Credit Risk and Contagion

Diogo Duarte\textsuperscript{a}, Rodolfo Prieto\textsuperscript{b}, Marcel Rindisbacher\textsuperscript{b}, Yuri F. Saporito\textsuperscript{c}

\textsuperscript{a}Florida International University
\textsuperscript{b}Boston University
\textsuperscript{c}FGV/EMAp

Abstract

We study contagion caused by default in a multi-firm equilibrium setting. Defaults can occur at predictable stopping times as in structural models or as a surprise at totally inaccessible stopping times as in reduced form credit risk models. Both types of default are modeled to be consistent with the firm’s balance sheet and aggregation over firms. Market prices and quantities of risk are derived in closed form. Their dependence on the type of default, risk aversion, the debt covenant, and on the number of firms is explored. If the number of firms increases, the market prices of risk converge to a well defined limit. The endogenous volatility and jump size components of debt and equity measuring the local sensitivity to diffusive and jump type cash-flow shocks of other firms, however, vanish as the number of firms becomes large. As a consequence, credit spreads asymptotically only depend on the firms own cash flow risk. We conclude that, in contrast to a production economy where quantities of risk are specified exogenously, contagion through the equilibrium pricing kernel cannot be a sizeable component of credit spreads if the number of firms is large. This novel contagion result questions some recent claims in the literature that attribute credit spreads mostly to contagion.

Keywords: Credit Risk, Credit Spreads, Contagion, Exchange Economy, Equilibrium, Idiosyncratic shocks, Risk Premia Representation.

JEL: G12, G13, G30
1. Introduction

Contagion of defaults to other asset classes are major concerns during financial crisis. The global financial crisis of 2007-2008 is a prime example that has renewed interests to understand the origins and dynamics of default contagion, i.e., the effects of the default of one firm on the value of assets of other firms. Understanding contagion in credit market is imperative for financial regulators to prevent systemic firms to spread credit risk across financial markets.

Equilibrium models with default, where the effects of contagion on the credit risk premium emerges endogenously, are difficult to solve. To identify sources of diversifiable and systematic risk is therefore challenging. Most existing models are cast in a single-firm setting and therefore fail to address contagion through the pricing kernel. The few multi-firm equilibrium models in the literature (see discussion below) are inspired by production economies and hence ignore the effects of contagion on endogenous quantities. Debt and equity in production economies are not modeled as contingent claims on the firm’s assets and the exogenously specified assets are directly interpreted as financial assets. Consequently, only the risk free interest rate and the market prices of risk, both obtained from the dynamics of the marginal utility of the representative investor on aggregate wealth, are determined endogenously in equilibrium. The quantities of risk (e.g., volatilities and jump sizes of debt) are therefore specified exogenously as well. As a result, contagion can solely occur through aggregate wealth and impact prices of risk. But credit spreads are the product of both quantities and prices of risk. Thus, in these production-based models, any component of risk with a positive price will automatically be part of the credit spread. By construction, there cannot be an offsetting effect on credit spreads generated by quantities of risk in production economies. To put it differently, volatilities and jumps associated with risk of other firms cannot vanish. As a consequence, the magnitude of the contagion component crucially depends on the assumptions made about the exogenous quantities of risk.

The effects of contagion through quantities of risk that emerge endogenously in an exchange economy have not been studied in the previous literature and are the focus of our paper. We study credit risk in a continuous time exchange economy with multiple firms. Agents have incomplete information about the regime that governs default intensities in recessions and booms. Cash flows of each firm are subject to default, consistently with the modeling of default in both structural and reduced form credit risk models. Locally predictable default occurs as in structural models introduced by Black and Cox (1976). As the firm’s cash flow approaches the covenant, the aggregate wealth moves accordingly, making the state price density to adjust smoothly in a locally predictable way. On the
other hand, if the firm’s cash flow experience an unexpected disaster event, default occurs at a totally inaccessible stopping time, aggregate wealth jumps and the state price density process displays a discontinuity. In our setting, both types of default are modeled such that the cash flow consequences result in a consistent balance sheet at the firm level. In equilibrium, market clearing requires that, given possible defaults, the sum of all firm cash flows correspond to aggregate wealth. At the same time, as both debt and equity of a firm are claims on underlying assets, payoffs to both debt and equity holders in aggregate must also correspond to the total wealth. Aggregate wealth itself then determines the equilibrium state price density through the marginal utility of a representative agent. To satisfy balance sheet constraints in a multi-firm setting for both types of default is a key novel feature of our equilibrium credit risk model.

