From this the proposition follows.  $\square$

With this choice of  $f$ , the Hamiltonian in (3.4) becomes

$$(3.13) \quad \exp(\epsilon \text{ad}_f) H = H_0 + \epsilon \langle H_1 \rangle + \epsilon^2 \left( H_2 + \text{ad}_f H_1 + \frac{1}{2} \text{ad}_f^2 H_0 \right) + O(\epsilon^3),$$

which is in first-order normal form. Notice that if we wish to bring the Hamiltonian into normal form only up to first order, then there is no need to explicitly calculate the generating function  $f$ .

To bring the function into normal form up to second order we repeat the process, now on the once transformed Hamiltonian. This time we seek a generating function of the form  $\epsilon g$ . Application of the corresponding change of coordinates results in

$$\begin{aligned}
\exp(\epsilon \text{ad}_{\epsilon g}) (\exp(\epsilon \text{ad}_f) H) &= \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \text{ad}_{\epsilon g}^i (\exp(\epsilon \text{ad}_f) H) \\
(3.14) \quad &= H_0 + \epsilon \langle H_1 \rangle + \epsilon^2 \left( H_2 + \text{ad}_f H_1 + \frac{1}{2} \text{ad}_f^2 H_0 + \text{ad}_g H_0 \right) + O(\epsilon^3).
\end{aligned}$$

The homological equation which needs to be solved is

$$(3.15) \quad \text{ad}_g H_0 = - \left( H_2 + \text{ad}_f H_1 + \frac{1}{2} \text{ad}_f^2 H_0 - \left( \langle H_2 \rangle + \langle \text{ad}_f H_1 \rangle + \left\langle \frac{1}{2} \text{ad}_f^2 H_0 \right\rangle \right) \right).$$

From Proposition 3.8, the solution to this equation is

$$\begin{aligned}
g &= \frac{1}{T} \int_0^T t \left( \phi_t^{H_0} \right)^* \left[ H_2 - \langle H_2 \rangle + \text{ad}_f H_1 - \langle \text{ad}_f H_1 \rangle \right. \\
(3.16) \quad &\quad \left. + \frac{1}{2} (\text{ad}_f^2 H_0 - \langle \text{ad}_f^2 H_0 \rangle) \right] dt.
\end{aligned}$$

With this choice our transformed Hamiltonian becomes

$$\begin{aligned}
\exp(\epsilon \text{ad}_{\epsilon g}) (\exp(\epsilon \text{ad}_f) H) \\
(3.17) \quad &= H_0 + \epsilon \langle H_1 \rangle + \epsilon^2 \left( \langle H_2 \rangle + \langle \text{ad}_f H_1 \rangle + \frac{1}{2} \langle \text{ad}_f^2 H_0 \rangle \right) + O(\epsilon^3).
\end{aligned}$$

The Hamiltonian can be placed into normal form up to arbitrary order  $n$  by repeating this process.

In practice one places the system into normal form up to some desired order and then truncates the higher-order terms. The truncated Hamiltonian gives an approximation to the original system. Since the coefficients of  $\epsilon^i$  in the transformed Hamiltonian all commute with  $H_0$  for  $i = 1, 2, \dots, n$ , the flow of the corresponding Hamiltonian vector field of the higher-order terms also commutes with the flow of the nominal system. Thus for a Hamiltonian in first-order normal form we have

$$(3.18) \quad \phi_t^{H_0 + \epsilon \langle H_1 \rangle}(m) = \phi_t^{\epsilon \langle H_1 \rangle} \circ \phi_t^{H_0}(m), \quad m \in M,$$

and the first-order terms give rise naturally to a first-order correcting symplectic map given by the flow of the Hamiltonian system  $\epsilon \langle H_1 \rangle$ . For systems in higher-order normal form, however, while the functions at each order do Poisson commute with  $H_0$ , they do not in general commute with each other, and thus a system in  $n$ th-order normal form defines a single  $n$ th-order correcting symplectic map.

**4. Normal forms and the Hannay–Berry phase.** In the setting of the moving systems approach the Poisson manifold is  $T^*Q$  together with the canonical Poisson bracket defined by

$$(4.1) \quad \{f, g\} = \sum_{i=1}^n \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i}, \quad f, g \in C^\infty(M).$$

To apply Hamiltonian normal form theory we make a few additional assumptions on the Hamiltonian of a moving system, (2.10). We first assume that the potential  $U$  on  $S$  is constant and drop it from the Hamiltonian. Next we assume that  $\mathcal{Z}_t(m_t(q))$  can be written in the form

$$(4.2) \quad \mathcal{Z}_t(m_t(q)) = \epsilon \hat{\mathcal{Z}}_t(m_t(q))$$

for some parameter  $\epsilon$ . For example, if  $M$  is a Riemannian manifold and  $\mathcal{Z}_t$  is a constant magnitude vector field, then we may take  $\epsilon = \|\mathcal{Z}_t\|$  and  $\hat{\mathcal{Z}}_t = \frac{\mathcal{Z}_t}{\|\mathcal{Z}_t\|}$ . If  $\|\mathcal{Z}_t\|$  is not constant, then one could take  $\epsilon$  to be the average magnitude of  $\|\mathcal{Z}_t\|$  over the loop in  $M$  starting at the given initial condition. However, the form of  $\mathcal{Z}_t$  in (4.2) is often natural to the problem, and in general  $\hat{\mathcal{Z}}_t$  is not a unit vector. Under these assumptions the Hamiltonian can be written as

$$(4.3) \quad H(q, p) = H_0(q, p) + \epsilon H_1(q, p) + \epsilon^2 H_2(q, p),$$

where

$$(4.4) \quad H_0(q, p) = \frac{1}{2} \|p\|^2 + V(q),$$

$$(4.5) \quad H_1(q, p) = -\mathcal{P}(\hat{\mathcal{Z}}_t),$$

$$(4.6) \quad H_2(q, p) = -\frac{1}{2} \|\hat{\mathcal{Z}}_t^\perp\|^2.$$

Finally we assume that  $H_0$  has periodic flow with period  $T$ . We then have a natural action of  $S^1$  on  $T^*Q$ , given by  $\phi_t^{H_0}$ , the flow of  $X_{H_0}$ . Let  $\langle \cdot \rangle$  denote the average with respect to this group action; i.e., for a smooth function  $f$  on  $T^*Q$  we have

$$(4.7) \quad \langle f \rangle = \frac{1}{T} \int_0^T \left( \phi_t^{H_0} \right)^* f dt.$$

In general, the parameter  $\epsilon$  captures the rate of the imposed motion on the system. In the adiabatic limit, then,  $\epsilon$  goes to zero and the terms in  $\epsilon^2$  are negligible. (Note that the terms in  $\epsilon$  are not negligible in the adiabatic limit since the time parametrization of the (slow) imposed motion scales as  $\frac{1}{\epsilon}$ . See section 2.2.) In what follows we are interested in relaxing the adiabatic condition; i.e., we assume that while  $\epsilon$  is small, the terms in  $\epsilon^2$  are not negligible. The truncated Hamiltonian defined by

$$(4.8) \quad \langle H \rangle^{(1)}(q, p) = H_0(q, p) + \epsilon \langle H_1 \rangle(q, p) = H_0(q, p) - \epsilon \langle \mathcal{P}(\hat{Z}_t) \rangle$$

is in first-order normal form. The flow of the system to first order is then given by

$$(4.9) \quad \phi_t^{H_0 - \epsilon \langle \mathcal{P}(\hat{Z}_t) \rangle}(q, p) = \phi_t^{-\epsilon \langle \mathcal{P}(\hat{Z}_t) \rangle} \circ \phi_t^{H_0}(q, p),$$

and the flow of  $-\langle \mathcal{P}(\hat{Z}_t) \rangle$  defines the correcting symplectic map to first order. Thus, in the setting where the group action on  $T^*Q$  is given by the flow of the nominal dynamics, the Hannay–Berry phase is the first-order correction at the completion of a closed loop in parameter space.

