Supplementary Material for "Labor Market Dynamics under Long-Term Wage Contracting"

Leena Rudanko*

December 15, 2008

Appendix B Details of Computational Approach

Solving for equilibrium amounts to solving a fixed point problem in the value of unemployment $V^u(z)$ and contract values $V(z)$ for all $z = z_1 \ldots z_n$. Taking as given a Pareto-frontier, the equilibrium conditions on these variables can be written as[1]

$$\left(\frac{\alpha}{1-\alpha}\right)\frac{f(V(z),z,V^u)}{-f_V(V(z),z,V^u)} = V(z) - V^u(z), \tag{B.1}$$

$$V^u(z) = u(b) + \beta\sum_{z'}\pi(z'|z)[V^u(z') + K^{\frac{1}{\alpha}}k^{\frac{\alpha-1}{\alpha}}f(V(z'),z',V^u)^{\frac{1-\alpha}{\alpha}}(V(z') - V^u(z'))]. \tag{B.2}$$

Equation (B.1) is the first order optimality condition of problem (4) and equation (B.2) the dynamic equation for the value of unemployment from the definition of equilibrium (market tightness $\theta(z)$ has been substituted out using the zero profit condition on vacancy-creation). Once one has found the Pareto-frontier, one can solve for the equilibrium values of $V^u(z)$ and $V(z)$, for $z = z_1 \ldots z_n$, from this system of $2n$ nonlinear equations. Other variables can then be calculated based on these.

Finding the Pareto Frontier

It is convenient to parameterize the Pareto-frontier with the starting wage instead of the contract value, and write equations (B.1) and (B.2) in terms of that wage instead.

With full commitment contracting, the contract wage is constant and the worker and firm values can be parameterized by that wage w as follows (in vector notation)

$$V(w, V^u) = (1-\beta(1-\delta)\Pi)^{-1}(u(w)\mathbf{i} + \delta\beta\Pi V^u) \tag{B.3}$$

*Address for correspondence: Boston University, Department of Economics, 270 Bay State Road, Boston, MA 02215, rudanko@bu.edu, (617) 353-7082.

[1] I use V^u to denote the vector of values $V^u(z)$ for all $z = z_1 \ldots z_n$.

$$F(w) = (I - \beta(1-\delta)\Pi)^{-1}(z - w\mathbf{i}), \tag{B.4}$$

where z stands for the vector of productivities. Using this representation, equations (B.1) and (B.2) can be written as

$$\left(\frac{\alpha}{1-\alpha}\right) F(w(z), z)u'(w(z)) = V(w(z), z, V^u) - V^u(z), \tag{B.5}$$

$$V^u(z) = u(b) + \beta \sum_{z'} \pi(z'|z)[V^u(z') + K^{\frac{1}{\alpha}} k^{\frac{\alpha-1}{\alpha}} F(w(z'), z')^{\frac{1-\alpha}{\alpha}} (V(w(z'), z', V^u) - V^u(z'))]. \tag{B.6}$$

Equations (B.3), (B.4), and (B.6) can be solved for expressions for worker values parameterized by wages: $V(w_1, \ldots, w_n, z)$ and $V^u(w_1, \ldots, w_n, z)$, where $w_1, \ldots, w_n$ are the (fixed) contract wages for productivity states $z_1, \ldots, z_n$. Substituting these expressions for worker values along with the firm values from equation (B.4) into equation (B.5) yields n equations in equilibrium wages $w_1, \ldots, w_n$.

With one-sided limited-commitment contracting, the problem is complicated slightly because in constructing the frontiers $F(w, z, V^u), V(w, z, V^u)$ one needs to take into account that instead of being forever constant, the contract wage may rise when productivity increases enough to make the worker's outside option binding. One needs to determine the boundary wages which make workers indifferent between staying and quitting in each state.

Consider a two-state example for simplicity: $z \in \{z_L, z_H\}$. One can proceed as follows: 1) Pick an initial guess for the vector V^u, with values increasing in productivity. Starting from the highest productivity state, find the lowest constant contract wage which gives the worker the utility equivalent of the value of unemployment in the highest productivity state, $V^u(z_H)$. This is the first boundary wage $\underline{w}_H$. 2) Then consider contracts with a constant contract wage below $\underline{w}_H$ while in state z_L. This wage switches permanently to $\underline{w}_H$ when state z_H occurs. Find the lowest such wage that gives the worker the utility equivalent of the value of unemployment in state z_L, $V^u(z_L)$. This is the second boundary wage $\underline{w}_L$. 3) Given the wage bounds $\underline{w}_L, \underline{w}_H$, one can compute values $V(w, z, V^u), F(w, z, V^u)$ for $w \geq \underline{w}_L$, taking into account that productivity shifts lead to the wage changing according to the boundary wages. Given these functions, one can solve for wages from equation (B.5) and update the value of V^u using equation (B.6). This yields a fixed point problem in the vector V^u.