The credit risk premium in our model is measured as the excess return of a single name credit default swap (CDS). Quantities of diffusive and jump risk can be separated in two parts. The first one, referred to as the contagion component, measures the exposure to risk from other firms’ cash flow risk and default, while the second component captures the firm-specific cash flow and default risk. This separation allows us to identify a contagion and an own cash flow component of the credit spread despite the fact that quantities of risk of each type depend on all sources of risk. A key result shows that if the number of firms grows, the exposure of sources of risk associated with other firms as measured by the corresponding quantities of risk vanishes, whereas the market prices of all sources of risk converge to a well-defined limit. As a consequence, the contagion component in the firms credit spread disappears if the number of firms is large, so that the credit spread only depends on the firm’s own cash flow component. Equilibrium prices of bonds and equity attain the lowest level in the limit, independent of whether the unknown true state of the economy is a boom or a recession. This occurs because, as the number of firms in the economy becomes large, the limit equilibrium pricing measure, referred to as the Contagion Measure, puts all its weight on the bad state. Asymptotically, the contagion channel is shut down and credit spreads are solely driven by shocks affecting the firm’s own cash flow, similar to the case where the regime of the economy is known.\footnote{For CARA preferences and identically, independently distributed firm cash-flows, and defaults, there is no contagion the model if the regime is known. The value of debt and equity in such a setting only depends on the firm’s own cash-flows. This follows as given the independence and CARA preference assumption, the relevant state price density is determined by the marginal utility evaluated at the firm’s own cash-flows. The same phenomena occurs if the number of firms is large as in this case only the bad state matters such that cash flow risk of other firms in the state price density becomes conditionally independent.}

All limit results for credit spreads in an exchange economy depend on the quantities of risk which are exogenously specified in a production economy. As a consequence, the effects
of contagion on credit spreads in an exchange and a production economy fundamentally differ.

Given our closed form equilibrium expressions, the impact of risk aversion, the face value of debt and the debt covenant are illustrated numerically. The sensitivity of endogenous firm leverage in a multi-firm setting is discussed. The tractability of our equilibrium expressions also allows us to derive closed-form expressions for basket of derivatives that became popular in the last decades. As the payoff of such securities depend on credit events of multiple firms, our framework is well suited to study the effects of contagion on such securities. We derive the equilibrium price and spread for first to default contract. The price of a collateralized debt obligation at time zero is also obtained in closed form. We investigate how the price levels and spreads of these contracts respond to changes with the model parameters such as risk aversion, the debt covenant and the initial cash flow level.

Our paper is closely related to Bai et al. (2015) and Benzoni et al. (2015). Bai et al. (2015) argue that the majority of the premium observed in the bond market corresponds to a contagion premium that emerges from the possibility of a firm’s default distress other assets. The authors conclude that firm’s own credit-event risk premium is only a small fraction of the credit spread and it has an upper bound of few basis points if the number of firms is large. By embedding a reduced-form credit risk model into a production economy, the authors argue that contagion effects triggered by credit events of other firms are the largest fraction of firm credit spreads observed in the market. The idea is that when firms experience a default, their cash flow at the terminal date, which represents the asset value at maturity, is negatively affected. Consequently, a risk averse agent discounts the payoff of the firm in these states at a higher rate, which, in turn, affects the price of debt and equity. Quantities of risk in production economy are fixed in terms of volatilities and jump sizes of the underlying assets and therefore cannot offset the effects of other firm’s defaults on the price of credit risk. In contrast, we show that in an exchange economy, if quantities are determined endogenously, the contagion component of credit spread vanishes because the quantities of risk associated with sources of risk from other firms vanish. The difference in conclusions about the contagion effect in a production and exchange economy are striking. Our results suggest that to assume that quantities of risk are exogenously given by assets and not determined endogenously by the delta of financial contracts like credit default swaps, fundamentally changes the effects of contagion if the number of firms is large.²