To find a more accurate expression we express the Hamiltonian in normal form to a higher order before truncating. Let  $G$  be the generator of a change of coordinates bringing the original Hamiltonian into first-order normal form. From Proposition 3.8 and the form of  $H_1$  in (4.5),  $G$  is given by

$$(4.10) \quad G = \frac{1}{T} \int_0^T t \left( \phi_t^{H_0} \right)^* \left[ \langle \mathcal{P}(\hat{Z}_t) \rangle - \mathcal{P}(\hat{Z}_t) \right] dt.$$

From (3.17) and the form of  $H_2$  in (4.6), the second-order truncated normal form is

$$(4.11) \quad \begin{aligned} \langle H \rangle^{(2)}(q, p) &= H_0(q, p) - \epsilon \langle \mathcal{P}(\hat{Z}_t) \rangle \\ &- \epsilon^2 \left( \frac{1}{2} \langle \|\hat{Z}_t^\perp\|^2 \rangle + \langle \text{ad}_G \mathcal{P}(\hat{Z}_t) \rangle - \frac{1}{2} \langle \text{ad}_G^2 H_0 \rangle \right). \end{aligned}$$

Notice that the terms at second order in the Hamiltonian not only account for the average effect of the centrifugal force but also include additional terms involving the first-order change of coordinates. The flow of the system to second order is

$$(4.12) \quad \begin{aligned} &\phi_t^{H_0 - \epsilon \langle \mathcal{P}(\hat{Z}_t) \rangle - \epsilon^2 \left( \frac{1}{2} \langle \|\hat{Z}_t^\perp\|^2 \rangle + \langle \text{ad}_G \mathcal{P}(\hat{Z}_t) \rangle - \frac{1}{2} \langle \text{ad}_G^2 H_0 \rangle \right)}(q, p) \\ &= \phi_t^{-\epsilon \langle \mathcal{P}(\hat{Z}_t) \rangle - \epsilon^2 \left( \frac{1}{2} \langle \|\hat{Z}_t^\perp\|^2 \rangle + \langle \text{ad}_G \mathcal{P}(\hat{Z}_t) \rangle - \frac{1}{2} \langle \text{ad}_G^2 H_0 \rangle \right)} \circ \phi_t^{H_0}(q, p), \end{aligned}$$

and this in general defines a correcting symplectic map to second order. If in addition the terms in  $\epsilon$  Poisson commute with the terms in  $\epsilon^2$ , then the second-order terms define a second-order correcting symplectic map. In this case the three Hamiltonian systems can be solved independently and their flows composed to obtain the second-order solution. This is captured in the following lemma.

LEMMA 4.1. *If*

$$(4.13) \quad \left\{ \langle \mathcal{P}(\hat{Z}_t) \rangle, \frac{1}{2} \langle \|\hat{Z}_t^\perp\|^2 \rangle + \langle \text{ad}_G \mathcal{P}(\hat{Z}_t) \rangle - \frac{1}{2} \langle \text{ad}_G^2 H_0 \rangle \right\} = 0,$$

then

$$(4.14) \quad \begin{aligned} &\phi_t^{H_0 - \epsilon \langle \mathcal{P}(\hat{Z}_t) \rangle - \epsilon^2 \left( \frac{1}{2} \langle \|\hat{Z}_t^\perp\|^2 \rangle + \langle \text{ad}_G \mathcal{P}(\hat{Z}_t) \rangle - \frac{1}{2} \langle \text{ad}_G^2 H_0 \rangle \right)}(q, p) \\ &= \phi_t^{-\epsilon^2 \left( \frac{1}{2} \langle \|\hat{Z}_t^\perp\|^2 \rangle + \langle \text{ad}_G \mathcal{P}(\hat{Z}_t) \rangle - \frac{1}{2} \langle \text{ad}_G^2 H_0 \rangle \right)} \circ \phi_t^{-\epsilon \langle \mathcal{P}(\hat{Z}_t) \rangle} \circ \phi_t^{H_0}(q, p). \end{aligned}$$

*Proof.* The proof is immediate by the assumption of the Poisson commutativity of the functions.  $\square$

**4.1. Time-dependence of nonadiabatic corrections.** In section 2.2 we showed that the Hannay–Berry phase is a geometric phenomenon by showing that the corresponding ordinary differential equation is independent of the time parametrization. We now show that the terms in  $\epsilon^2$  in the moving systems Hamiltonian do not result in a geometric effect. Consider (4.11). For simplicity assume that the generating function for the change of coordinates is  $G = 0$  and that  $\{\langle H_1 \rangle, \langle H_2 \rangle\} = 0$  so that we can calculate the effect on the system from these two terms separately. Denote points in  $T^*Q$  by  $z$ . Noticing that  $X_{\langle \|Z_t^\perp\|^2 \rangle}$  is a quadratic form in the vector field  $Z$  on the base space, we define

$$(4.15) \quad Y(\mathcal{Z}_t, z) = -X_{\langle \|Z_t^\perp\|^2 \rangle},$$

where  $Y(a\mathcal{Z}_t, z) = a^2 Y(\mathcal{Z}_t, z)$ . The corresponding ordinary differential equation is

$$(4.16) \quad \dot{z} = Y(\mathcal{Z}_t, z).$$

We now change the time parametrization (as in section 2.2) by taking  $t \mapsto \tau(t)$  with  $\frac{d\tau}{dt}$  strictly positive. Under this parametrization, the vector field  $\mathcal{Z}_t$  is scaled by  $\frac{d\tau}{dt}$  and thus

$$(4.17) \quad \frac{dz}{dt} = \frac{dz}{d\tau} \frac{d\tau}{dt} = Y\left(\frac{d\tau}{dt} \mathcal{Z}_\tau, z\right) = \left(\frac{d\tau}{dt}\right)^2 Y(\mathcal{Z}_\tau, z).$$

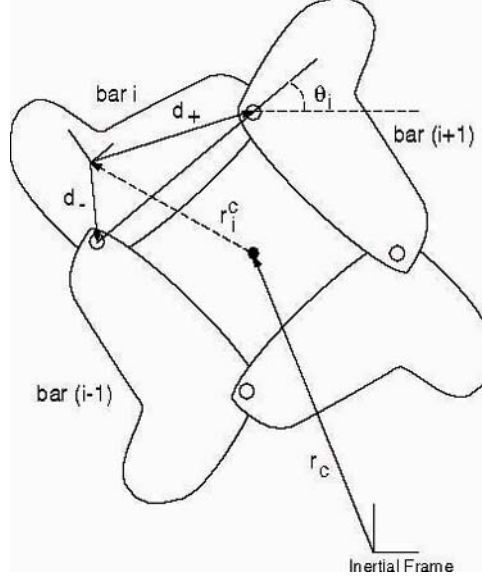
From this we have

$$(4.18) \quad \frac{dz}{d\tau} = \frac{d\tau}{dt} Y(\mathcal{Z}_\tau, z),$$

which shows the dependence on the time parametrization.