With two-sided limited-commitment contracting, the problem is complicated more significantly. There are (at least) two ways to approach the problem. In both, the outer loop is a fixed point iteration in a vector V^u. The two approaches to finding the Pareto-frontiers are as follows: 1) One can iterate on the recursive representation for $f(V, z, V^u)$ from an initial function. This is slow because the iteration takes place inside a fixed point iteration for the vector V^u. The recursive representation is not a contraction, but starting from an initial f corresponding to the unconstrained case should lead to the iteration converging to the Pareto-frontier (see Alvarez and Jermann (2001)). 2) One can use an alternative approach discussed in Alvarez and Jermann (2001). This approach involves taking advantage of the form of optimal contracts to write down a set of systems of equations in equilibrium wages and values. These systems are straightforward to solve and the approach is both accurate and fast, but the set of systems of equations grows quickly in the dimensionality of z. The numerical results presented in this paper are based on this latter approach, described next.

Recall that the contract is defined over value intervals $[V^u(z), \overline{V}(z, V^u)]$, which translate into wage intervals $[\underline{w}(z, V^u), \overline{w}(z, V^u)]$. The contract wage remains constant from one period to the next if the wage remains inside the corresponding interval. If a change in productivity causes the wage to lie outside the interval, the wage adjusts to the closest boundary. Relative to the one-sided case above, one now needs to find both the lower and upper wage bounds for each productivity state.

Consider again the two-state example with $z \in \{z_L, z_H\}$. The equilibrium can take two forms: In the first, the wage intervals $[\underline{w}(z, V^u), \overline{w}(z, V^u)]$ do not overlap for the two productivity levels, so there is no wage w that lies in both intervals. This means that when productivity changes, the contract wage must always change in response. In the second, the intervals do overlap, so for some wage levels a change in productivity causes no wage change. Consider these two forms in turn:

Case 1): Suppose that the intervals do not overlap, and the wage shifts whenever the state changes. The starting wage in a contract is generally some intermediate value within the wage interval, but once productivity switches, the wage switches to the closest feasible boundary value. After this initial switch, the wage will always be at one of two boundaries: If $z = z_L$, the wage is at the upper bound of the low state interval $\overline{w}_L$ where firm value is zero. If $z = z_H$, the wage is at the lower bound of the high state interval $\underline{w}_H$ where worker value equals $V^u(z_H)$. One can write down Bellman equations for the firm and worker values at these points, imposing the rule of switching above. This yields a system of equations in the values

$$F(\underline{w}_H, z_H; [\overline{w}_L, \underline{w}_H]), V(\underline{w}_H, z_H; V^u, [\overline{w}_L, \underline{w}_H]), F(\overline{w}_L, z_L; [\overline{w}_L, \underline{w}_H]), V(\overline{w}_L, z_L; V^u, [\overline{w}_L, \underline{w}_H]).$$

Similarly, one can write Bellman equations at the bounds $\underline{w}_L, \overline{w}_H$ for the values

$$F(\overline{w}_H, z_H; [\overline{w}_L, \underline{w}_H]), V(\overline{w}_H, z_H; V^u, [\overline{w}_L, \underline{w}_H]), F(\underline{w}_L, z_L; [\overline{w}_L, \underline{w}_H]), V(\underline{w}_L, z_L; V^u, [\overline{w}_L, \underline{w}_H])$$

Finally, one can write Bellman equations at the starting wages for the values

$$F(w_H^0, z_H; [\overline{w}_L, \underline{w}_H]), V(w_H^0, z_H; V^u, [\overline{w}_L, \underline{w}_H]), F(w_L^0, z_L; [\overline{w}_L, \underline{w}_H]), V(w_L^0, z_L; V^u, [\overline{w}_L, \underline{w}_H]).$$

In addition to these Bellman equations, in equilibrium we must have holding the two equations in (B.1), the two equations in (B.2), $V(\underline{w}_H, z_H; V^u, [\overline{w}_L, \underline{w}_H]) = V^u(z_H)$, $V(\underline{w}_L, z_L; V^u, [\overline{w}_L, \underline{w}_H]) = V^u(z_L)$, $F(\overline{w}_H, z_H; [\overline{w}_L, \underline{w}_H]) = 0$, and $F(\overline{w}_L, z_L; [\overline{w}_L, \underline{w}_H]) = 0$. The resulting system of equations can be reduced by hand to a small non-linear system, which can be solved numerically. If the solution to this system does not have the property that $\overline{w}_L < \underline{w}_H$, the equilibrium does not have the postulated property that the wage intervals do not overlap, and one proceeds to Case 2. If it does, we have found an equilibrium.

Case 2): Suppose we have $\underline{w}_L \leq \underline{w}_H \leq \overline{w}_L \leq \overline{w}_H$. Under this assumption, one can again write down a set of equations in the wage bounds, starting wages and values, imposing the appropriate wage rule in doing so. Depending on where the initial wages lie, there are several cases. Solving these systems, one can then verify whether the solutions are consistent with the assumptions made about the form of equilibrium.

This solution method becomes tedious when the dimension of z increases. With more wage intervals, there are more ways in which they may overlap. One thus needs to consider more systems of equations as possible equilibria.

References

Alvarez, F., and U. Jermann (2001): "Quantitative Asset Pricing Implications of Endogenous Solvency Constraints," *Review of Financial Studies*, pp. 1117– 1154.