²In an online appendix, Bai et al. (2015) consider an exchange economy with reduced-form type of default only. Limits for quantities of risk relevant for contagion effects in credit risk premia also vanish in
Our model also complements the findings of Capponi and Larsson (2015) who investigate the contagion effect through the pricing kernel in a setting with both debt and equity. In contrast to our multi-firm setting, the authors focus on an economy with a single firm. As a consequence, contagion effects across the same class of assets cannot be studied in their framework. In addition, equity and bonds in their model are modeled as claims on the firm’s dividend that are completely detached from each other. Balance sheet identities do not hold. In their setup, it is possible that the firm is performing extremely well and paying high dividend yield and, simultaneously, will default on its debt. In our model, equity is considered a claim on cash flow after the payment of debt services, which gives us a consistent balance sheet and, as a consequence, bad shocks to cash flow impact both securities simultaneously given that their cash flow is linked through the balance sheet. In contrast to their model, our setting allows to study the co-movement of bonds and stocks due to cash flow shocks in a general equilibrium framework.

Our result shows that both diffusive and jump quantities of risk, i.e., volatility and jump sizes, play an important role in understanding credit risk premia. This is consistent with the empirical literature. Empirically, in contrast to assumptions in Bai et al. (2015), both jump size and volatility are not constant and allow to predict the dynamics credit spreads. Campbell and Taksler (2003) find that the effects of jump risk and credit spreads are time-varying and non-linear. Zhang et al. (2009) show that realized jump measures explain 19% of the total variations of credit spreads. Volatility and jump risk together can explain 53% of the total variation of credit spreads. Adding other firm specific and macro variables only increases the R-squared to 71%. Tauchen and Zhou (2011) demonstrate that jumps identified using high frequency data have strong explanatory power for CDS spreads in addition to realized volatility measures. These empirical results provide evidence for the importance of quantities of jump and diffusion risk associated with different forms of default in understanding credit spreads. To simply ignore the endogeneity of quantities of risk in equilibrium, as it is done in a multi-firm production economy setting in order to study contagion, is unlikely to provide models that fit both quantities and prices of risk, and, therefore, explain credit spreads, the volatility, and jumps of credit instruments like the CDS simultaneously.

The rest of this paper is organized as follows. Section 2 introduces our model by describing agents, firms, asset markets and the information structure. Section 3 presents the general equilibrium results and discusses the main features of the model. Section 4 presents the asymptotic behavior of equilibrium quantities when the economy is populated

their model. The result documented here is not an artifact of the two types of defaults and the balance sheet constraint imposed here.
by a large number of firms. Section 5 presents the expressions for the basket of derivatives. Section 6 investigates comparative static result numerically, and Section 7 concludes. All proofs are presented in the appendix.

2. Model

2.1. Firms

We consider a continuous-time economy on \([0, T]\). At initial time, there are \(N\) firms with same initial assets denoted by \(D_0\). We use subindex \(i\) to associate quantities to Firm \(i\), \(i \in \{1, 2, ..., N\}\). Without loss of generality, all firms are financed with both debt, denoted by \(B_i\), and equity, denoted by \(S_i\). Both securities are in positive supply of one unit and represent claims paid at time \(T\), which we detail in Table B.1.

\[\text{Table 1 about here.}\]

The first line displays the payoff of Firm \(i\) at time \(T\) when firm does not default. If Firm \(i\) does not default, the firm value is equal to the terminal asset value \(D_i(T)\). On the other hand, if Firm \(i\) defaults at time \(t \in (0, T)\) prior to the terminal date, debt holders receive the covenant \(L\) at the terminal date and equity drops to zero, as illustrated by the second row.

We model default following two traditional approaches. The first possibility is that a sequence of bad cash-flow news about the firm erodes its asset value until it reaches a covenant \(L\). The second possibility is that the firm’s asset experiences a disaster event that drops its asset value to the level \(L\).

Our model produces a coherent balance sheet where the asset is the sum of debt an equity at all times and across all states.\(^3\)

2.2. Investors

The economy is populated by investors with CARA utility function on terminal wealth, \(U(w) = -e^{-\gamma w}/\gamma\), where \(\gamma > 0\) is the absolute risk aversion parameter.