**5. The equal-sided spring-jointed free-floating four-bar mechanism.** In this section we apply the method developed in section 4 to an equal-sided spring-jointed free-floating four-bar mechanism which is being rotated about its center of mass. The study of the four-bar mechanism has a long history, dating at least back to the work of Grashof in the mid-nineteenth century [22]. (See also [29] and references therein.) Building upon an analysis of four-bar linkages due to Yang and Krishnaprasad [37], this system is analyzed in [4, 3] using the moving system approach, and it is shown that the Hannay–Berry phase is zero. After briefly recalling this result, we will show that the second-order effects of the imposed rotation are also zero.

Consider an equal-sided four-bar mechanism as shown in Figure 5.1. By a “bar” we mean a planar rigid body on which the center of mass and pin joints are arbitrarily located. The identical bars are labeled sequentially from 0 to 3, and on each a body-fixed frame is defined such that its origin is at the body center of mass and the  $x$ -axis is parallel to the line connecting the pin joints. The positive direction of the  $x$ -axis of the  $i$ th bar is defined to be towards the  $(i + 1)$ th bar for  $i = 0, 1, 2, 3$ , where we adopt

FIG. 5.1. *Equal-sided four-bar mechanism.*

the convention of modulo four addition for subscripts. We define the following:

- $\mathbf{d}_+$  the vector from the body center of mass of the  $i$ th bar to the pin joint with the  $(i + 1)$ th bar,
- $\mathbf{d}_-$  the vector from the body center of mass of the  $i$ th bar to the pin joint with the  $(i - 1)$ th bar,
- $l$  the length of each bar  $\|\mathbf{d}_+ - \mathbf{d}_-\|$ , where  $\|\cdot\|$  is the standard Euclidean norm,
- $\mathbf{r}_i^c$  the vector from the system center of mass to the  $i$ th body center of mass,
- $\mathbf{r}_c$  the vector from the origin of the inertial system to the system center of mass,
- $\theta_i$  the angle between the  $i$ th bar frame and the inertial frame,
- $\theta_{i,j}$  the angle  $\theta_i - \theta_j$  between the  $i$ th and  $j$ th bars,
- $I, m$  the moment of inertia and mass of each bar.

From Figure 5.1 we have that

$$(5.1) \quad r_{i+1}^c = r_i^c + R(\theta_i)\mathbf{d}_+ - R(\theta_{i+1})\mathbf{d}_-, \quad i = 0, 1, 2, 3,$$

where  $R(\theta_i)$  is the rotation matrix given by

$$R(\theta_i) = \begin{pmatrix} \cos(\theta_i) & -\sin(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i) \end{pmatrix}.$$

We use (5.1) recursively to define the loop closure constraint

$$(5.2) \quad F(r) = \sum_{i=0}^3 R(\theta_i)(\mathbf{d}_+ - \mathbf{d}_-) = 0.$$

In [32] it is shown that the configuration space for a free-floating four-link open chain is  $\mathcal{R} = \mathbb{R}^2 \times S^1 \times S^1 \times S^1 \times S^1$ . The configuration space for a general four-bar mechanism is thus  $S_{\text{gen}} = \{r \in \mathcal{R} | F(r) = 0\}$ . For a four-bar mechanism with identical bars it can be shown that if the system is not allowed to pass through any singularities (joint angles of 0 or  $2\pi$ ), then the configuration space  $S = \{r \in \mathcal{R} | F(r) = 0, \theta_{i+1,i} \neq 0, \pi\}$  is a smooth submanifold of  $\mathcal{R}$  [3].

While in the general four-bar mechanism the relations between the angles  $\theta_i$  can be quite complicated (see, for example, [29]), they have a particularly simple form for the equal-sided case, namely

$$(5.3) \quad \theta_2 = \theta_0 + \pi, \quad \theta_3 = \theta_0 - \pi,$$

which leads to the following equalities:

$$(5.4) \quad \theta_{32} = \theta_{10}, \quad \theta_{21} = \theta_{03} = \pi - \theta_{10}, \quad \theta_{13} = \theta_{20} = \pi.$$

For the free-floating equal-sided four-bar linkage the configuration is completely specified by the choice of one global angle and one joint angle. We arbitrarily choose  $\theta_0$  and  $\theta_{10}$ . After removing the singular points  $\theta_{10} = 0, \pi$ , the configuration space is given by  $S^1 \times \{(0, \pi) \cup (0, -\pi)\}$ . Since the joint angle is not allowed to pass through the singular points, we may arbitrarily choose either one of the connected components of this space to describe the configuration of our system, with the additional requirement that the initial condition lie in the component we have chosen. Without loss of generality, then, we take  $S = S^1 \times (0, \pi)$  as the configuration space of the free-floating equal-sided four-bar mechanism.

The total kinetic energy of the system in the center of mass frame is given by

$$(5.5) \quad T = \frac{1}{2}I \sum_{i=0}^3 \omega_i^2 + \frac{1}{2}m \sum_{i=0}^3 \|\dot{r}_i^c\|^2.$$

Following [37], this can be written

$$(5.6) \quad T = \frac{1}{2} \langle \tilde{\omega}, \tilde{M} \tilde{\omega} \rangle,$$

where  $\tilde{\omega} = (\omega_0, \omega_1, \omega_2, \omega_3)'$  and  $\tilde{M}$  is a  $4 \times 4$  symmetric matrix whose elements for the equal-sided four-bar system are

$$(5.7) \quad \tilde{M}_{ii} = I + \frac{3m}{8} (\|\mathbf{d}_+\|^2 + \|\mathbf{d}_-\|^2),$$

$$(5.8) \quad \tilde{M}_{i,i+1} = \frac{m}{8} (\mathbf{d}'_- R_{i+1,i} \mathbf{d}_+ - 3\mathbf{d}'_+ R_{i+1,i} \mathbf{d}_-),$$

$$(5.9) \quad \tilde{M}_{i,i+2} = -\frac{m}{8} (\mathbf{d}'_+ R_{i+2,i} \mathbf{d}_+ + \mathbf{d}'_- R_{i+2,i} \mathbf{d}_-),$$

where  $'$  indicates transpose and  $R_{i,j} = R(\theta_i - \theta_j)$ . From (5.3) we have

$$(5.10) \quad \begin{pmatrix} \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \omega_0 \\ \omega_1 \end{pmatrix},$$

where  $\omega_i = \dot{\theta}_i$ . Let  $\mathbb{I}$  denote the identity matrix. Define

$$(5.11) \quad M = \begin{pmatrix} \mathbb{I} & \mathbb{I} \\ & \mathbb{I} \end{pmatrix} \tilde{M} \begin{pmatrix} \mathbb{I} \\ \mathbb{I} \end{pmatrix}.$$



and so

$$(5.17) \quad M_{10} = 2ml\delta_y \sin(\theta_{10}).$$

The kinetic energy defines a Riemannian metric  $K$  on  $S$  given by

$$(5.18) \quad K(\theta_{10})(X, W) = X' \hat{M}(\theta_{10}) W, \quad X, W \in T_{(\theta_0, \theta_{10})} S.$$