\(^3\)As surveyed in the related literature, previous models addressing the firms’ capital structure under a general equilibrium framework fail to generate this coherent balance sheet. Some researchers are able to circumvent this issue by assuming that both securities as independent contracts as claims on different underlying asset. In our opinion, the fact that equity is not a residual claim on the assets of the firm represents a drawback since it cannot help us to understand the joint movements of debt and equity generated by shocks to the cashflow.
2.3. Information Structure

Consider an economy with an underlying probability space \((\Omega, \mathcal{H}, \mathbb{P})\). There exists a set of fundamental shocks \((\epsilon, J, Y)\) that generate the natural filtration \(G_t = \sigma\left(\{Y(u)\}_{0 \leq u \leq t}\right) \vee \sigma\left(\{J(u)\}_{0 \leq u \leq t}\right) \vee \sigma(\epsilon)\). The random variable \(\epsilon\) is a binary random variable whose realization, \(\{b, g\}\), determines the regime of the cash flows news and defaults. We assume that the regime \(\epsilon\) is not known to investors, i.e., it is not measurable with respect to \(\mathcal{F}_t\), the filtration observed by investors, which will be defined below. The stochastic process \(Y\) is an \(\mathbb{R}^N\)-valued standard Brownian motion and \(J\) is a vector of default indicators, \(J_i(t) = 1_{\{\tau_i \leq t\}}, \ i \in \{1, ..., N\}\). The \(G_t\)-totally inaccessible stopping times \(\tau_i^r\) are defined by

\[
\tau_i^r = \inf\{t : \Theta_i \leq \lambda_i t\},
\]

where the \(\mathcal{H}\)-measurable random variables \(\Theta_i, i \in \{1, ..., N\}\), are i.i.d. and exponentially distributed (with mean one), so that given the regime \(\epsilon\), \(\tau_i^r\) is exponentially distributed with density \(\lambda_i e^{-\lambda_i t}, \ t > 0\), where \(\lambda_i \in \{\lambda_b, \lambda_g\}\), with \(0 < \lambda_g < \lambda_b\).

Investors observe the firms’ cash-flows news. These are composed of two parts, which account for their value pre- and post-default. Prior to default, the information flow about Firm \(i\), represented by \(D_i(t)\), evolves according to the following stochastic differential equation:

\[
dD_i(t) = \mu_\epsilon dt + \sigma dY_i(t), \quad D_i(0) = D_0.
\]

where \(\sigma > 0\) and \(\mu_\epsilon \in \{\mu_b, \mu_g\}\) with \(\mu_b < \mu_g\). The filtration \(\mathcal{F}_t^{D_i}\) denotes the natural filtration of the firm specific process \(D_i\). Post-default, assets are fixed at the covenant level \(L\). Default is the result of a locally non-predictable disaster event (natural disasters, wars, and operational losses), whose arrival time is given by \(\tau_i^r\), or structural (see Black and Cox (1976) and Leland (1994)), represented by the \(\mathcal{F}_t^{D_i}\)-predictable stopping time \(\tau_i^s\) defined by

\[
\tau_i^s = \inf\{t > 0 : D_i(t) = L\}.
\]

The default time of Firm \(i\) is thus given by the stopping time \(\tau_i \equiv \tau_i^s \wedge \tau_i^r\). Default can occur as a surprise or is locally predictable given the trajectory of \(D_i\).\(^4\) Investors can

\(^4\)Default times occurring at predictable stopping times correspond to the type of default captured by structural credit risk models whereas default times occurring at totally inaccessible stopping times correspond to the type of default considered in reduced form credit risk models. In the real economy, the default timing of a firm encompasses both of these aspects captured by our modeling choice through...
identify which type of default led the firm to declare bankruptcy. At any point in time, the cash-flow news of Firm $i$ can be represented by

$$CF_i(t) = D_i(t)1_{\{\tau_i > t\}} + L1_{\{\tau_i \leq t\}}.$$ 

Since investors do not observe the realization of $\epsilon$, they assign probabilities $P(\epsilon = g) = 1 - P(\epsilon = b) = p$, with $p \in (0, 1)$. Beliefs are dogmatic. Their information filtration $\mathcal{F}_t$ is the natural filtration generated by observing $D(t)$ and $J(t)$, so that $\mathcal{F}_t = \sigma(\{D(u)\}_{0 \leq u \leq t}) \lor \sigma(\{J(u)\}_{0 \leq u \leq t})$. Note that under filtration $\mathcal{F}_t$, the innovations are

$$dZ_i(t) = dY_i(t) + \frac{\mu - \bar{\mu}}{\sigma} dt$$
$$dm_i(t) = d1_{\{\tau^*_i \leq t\}} - \bar{\lambda}1_{\{\tau^*_i > t\}} dt,$$

where $\bar{\mu} = p\mu_g + (1 - p)\mu_b$ and $\bar{\lambda} = p\lambda_g + (1 - p)\lambda_b$. The vectors $(Z, m)$ define a standard Brownian motion and a vector of independent compensated Poisson processes with unit jump size with respect to the filtration $\mathcal{F}$. It worth noticing that, given the realization $\epsilon$, the filtration $\mathcal{F}$ and $\mathcal{G}$ have the same information.

We use the following definition throughout

**Definition 1.** The expectation $E[\cdot]$ is taken with respect to the state of the economy $\epsilon$, i.e., given any function of the form $\mathcal{G}_i(t, x, y)$, we denote $E[\mathcal{G}_i(t, x, y)] = p \mathcal{G}_g(t, x, y) + (1 - p) \mathcal{G}_b(t, x, y)$.

### 2.4. Securities Market

There are three classes of investment opportunities in our economy: a money market fund, debt and equity securities. The usual representation of money market funds dynamics is

$$dB_0(t) = rB_0(t)dt, \quad B_0(0) = 1,$$

the stopping time $\tau$. If a firm defaults because it misses a coupon payment, default occurs at a predictable stopping time given by the say semi-annual coupon date. At other times, default can occur as a surprise given cash flow information. Cases where from cash flow information alone default is not locally predictable and arrives as a true surprise are not rare given that cash flow or earnings information is self-reported and often smoothed over time. Our default time model structure is able to capture both types of defaults. The distinction between the two is further discussed in the Appendix A. Furthermore, in contrast to many reduced form models, default remains consistent with the balance sheet of the firm. This is crucial in order to aggregate firms cash flows and to consider default risk premia in a multi-firm setting in equilibrium.
and we set $r = 0$, so that $B_0(t) = 1, \forall t \in [0, T]$.

The two other securities are firms’ debt $B_i(t)$ and equity $S_i(t)$ which are posited to satisfy the following stochastic differential equations, respectively,

$$
\begin{align*}
    dB_i(t) &= \mu_i B_i(t) dt + \sum_{j=1}^{N} \sigma_{ij} B_i(t) dZ_j(t) + \sum_{j=1}^{N} J_{ij} B_i(t) dm_j(t), \\
    dS_i(t) &= \mu_i S_i(t) dt + \sum_{j=1}^{N} \sigma_{ij} S_i(t) dZ_j(t) + \sum_{j=1}^{N} J_{ij} S_i(t) dm_j(t).
\end{align*}
$$

All coefficients in equation (3) are determined endogenously in equilibrium.

### 2.5. Equilibrium

To construct the equilibrium, we rely on the representative agent representation and the martingale methods of Cox and Huang (1989) and Karatzas et al. (1990). Given the information structure, the state price density is conjectured to satisfy the following stochastic differential equation,

$$
\frac{dM(t)}{M(t^-)} = -\sum_{i=1}^{N} \theta_i(t^-) dZ_i(t) + \sum_{i=1}^{N} (\psi_i(t^-) - 1) dm_i(t),
$$

where $\theta_i(t)$ and $\psi_i(t)$ represent, respectively, the market price of diffusion and jump risk.

The representative agent is endowed with all assets in positive supply, providing her an initial wealth of $W(0) = W_0$. Consequently, the static optimization problem of the representative agent can be stated as

$$
\max_{W(T)} \mathbb{E} \left[ -\frac{e^{-\gamma W(T)}}{\gamma} \right] \quad \text{s.t.} \quad \mathbb{E} [M(T)W(T)] \leq W_0.
$$

The definition of a competitive equilibrium in this economy is presented as follows:

**Definition 2.** The representative agent equilibrium is a set of market prices of diffusion and jump risk represented by the functions $\{\theta_i(t), \psi_i(t)\}_{i \in \{1, \ldots, N\}}$, and a set of prices of debt and equity securities $\{B_i(t), S_i(t)\}_{i \in \{1, \ldots, N\}}$ such that:

(i) The representative agent maximizes the expected utility in (4).

(ii) Asset markets clear: $\sum_{i=1}^{N} (B_i(t) + S_i(t)) = W(t), \forall t \in [0, T]$.