Each joint is equipped with an identical spring. Let the spring potential for each be given by  $V_s(\theta_{i+1, i})$ ,  $i = 0, 1, 2, 3$ , with  $V_s$  twice continuously differentiable. The total potential energy is then

$$(5.19) \quad V(\theta_0, \theta_{10}) = 2(V_s(\theta_{10}) + V_s(\pi - \theta_{10})) \triangleq V(\theta_{10}),$$

where the relations in (5.4) have been used to simplify the expression. We assume that the potential energy is such that there exists  $\alpha \in \{0, \pi\}$  such that

$$(5.20) \quad \left. \frac{\partial V}{\partial \theta_{10}} \right|_{\alpha} = 0, \quad \left. \frac{\partial^2 V}{\partial \theta_{10}^2} \right|_{\alpha} > 0,$$

and without loss of generality we take  $V(\alpha) = 0$ . The standard Lagrangian is then given by

$$(5.21) \quad L(\theta_{10}, \omega_{10}) = \frac{1}{2} \begin{pmatrix} \omega_0 & \omega_{10} \end{pmatrix} \hat{M}(\theta_{10}) \begin{pmatrix} \omega_0 \\ \omega_{10} \end{pmatrix} - V(\theta_{10}).$$

Consider now the following action  $\Phi_g$  of the Lie group  $S^1$  on  $S$ :

$$(5.22) \quad \Phi_g(\theta_0, \theta_{10}) = (\theta_0 + g, \theta_{10}).$$

The quadruple  $(S, K, V, S^1)$  is a simple mechanical system with symmetry, where the action of  $S^1$  on  $S$  is given by (5.22). (For a definition and discussion of simple mechanical systems with symmetry, see [1].) Since the action is both free and proper, the reduced space is a manifold. Recall that we defined  $S = S^1 \times (0, \pi)$ ; the reduced (or shape) space is then  $Q = (0, \pi)$  with the coordinate  $\theta_{10}$ .

In the language of the moving systems approach,  $Q$  is the configuration space and  $S$  is the ambient space. To slowly rotate the mechanism set  $\theta_0 = \Omega t + \hat{\theta}_0$  for some fixed initial offset  $\hat{\theta}_0$ . (Note that  $\theta_0$  and  $\theta_0 + 2\pi$  are identified.) The imposed motion on the four-bar system is captured by the parametrized family of embeddings from  $Q$  into  $S$  given by

$$(5.23) \quad m_t(\theta_{10}) = \begin{pmatrix} \Omega t + \hat{\theta}_0 \\ \theta_{10} \end{pmatrix}.$$

**5.1. The nominal dynamics.** To apply the method developed in section 4 the dynamics of the system in the absence of any imposed motion must be periodic. Consider the nominal Lagrangian for the four-bar mechanism, defined by setting  $\Omega = 0$  in (5.21):

$$(5.24) \quad L_0(\theta_{10}, \omega_{10}) = \frac{M_{11}}{2} \omega_{10}^2 - V(\theta_{10}).$$

Applying the Legendre transform, the conjugate momentum is found to be

$$p_{10} = M_{11} \omega_{10},$$

and thus the Hamiltonian for the nominal system is

$$(5.25) \quad H_0(\theta_{10}, p_{10}) = \frac{p_{10}^2}{2M_{11}} + V(\theta_{10}).$$

The nominal dynamics is

$$(5.26) \quad \dot{\theta}_{10} = \frac{p_{10}}{M_{11}}, \quad \dot{p}_{10} = -\frac{\partial V}{\partial \theta_{10}}.$$

From (5.20) and (5.26), we see that there is an equilibrium point at  $(\theta_{10} = \alpha, p_{10} = 0)$ . The existence of periodic solutions near this equilibrium point can be assured by appealing to the following theorem by Weinstein.

**THEOREM 5.1** (see [36]). *Consider  $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ . If  $H$  is twice continuously differentiable near an equilibrium point  $z$  and the Hessian matrix at the equilibrium point is positive definite, then for sufficiently small  $\epsilon$  any energy surface  $H(z) = H(0) + \epsilon^2$  contains at least  $n$  periodic orbits of the associated Hamiltonian system.*

The Hessian matrix of the nominal system, evaluated at the equilibrium  $(\alpha, 0)$ , is

$$(5.27) \quad \left( \begin{array}{cc} \frac{\partial^2 H}{\partial \theta_{10}^2} & 0 \\ 0 & \frac{\partial^2 H}{\partial p_{10}^2} \end{array} \right) \Big|_{(\alpha, 0)} = \left( \begin{array}{cc} \frac{\partial^2 V}{\partial \theta_{10}^2} & 0 \\ 0 & M_{11}^{-1} \end{array} \right) \Big|_{(\alpha, 0)}.$$

$M$  is positive definite by its construction, and so  $M_{11} > 0$ . Combining this with the assumption in (5.20), we see that the Hessian matrix of the Hamiltonian at the equilibrium point is positive definite, and therefore by Theorem (5.1) there is a periodic solution around the equilibrium if the energy is sufficiently small.

Since this is a one-degree-of-freedom system, it is integrable, and thus there exist action-angle coordinates  $(J, \psi)$  [6]. These coordinates will prove particularly convenient for evaluating both the Hannay–Berry phase and the higher-order corrections. Let  $\Gamma(h)$  be the trajectory in phase space corresponding to the energy  $h$ . Then

$$(5.28) \quad J = \frac{1}{2\pi} \oint_{\Gamma(h)} p_{10} d\theta_{10}.$$

The trajectory  $\Gamma(h)$ , and thus the action, depends on the form of  $V(\theta_{10})$ . We can write in general

$$(5.29) \quad \begin{aligned} J &= g_1(\theta_{10}, p_{10}), & \theta_{10} &= f_1(J, \psi), \\ \psi &= g_2(\theta_{10}, p_{10}), & p_{10} &= f_2(J, \psi). \end{aligned}$$

For the remainder of this paper we will assume that the initial conditions are such that the periodic solutions of the nominal system are of small amplitude. The expansion of  $V(\theta_{10})$  about the equilibrium point  $(\alpha, 0)$  yields

$$(5.30) \quad V(\theta_{10}) = \frac{1}{2} \frac{\partial^2 V}{\partial \theta_{10}^2} \Big|_{\alpha} (\theta_{10} - \alpha)^2 + O((\theta_{10} - \alpha)^3).$$