---

5In the absence of intermediate consumption, the short-term interest rate is undetermined and the riskless bond serves as the numéraire. Without loss of generality, we set $r = 0$, for simplicity.
Remark 1. The positive cash covenant ensures a positive wealth process, e.g. \( W(T) = \ln + \sum_{i=1}^{N}(D_i(T) - L) \mathbb{1}_{\{\tau_i > T\}} > 0 \), and partially compensates the lack of wealth effects due to the CARA utility.

3. Analysis of equilibrium quantities

3.1. State price density and market prices of risk

We characterize the state price density in the next proposition.

**Proposition 1.** Define the \( N \)-dimensional vectors \( \mathbf{D}(t) \) and \( \mathbf{1}_d(t) \) as
\[
\mathbf{D}(t) \equiv (D_1(t), D_2(t), ..., D_N(t)), \\
\mathbf{1}_d(t) \equiv (\mathbb{1}_{\{\tau_1 > t\}}, \mathbb{1}_{\{\tau_2 > t\}}, ..., \mathbb{1}_{\{\tau_N > t\}}).
\]

At time \( t \), the unique state price density process in this economy is given by
\[
M(t) = M(t, \mathbf{D}(t), \mathbf{1}_d(t)),
\]
where the function \( M : [0, T] \times \mathbb{R}_+^N \times \{0, 1\}^N \rightarrow \mathbb{R} \) is given by
\[
M(t, x, y) = \frac{\mathbb{E}[F_\epsilon(t, x, y)]}{\mathbb{E}[F_\epsilon(0, \mathbf{D}(0), \mathbf{1}_d(0))]},
\]
and
\[
F_\epsilon(t, x, y) = \prod_{i=1}^{N} f_\epsilon(t, x_i, y_i).
\]

The next proposition outlines the details of the function \( f_\epsilon(t, x_i, y_i) \).

**Proposition 2.** The function \( f_\epsilon : [0, T] \times [L, \infty) \times \{0, 1\} \rightarrow \mathbb{R} \) defined by
\[
f_\epsilon(t, D_i(t), \mathbb{1}_{\{\tau_i > t\}}) = \mathbb{E} \left[ e^{-\gamma(D_i(T) - L)\mathbb{1}_{\{\tau_i > T\}}} \mid \mathcal{F}_t, \epsilon \right],
\]
is given by
\[
f_\epsilon(t, x, y) = 1 + ye^{-\lambda(T-t)}(a_\epsilon(t, x, \gamma) - a_\epsilon(t, x, 0)),
\]
where
\[
a_\epsilon(t, x, \gamma) = e^{-\gamma(x - L + (\mu - \frac{\sigma^2}{2})(T-t))} \left( \Phi\left(-d_\epsilon^{-}\right) - e^{\frac{2(L-x)(\mu - \gamma\sigma^2)}{\sigma^2}} \Phi\left(d_\epsilon^{+}\right) \right),
\]
\(^6\text{Let } V(t) \text{ denote a Markovian stochastic process adapted to the filtration } \mathcal{F}_t. \text{ We then consider the function } V(t, x, y) \text{ that satisfies } V(t) = V(t, \mathbf{D}(t), \mathbf{1}_d(t)). \text{ We will use this notation throughout the paper.}
Φ is the CDF of the standard Normal distribution, $d_{t+}^\epsilon = d_{t-}^\epsilon (T - t, L - x, \gamma)$, and
\begin{equation}
    d_{t+}^\epsilon(u, z, \beta) = \frac{z \pm (\mu_r - \beta \sigma^2)u}{\sigma \sqrt{t}}. \tag{8}
\end{equation}
Furthermore, $f_\epsilon$ satisfies the following properties:

(i) $f_\epsilon(t, x, y) \in [0, 1]$;

(ii) $\partial_x f_\epsilon(t, x, y) \leq 0$;

(iii) $f_g(t, x, y) < f_b(t, x, y)$;

(iv) there exists $\Delta L > 0$ such that $\partial_x \log f_g(t, x, y) < \partial_x \log f_b(t, x, y)$, for $x \in (L, L + \Delta L)$.