In the small angle limit the potential is taken only to second order. Since the springs on each bar are identical, this is equivalent to taking

$$(5.31) \quad V_s(\theta_{10}) = \frac{k_s}{2} (\theta_{10} - \alpha_s)^2$$

for the potential of each spring. Here  $\alpha_s \in S^1$ . From (5.19) the total potential is

$$(5.32) \quad V(\theta_{10}) = k_s [(\theta_{10} - \alpha_s)^2 + (\pi - \theta_{10} - \alpha_s)^2].$$

The equilibrium point  $\alpha$  is given by setting the derivative of  $V$  with respect to  $\theta_{10}$  to zero. This yields

$$(5.33) \quad 0 = \left. \frac{\partial V}{\partial \theta_{10}} \right|_{\theta_{10}=\alpha} = 2k_s (2\alpha - \pi)$$

and thus  $\alpha = \frac{\pi}{2}$ . The second derivative of  $V$  with respect to  $\theta_{10}$  is

$$(5.34) \quad \frac{\partial^2 V}{\partial \theta_{10}^2} = 4k_s \triangleq k.$$

With this choice of spring potential the nominal Hamiltonian is given by

$$(5.35) \quad H_0 = \frac{p_{10}^2}{2M_{11}} + \frac{k}{2}(\theta_{10} - \alpha)^2.$$

This is the Hamiltonian for a harmonic oscillator. From [6] the angle variable is the phase of the oscillation, and the action is

$$(5.36) \quad J = \frac{h}{\omega},$$

where  $\omega = \sqrt{\frac{k}{M_{11}}}$  is the frequency of oscillation and  $h$  is the energy corresponding to a given initial condition. From (5.35) and (5.36)

$$(5.37) \quad J = \frac{p_{10}^2 + kM_{11}(\theta_{10} - \alpha)^2}{2\sqrt{kM_{11}}}.$$

Therefore

$$(5.38) \quad \theta_{10} = \alpha + \left[ \frac{2J}{\sqrt{kM_{11}}} \right]^{\frac{1}{2}} \cos \psi = f_1(J, \psi),$$

$$(5.39) \quad p_{10} = - \left[ 2J\sqrt{kM_{11}} \right]^{\frac{1}{2}} \sin \psi = f_2(J, \psi).$$

The use of (5.38), (5.39) in (5.35) yields

$$(5.40) \quad H_0(\theta_{10}, p_{10}) = H_0(J) = \sqrt{\frac{k}{M_{11}}} J,$$

leading to the following dynamics in the action-angle variables.

$$(5.41) \quad \dot{\psi} = \sqrt{\frac{k}{M_{11}}}, \quad \dot{J} = 0.$$

The average  $\langle \cdot \rangle$  is then the average over one cycle of the angle variable  $\psi$ .

**5.2. The Hannay–Berry phase of the four-bar linkage.** Recall that the Hannay–Berry phase is the holonomy of the Cartan–Hannay–Berry connection and is determined by solving the Hamiltonian system associated to the averaged momentum function defined in (2.3). From (5.23), the velocity vector of the motion in  $S$  is

$$\frac{d}{dt}(m_t(\theta_{10})) = \begin{pmatrix} 0 \\ \omega_{10} \end{pmatrix} + \begin{pmatrix} \Omega \\ 0 \end{pmatrix},$$

and thus the tangent vector which must be projected to  $T_{m_t(q)}m_t(Q)$  is

$$(5.42) \quad \mathcal{Z} \triangleq \mathcal{Z}_t(m_t(q(t))) = \begin{pmatrix} \Omega \\ 0 \end{pmatrix}.$$

The projection of  $\mathcal{Z}$  to  $T_{m_t(q)}m_t(Q)$  with respect to the kinetic energy metric on  $S$  is given by  $\mathcal{Z}^T = \mathcal{Z} - \mathcal{Z}^\perp$ , where  $\mathcal{Z}^\perp$  satisfies the orthogonality condition

$$K(\theta_{10})(\mathcal{Z}^\perp, X) = 0 \quad \forall X \in T_{m_t(q)}m_t(Q).$$

Application of the orthogonality condition yields

$$(5.43) \quad \mathcal{Z}^\perp = \begin{pmatrix} \Omega \\ -\Omega \left[ \frac{M_{10}(\theta_{10}) + M_{11}}{M_{11}} \right] \end{pmatrix}$$

and thus

$$(5.44) \quad \mathcal{Z}^T = \begin{pmatrix} 0 \\ \Omega \left[ \frac{M_{10}(\theta_{10}) + M_{11}}{M_{11}} \right] \end{pmatrix}.$$

The pull-back of  $\mathcal{Z}^T$  to  $T_qQ$  by  $[Tm]^{-1}$  is given by projection onto the second factor,

$$(5.45) \quad Z \triangleq [Tm]^{-1} \mathcal{Z}^T = \Omega \left[ \frac{M_{10}(\theta_{10}) + M_{11}}{M_{11}} \right].$$

The function  $\mathcal{P}(Z)$  is then (following (2.3))

$$(5.46) \quad \begin{aligned} \mathcal{P}(Z)(J, \psi) &= \Omega \left[ \frac{M_{10}(\theta_{10}(J, \psi)) + M_{11}}{M_{11}} \right] p_{10}(J, \psi) \\ &= -\Omega \left[ 1 + \frac{2ml\delta_y}{M_{11}} \sin \left( \alpha + \left[ \frac{2J}{\sqrt{kM_{11}}} \right]^{\frac{1}{2}} \cos \psi \right) \right] \left[ 2J\sqrt{kM_{11}} \right]^{\frac{1}{2}} \sin \psi, \end{aligned}$$

where we have expressed the function in terms of the action-angle coordinates given by (5.38), (5.39) and substituted for  $M_{10}$  using (5.17).

The flow of the nominal system induces an  $S^1$  action on  $T^*Q$ , and the average with respect to this action is simply the average over one cycle of the angle coordinate  $\psi$ . Thus the Hamiltonian function defining the lift with respect to the Cartan–Hannay–Berry connection is

$$(5.47) \quad \begin{aligned} \langle P(Z) \rangle &= \frac{1}{2\pi} \int_0^{2\pi} P(Z)(J, \psi) d\psi \\ &= -\frac{\Omega \left[ 2J\sqrt{kM_{11}} \right]^{\frac{1}{2}}}{2\pi} \left[ \int_0^{2\pi} \sin \psi d\psi + \int_0^{2\pi} \frac{2ml\delta_y}{M_{11}} \sin \left( \alpha + \left[ \frac{2J}{\sqrt{kM_{11}}} \right]^{\frac{1}{2}} \cos \psi \right) \sin \psi d\psi \right] \\ &= -\frac{\Omega\sqrt{kM_{11}}}{2\pi} \left[ \frac{2ml\delta_y}{M_{11}} \right] \cos \left( \alpha + \left[ \frac{2J}{\sqrt{kM_{11}}} \right]^{\frac{1}{2}} \cos \psi \right) \Big|_0^{2\pi} = 0. \end{aligned}$$

Therefore, under the assumption of linear springs, the Hannay–Berry phase for the equal-sided spring-jointed four-bar mechanism is zero.

**5.3. Nonadiabatic corrections.** From the kinetic energy metric in (5.18) and the form of  $\mathcal{Z}^\perp$  in (5.43) we have

$$(5.48) \quad \|\mathcal{Z}^\perp\|^2 = \Omega^2 \left[ \frac{M_{11}^2 - M_{10}^2(J, \psi)}{M_{11}} \right].$$

Using this and the form of  $\mathcal{P}(Z)$  for the four-bar in (5.46), the Hamiltonian for the rotating four-bar may be written as

$$(5.49) \quad H(J, \psi) = H_0(J) + \Omega H_1(J, \psi) + \Omega^2 H_2(J, \psi),$$

where

$$(5.50) \quad H_0(J) = \sqrt{\frac{k}{M_{11}}} J,$$

$$(5.51) \quad H_1(J, \psi) = \left[ \frac{b \sin \left( \alpha + \left[ \frac{2J}{\sqrt{kM_{11}}} \right]^{\frac{1}{2}} \cos \psi \right) + M_{11}}{M_{11}} \right] \left[ 2J \sqrt{kM_{11}} \right]^{\frac{1}{2}} \sin \psi,$$

$$(5.52) \quad H_2(J, \psi) = -\frac{1}{2} \left[ \frac{M_{11}^2 - b^2 \sin^2 \left( \alpha + \left[ \frac{2J}{\sqrt{kM_{11}}} \right]^{\frac{1}{2}} \cos \psi \right)}{M_{11}} \right],$$

where we have defined the constant  $b = 2ml\delta_y$  to ease the notation. To find the effect of the imposed rotation to second order we must first find the generating function for the change of coordinates bringing the system into first-order normal form. From Proposition 3.8 we have

$$(5.53) \quad \begin{aligned} G(J) &= \frac{1}{2\pi} \int_0^{2\pi} \psi (H_1(J, \psi) - \langle H_1(J, \psi) \rangle) d\psi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \psi \left[ \frac{b \sin \left( \alpha + \left[ \frac{2J}{\sqrt{kM_{11}}} \right]^{\frac{1}{2}} \cos \psi \right) + M_{11}}{M_{11}} \right] \left[ 2J \sqrt{kM_{11}} \right]^{\frac{1}{2}} \sin \psi d\psi, \end{aligned}$$

where we have used (5.47). Define

$$(5.54) \quad c_1(J) = \left[ \frac{2J}{\sqrt{kM_{11}}} \right]^{\frac{1}{2}}, \quad c_2(J) = \left[ 2J \sqrt{kM_{11}} \right]^{\frac{1}{2}}.$$

Then

$$(5.55) \quad G(J) = \frac{c_2(J)b}{2\pi M_{11}} \int_0^{2\pi} \psi \sin \psi \sin(\alpha + c_1(J) \cos \psi) d\psi + \frac{c_2(J)}{2\pi} \int_0^{2\pi} \psi \sin \psi d\psi.$$

A simple integration by parts shows that

$$(5.56) \quad \int_0^{2\pi} \psi \sin \psi d\psi = -2\pi.$$

Similarly

$$\begin{aligned}
 & \int_0^{2\pi} \psi \sin \psi \sin(\alpha + c_1(J) \cos \psi) d\psi \\
 &= \frac{2\pi}{c_1(J)} \cos(\alpha + c_1(J)) - \frac{1}{c_1(J)} \int_0^{2\pi} \cos(\alpha + c_1(J) \cos \psi) d\psi \\
 (5.57) \quad &= \frac{2\pi}{c_1(J)} [\cos(\alpha + c_1(J)) - \cos(\alpha) J_0(c_1(J))],
 \end{aligned}$$

where  $J_0$  is a Bessel function of the first kind. Therefore

$$(5.58) \quad G(J) = \frac{c_2(J)b}{c_1(J)} [\cos(\alpha + c_1(J)) - \cos(\alpha) J_0(c_1(J))] - c_2(J).$$

To determine the second-order terms in the truncated second-order normal form Hamiltonian for a moving system, as in (4.11), we must find the terms  $\text{ad}_G^2 H_0$  and  $\text{ad}_G H_1$ . Using the canonical bracket (4.1), we have

$$(5.59) \quad \text{ad}_G H_0 = \frac{\partial H_0}{\partial \psi} \frac{\partial G}{\partial J} - \frac{\partial H_0}{\partial J} \frac{\partial G}{\partial \psi} = 0,$$

since  $H_0$  and  $G$  are each independent of  $\psi$ . Thus

$$(5.60) \quad \text{ad}_G^2 H_0 = 0.$$

For the next term we have

$$(5.61) \quad \text{ad}_G H_1 = \frac{\partial H_1}{\partial \psi} \frac{\partial G}{\partial J} - \frac{\partial H_1}{\partial J} \frac{\partial G}{\partial \psi} = \frac{\partial H_1}{\partial \psi} \frac{\partial G}{\partial J}.$$

From the form of  $G$  in (5.58) we have

$$(5.62) \quad \frac{\partial G}{\partial J} = \frac{1}{c_1(J)} (b \cos(\alpha) J_1(c_1(J)) - b \sin(\alpha + c_1(J)) - 1),$$

and from (5.51) we have

$$\begin{aligned}
 (5.63) \quad \frac{\partial H_1}{\partial \psi} = c_2(J) & \left[ \cos \psi \left( \frac{b \sin(\alpha + c_1(J) \cos \psi) + M_{11}}{M_{11}} \right) \right. \\
 & \left. - \frac{bc_1(J)}{M_{11}} \sin^2 \psi \cos(\alpha + c_1(J) \cos \psi) \right].
 \end{aligned}$$

Since the average  $\langle \cdot \rangle$  is over the variable  $\psi$ , we simplify the notation by defining

$$(5.64) \quad A_1 = \frac{c_2(J)}{c_1(J)M_{11}} (b \cos \alpha J_1(c_1(J)) - b \sin(\alpha + c_1(J)) - 1).$$

With this definition we may write

$$(5.65) \quad \text{ad}_G H_1 = A_1 [b \cos \psi \sin(\alpha + c_1(J) \cos \psi) + \cos \psi - bc_1(J) \sin^2 \psi \cos(\alpha + c_1(J) \cos \psi)].$$

We now calculate  $\langle \text{ad}_G H_1 \rangle$ . Since the average of  $\cos \psi$  over a full cycle of  $\psi$  is zero, we have

(5.66)

$$\langle \text{ad}_G H_1 \rangle = A_1 b \langle \cos \psi \sin(\alpha + c_1(J) \cos \psi) \rangle - A_1 b c_1(J) \langle \sin^2 \psi \cos(\alpha + c_1(J) \cos \psi) \rangle.$$