The function $f_\epsilon$ encapsulates the effects of cash-flow level on the state price density and also the impact of the firm’s default probability on it. If a firm is inactive at time $t$, then $y = 0$ and $f_\epsilon(t, x, y) = 1$. On the other hand, if a firm is still active at time $t$, $y = 1$ and the marginal sensitivity of the default probability to changes in the cash-flow level $x$ can be calculated by calculating $\partial_x f_\epsilon(t, x, y)$. The pair $(f_\epsilon, \partial f_\epsilon)$ along with $C_\epsilon(t, x_i, y_i) = F_\epsilon(t, x_i, y_i)/E[F_\epsilon(t, x, y)]$ characterize the market prices of risk. Note that since the terms $f_\epsilon(t, x_i, y_i)$ belong to the interval $[0, 1]$, their product also belongs to the same interval. By definition, $C_\epsilon(t, x, y)$ is the normalized product of such terms, assuming values in the interval $[0, 1]$ and integrating to one, so we use $C_\epsilon(t, x, y)$ to construct a new probability measure that we use to express prices. We label it the Contagion measure.

**Definition 3.** For each $(t, x_i, y_i) \in [0, T] \times [L, +\infty)^N \times \{0, 1\}^N$ and $A \in \sigma(\epsilon)$, define the Contagion Probability Measure on $(\Omega, \sigma(\epsilon))$ as
\begin{equation}
    C(A) \equiv C_{t,x,y}(A) = E[C_\epsilon(t, x_i, y_i)1_A].
\end{equation}

**Proposition 3.** The market prices of diffusion and jump risk processes can be expressed, respectively, as the following maps evaluated at $(t, D(t), 1_d(t))$
\begin{align}
    \theta_\epsilon(t, x, y) &= -\sigma \ E[C_\epsilon(t, x, y)\partial_x \log f_\epsilon(t, x_i, y_i)] = -\sigma \ E^C[\partial_x \log f_\epsilon(t, x_i, y_i)], \tag{9} \\
    \psi_\epsilon(t, x, y) &= E\left[\frac{C_\epsilon(t, x, y)}{f_\epsilon(t, x_i, y_i)}\right] = E^C\left[f_\epsilon(t, x_i, y_i)^{-1}\right]. \tag{10}
\end{align}

Proposition 3 contains several insightful results. Despite the apparent simplicity of the stochastic process describing the evolution of cash-flows and the CARA utility function
of the representative agent, both the market price of diffusion and jump risk are time-varying.

The market prices of risk associated to shocks of Firm $i$’s cash-flow are impacted by the shocks of all other firms’ cash-flow. The term $C_\epsilon(t, x, y)$ is the contagion channel through which firms’ idiosyncratic shocks propagate across all prices of risk in the economy. If the agent beliefs were $p = \{0, 1\}$, it would follow $C_\epsilon(t, x, y) = 1$. In this case, there would be no contagion effects due to the multiplicative structure of the utility function and market prices of risk would solely depend on shocks affecting the firm’s own cash-flow.

The market price of diffusion risk in (9) is inversely related to changes in the default probability of Firm $i$ due to a marginal cash-flow increase, captured by the term $\partial_x \log f_\epsilon$. The economic intuition is that a higher cash-flow news decrease the probability of a structural default which reduces the risk price associated with it. Consequently, the market price of diffusion risk is always positive since $\partial_x \log f_\epsilon \leq 0$, as proved in Proposition 2.

The market price of jump risk in (10) is inversely proportional to default probability captured by $f_\epsilon$. In fact, since $f_\epsilon \in [0, 1]$, the market price of jump risk is always bigger than one, indicating that the disaster events carry a positive premium. Since the market price of jump risk is the ratio between the jump frequency under the risk neutral measure and the physical jump frequency, our model reproduces the realistic feature of more frequent jumps under the risk neutral measure.

3.2. Prices

The pricing kernel presented in Proposition 1 allows us to compute the price of debt and equity securities at any point of time. The next proposition contains these results.