Using integration by parts, we have

$$\begin{aligned} \langle \sin^2 \psi \cos(\alpha + c_1(J) \cos \psi) \rangle &= \frac{1}{2\pi} \int_0^{2\pi} \sin^2 \psi \cos(\alpha + c_1(J) \cos \psi) d\psi \\ &= \frac{1}{2\pi c_1(J)} \int_0^{2\pi} \sin(\alpha + c_1(J) \cos \psi) \cos \psi d\psi \\ (5.67) \qquad \qquad \qquad &= \frac{1}{c_1(J)} \langle \cos \psi \sin(\alpha + c_1(J) \cos \psi) \rangle, \end{aligned}$$

and using this result in (5.65) yields

$$(5.68) \qquad \qquad \qquad \langle \text{ad}_G H_1 \rangle = 0.$$

Finally consider

$$\begin{aligned} \langle H_2 \rangle &= -\frac{1}{2} \left\langle \frac{M_{11}^2 - b^2 \sin^2(\alpha + c_1(J) \cos \psi)}{M_{11}} \right\rangle \\ (5.69) \qquad \qquad \qquad &= -\frac{M_{11}}{2} + \frac{b^2}{2M_{11}} \langle \sin^2(\alpha + c_1(J) \cos \psi) \rangle. \end{aligned}$$

Thus the second-order truncated normal-form Hamiltonian for the rotating four-bar is

$$\begin{aligned} H^{(2)} &= H_0 + \Omega^2 \langle H_2 \rangle \\ (5.70) \qquad \qquad \qquad &= \sqrt{\frac{k}{M_{11}}} J + \frac{b^2 \Omega^2}{2M_{11}} \langle \sin^2(\alpha + c_1(J) \cos \psi) \rangle, \end{aligned}$$

where a constant term has been dropped from the Hamiltonian. Since the system is now in second-order normal form, we have for the initial conditions  $(\bar{J}, \bar{\psi})$

$$(5.71) \qquad \qquad \phi_t^{H^{(2)}}(\bar{J}, \bar{\psi}) = \phi_t^{\Omega^2 \langle H_2 \rangle} \circ \phi_t^{H_0}(\bar{J}, \bar{\psi}).$$

From (5.41) the flow map for the nominal dynamics is

$$(5.72) \qquad \qquad \phi_t^{H_0}(\bar{J}, \bar{\psi}) = \left( \bar{J}, \sqrt{\frac{k}{M_{11}}} t + \bar{\psi} \right).$$

To determine the flow map for the second-order correction we need to solve the Hamiltonian vector field of  $\Omega^2 \langle H_2 \rangle$ . We have

$$\begin{aligned} \dot{\psi} &= \Omega^2 \frac{\partial}{\partial J} \langle \sin^2(\alpha + c_1(J) \cos \psi) \rangle \\ (5.73) \qquad \qquad \qquad &= \Omega^2 \left[ \frac{2}{J \sqrt{k M_{11}}} \right]^{\frac{1}{2}} \langle \sin(\alpha + c_1(J) \cos \psi) \cos(\alpha + c_1(J) \cos \psi) \rangle, \end{aligned}$$

$$(5.74) \qquad \dot{J} = 0.$$

Calculating the average, we find

$$\begin{aligned}
& \langle \sin(\alpha + c_1(J) \cos(\psi)) \cos(\alpha + c_1(J) \cos(\psi)) \rangle \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sin(\alpha + c_1(J) \cos \psi) \cos(\alpha + c_1(J) \cos \psi) d\psi \\
&= \frac{1}{2\pi} \left[ \int_0^\pi \sin(\alpha + c_1(J) \cos \psi) \cos(\alpha + c_1(J) \cos \psi) d\psi \right. \\
&\quad \left. + \int_\pi^{2\pi} \sin(\alpha + c_1(J) \cos \psi) \cos(\alpha + c_1(J) \cos \psi) d\psi \right] \\
&= \frac{1}{2\pi} \left[ \int_0^\pi \sin(\alpha + c_1(J) \cos \psi) \cos(\alpha + c_1(J) \cos \psi) \right. \\
(5.75) \quad & \left. + \sin(\alpha - c_1(J) \cos \psi) \cos(\alpha - c_1(J) \cos \psi) d\psi \right].
\end{aligned}$$

The use of the standard sum-angle formulas for  $\sin$  and  $\cos$  together with an expansion of the products in (5.75) yields

$$\begin{aligned}
& \frac{1}{2\pi} \left[ \int_0^\pi [(\sin \alpha \cos \alpha \cos^2(c_1(J) \cos \psi) - \sin^2 \alpha \cos(c_1(J) \cos \psi) \sin(c_1(J) \cos \psi) \right. \\
&\quad + \cos^2 \alpha \cos(c_1(J) \cos \psi) \sin(c_1(J) \cos \psi) - \sin \alpha \cos \alpha \sin^2(c_1(J) \cos \psi)) \\
&\quad + (\sin \alpha \cos \alpha \cos^2(c_1(J) \cos \psi) + \sin^2 \alpha \cos(c_1(J) \cos \psi) \sin(c_1(J) \cos \psi) \\
&\quad \left. - \cos^2 \alpha \cos(c_1(J) \cos \psi) \sin(c_1(J) \cos \psi) - \cos \alpha \sin \alpha \sin^2(c_1(J) \cos \psi))] d\psi \right] \\
&= \frac{\sin \alpha \cos \alpha}{\pi} \int_0^\pi (\cos^2(c_1(J) \cos \psi) - \sin^2(c_1(J) \cos \psi)) d\psi \\
&= \frac{\sin \alpha \cos \alpha}{\pi} \int_0^\pi (2 \cos^2(c_1(J) \cos \psi) - 1) d\psi \\
&= \frac{\sin \alpha \cos \alpha}{\pi} \int_0^\pi \cos(2c_1(J) \cos \psi) d\psi \\
(5.76) \quad &= \sin \alpha \cos \alpha J_0(2c_1(J)),
\end{aligned}$$

and thus

$$(5.77) \quad \dot{\psi} = \frac{\Omega^2 b^2}{2M_{11}} \left[ \frac{2}{J\sqrt{k}M_{11}} \right]^{\frac{1}{2}} \sin \alpha \cos \alpha J_0(2c_1(J)).$$

Recall now that the equilibrium point is  $\alpha = \frac{\pi}{2}$ . Insertion of this value into (5.77) yields

$$(5.78) \quad \dot{\psi} = 0.$$

The second-order correction is therefore the identity map, and we have

$$(5.79) \quad \phi_t^{H^{(2)}}(\bar{J}, \bar{\psi}) = \phi_t^{H_0}(\bar{J}, \bar{\psi}).$$

Thus to second order the effect of a slow imposed rotation on the four-bar is zero.

**6. Conclusions.** The approach developed by Marsden, Montgomery, and Ratiu provides a unified setting for understanding the role of the Coriolis force in moving systems. In this work we have extended the method through the use of Hamiltonian normal form theory to account for the nonadiabatic nature of the imposed motion. In particular we have shown that the Hannay–Berry phase can be viewed as arising from a first-order normal form approximation to the moving system. The moving systems approach is then naturally understood as a perturbation approach in the rate of the imposed motion. More accurate models can be determined by carrying the perturbation series out to higher orders. The method was applied to a rotating free-floating spring-jointed equal-sided four-bar mechanism, and it was shown that the effect of the imposed motion is zero to second order.

It is important to note that the approach developed here is perturbative, and it therefore relies on the assumption that the imposed motion, while not adiabatic, is slow with respect to the nominal dynamics. The resulting correction terms are second-order in the rate of the imposed motion, and the technique is thus useful only when extremely accurate modeling of the moving system is required. In addition, the resulting approximation is valid only on the time scale  $O(\frac{1}{\epsilon^2})$ , where  $\epsilon$  is the rate of the imposed motion. See [21] for more detailed comments along these lines.

**Acknowledgment.** The author gratefully acknowledges P. S. Krishnaprasad for fruitful discussion, critical comments, and invaluable aid during the course of this work.

#### REFERENCES

- [1] R. ABRAHAM AND J. MARSDEN, *Foundations of Mechanics*, 2nd ed., Perseus Press, New York, 1985.
- [2] A. ACZEL, *Pendulum. Léon Foucault and the Triumph of Science*, Atria Books, New York, 2003.
- [3] S. ANDERSSON, *Geometric Phases in Sensing and Control*, Ph.D. thesis, University of Maryland, College Park, MD, 2003.
- [4] S. ANDERSSON AND P. KRISHNAPRASAD, *The Berry–Hannay phase of the equal-sided spring-jointed four-bar mechanism*, in Proceedings of the 40th IEEE Conference on Decision and Control, 2001, IEEE Press, Piscataway, NJ, pp. 3406–3407.
- [5] V. ARNOLD, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, New York, 1988.
- [6] V. ARNOLD, *Mathematical Methods of Classical Mechanics*, 2nd ed., Springer-Verlag, New York, 1989.
- [7] M. BERRY, *Quantal phase factors accompanying adiabatic changes*, Proc. Roy. Soc. London, A, 392 (1984), pp. 45–57.
- [8] M. BERRY, *Quantum phase corrections from adiabatic iteration*, Proc. Roy. Soc. London, A, 414 (1987), pp. 31–46.
- [9] A. BHATTACHARJEE AND T. SEN, *Geometric angles in cyclic evolutions of a classical system*, Phys. Rev. A, 38 (1988), pp. 4389–4394.
- [10] G. BRYAN, *On the beats in the vibrations of a revolving cylinder or bell*, Proc. Cambridge Philos. Soc., 7 (1890), pp. 101–111.
- [11] E. CARTAN, *Léçons sur la Méthode de la Répère Mobile*, Gauthier-Villars, Paris, 1952.
- [12] J. CARY, *Lie transform perturbation theory for Hamiltonian systems*, Phys. Rep., 79 (1981), pp. 130–159.
- [13] R. CUSHMAN, *Normal form for Hamiltonian vectorfields with periodic flow*, in Differential Geometric Methods in Mathematical Physics, S. Sternberg, ed., Reidel, Boston, 1984, pp. 125–144.
- [14] R. CUSHMAN, *A survey of normalization techniques applied to perturbed Keplerian systems*, in Dynamics Reported, Expositions in Dynamical Systems, C. Jones, U. Kirchgraber, and H. Walther, eds., Springer-Verlag, New York, 1992, Vol. 1, pp. 54–112.
- [15] P. DE SOUSA GERBERT, *A systematic derivative expansion of the adiabatic phase*, Ann. Phys., 189 (1989), pp. 155–173.

- [16] R. FOX AND P. JUNG, *Quasiadiabatic time evolution, avoided level crossings, and Berry's phase*, Phys. Rev. A, 57 (1998), pp. 2339–2346.
- [17] T. FRANKEL, *The Geometry of Physics*, Cambridge University Press, Cambridge, UK, 1997.
- [18] S. FUJISHIMA, T. NAKAMURA, AND K. FUJIMOTO, *Piezoelectric vibratory gyroscope using flexural vibration of a triangular bar*, in Proceedings of the IEEE 45th Annual Symposium on Frequency Control, 1991, IEEE Press, Piscataway, NJ, pp. 261–265.
- [19] F. GAITAN, *Berry's phase in the presence of a non-adiabatic environment with an application to magnetic resonance*, J. Magnetic Resonance, 139 (1999), pp. 152–164.
- [20] I. GJAJA AND A. BHATTACHARJEE, *Divergences in the iterative and perturbative methods for computing Hannay's angle*, Phys. Rev. A, 41 (1990), pp. 5650–5665.
- [21] S. GOLIN AND S. MARMI, *A class of systems with measurable Hannay angles*, Nonlinearity, 3 (1990), pp. 507–518.
- [22] F. GRASHOF, *Theretische Maschinenlehre*, Verlag L. Voss, Leipzig, Germany, 1883.
- [23] J. HENRARD, *The adiabatic invariant in classical mechanics*, in Dynamics Reported, Springer-Verlag, New York, 1993, pp. 117–235.
- [24] A. LAWRENCE, *Modern Inertial Technology*, Springer-Verlag, New York, 1993.
- [25] J. MARSDEN, R. MONTGOMERY, AND T. RATIU, *Reduction, symmetry, and phases in mechanics*, Mem. Amer. Math. Soc., 88 (1990), pp. 248–260.
- [26] J. MARSDEN AND T. RATIU, *Introduction to Mechanics and Symmetry*, 2nd ed., Springer-Verlag, New York, 1999.
- [27] G. NEWTON, *Theory and practice in vibratory rate gyroscopes*, Control Engineering, (1963), pp. 95–99.
- [28] K. NOMIZU, *Lie Groups and Differential Geometry*, The Mathematical Society of Japan, Tokyo, 1956.
- [29] B. PAUL, *Kinematics and Dynamics of Planar Machinery*, Prentice–Hall, Englewood Cliffs, NJ, 1979.
- [30] M. PUTTY, *A Micromachined Vibrating Ring Gyroscope*, Ph.D. thesis, Department of Electrical Engineering, University of Michigan, Ann Arbor, MI, 1995.
- [31] A. SHAPER AND F. WILCZEK, EDS., *Geometric Phases in Physics*, World Scientific, River Edge, NJ, 1989.
- [32] N. SREENATH, *Modeling and Control of Multibody Systems*, Ph.D. thesis, University of Maryland, College Park, MD, 1987.
- [33] C. SUN, *High-order quantum adiabatic approximation and Berry's phase factor*, J. Phys. A, 21 (1988), pp. 1595–1599.
- [34] C. SUN, *High-order adiabatic approximations related to non-Abelian Berry's phase factors and nuclear quadropole resonance*, Phys. Rev. D, 41 (1990), pp. 1318–1323.
- [35] A. TOMITA AND R. CHIAO, *Observation of Berry's topological phase by use of an optical fiber*, Phys. Rev. Lett., 57 (1986), pp. 937–940.
- [36] A. WEINSTEIN, *Normal modes for nonlinear Hamiltonian systems*, Invent. Math., 20 (1973), pp. 47–57.
- [37] R. YANG AND P. KRISHNAPRASAD, *On the geometry and dynamics of floating four-bar linkages*, Dyn. Stab. Syst., 9 (1994), pp. 19–45.