**Proposition 4.** The price of debt and equity are nonnegative functions of the state variables and given explicitly by

$$S_i(t, x, y) = \mathbb{E}^C \left[ \frac{S_\epsilon(t, x_i, y_i)}{f_\epsilon(t, x_i, y_i)} \right],$$

$$B_i(t, x, y) = L + \mathbb{E}^C \left[ \frac{B_\epsilon(t, x_i, y_i)}{f_\epsilon(t, x_i, y_i)} \right].$$

Functions $(S_\epsilon, B_\epsilon)$ are defined by

$$B_\epsilon(t, x, y) = (P - L)F_\epsilon^0(t, x, y) - \partial_\gamma F_\epsilon^{P,0}(t, x, y),$$

$$S_\epsilon(t, x, y) = \partial_\gamma F_\epsilon^{P,0}(t, x, y) - \partial_\gamma F_\epsilon^0(t, x, y)$$
and

\[ F_{t}^{0} = ye^{\gamma(x-L)-(\lambda+\gamma(\mu-\gamma\sigma^{2}/2))(T-t)} \]

\[ \cdot \left\{ e^{2\gamma(L-x)} \left( \Phi(d_{p-}) - \Phi(d_{l-}) \right) - e^{2(L-x)\frac{\gamma^{2}}{2}} \left( \Phi(d_{p-}) - \Phi(d_{l+}) \right) \right\} \]

\[ F_{t} = ye^{\gamma(x-L)-(\lambda+\gamma(\mu-\gamma\sigma^{2}/2))(T-t)} \left( \Phi(-d_{l-}) - e^{2(L-x)\frac{\gamma^{2}}{2}} \Phi(d_{l+}) \right) \]

where \( d_{l\pm} = d_{l}(T-t, L-x, \gamma) \), \( d_{p-} = d_{p}(T-t, P-x, \gamma) \) and \( d_{p-} = d_{p}(T-t, P-2L+x, \gamma) \), with \( d_{l\pm} \) given in (8).

The value of Firm \( i \), \( V_{i}(t) = S_{i}(t) + B_{i}(t) \), given by

\[ V_{i}(t) = \mathbb{E} \left[ \frac{M(T)}{M(t)} \left( D_{i}(T)1_{\{\tau_{i}>T\}} + L1_{\{\tau_{i}\leq T\}} \right) \mid F_{t} \right] . \]

is independent of \( P \), as \( M(T)/M(t), D_{i}(T) \) and \( L \) are independent of \( P \). The leverage ratio, defined as

\[ \ell(t) = \frac{B_{i}(t)}{V_{i}(t)} , \]

carries the dependence on the face value of debt through the debt contract.\(^7\)

We complete the equilibrium characterization by describing the price of a CDS. As pointed out by Arora and Longstaff (2012), a CDS can be seen as a simple insurance contract against the default on the debt of a particular entity. A CDS on Firm \( i \) is thus a claim in zero net supply that pays one unit of the numeraire at maturity if the firm defaults prior to the terminal date. Formally, its price is given by

\[ \text{CDS}_{i}(t, D(t), 1_{d}(t)) = \mathbb{E} \left[ \frac{M(T)}{M(t)} 1_{\{\tau_{i}\leq T\}} \mid F_{t} \right] . \]

so that the cost of protection against the total loss generated by the default of Firm \( i \) is \( (P - L) \times \text{CDS}_{i}(t, D(t), 1_{d}(t)) \).

**Proposition 5.** The CDS price for Firm \( i \) is given by

\[ \text{CDS}_{i}(t, x, y) = 1 - \mathbb{E} \left[ C_{i}(t, x, y) \frac{\mathcal{F}_{t}^{0}(t, x_{i}, y_{i})}{f_{t}(t, x_{i}, y_{i})} \right] , \]

where \( \mathcal{F}_{t}^{0}(t, x_{i}, y_{i}) \) is given in (13).

\(^7\)Despite the expressions in Proposition 4 prevent one from directly accessing the impact of the par value of debt on equity and debt levels due to the high nonlinearity on \( P \), our numerical exercise in Section 6 helps to clarify some of these effects.
3.3. Risk premia and contagion

In the next result, we provide a general representation of the risk premium in equilibrium for bonds, stocks and CDSs.

Proposition 6. The risk premium of asset \( H \in \{B, S, CDS\} \) is given by

\[
\begin{align*}
\mu^i_H(t, x, y) &= \mu^i_{H, \text{contagion}}(t, x, y) + \mu^i_{H, \text{cash-flow}}(t, x, y), \quad \text{where} \\
\mu^i_{H, \text{contagion}}(t, x, y) &= \sum_{j \neq i} \sigma^i_{H}(t, x, y) \theta_j(t, x, y) - \lambda \sum_{j \neq i} J^i_{H}(t, x, y) y_j(\psi_j(t, x, y) - 1), \\
\mu^i_{H, \text{cash-flow}}(t, x, y) &= \sigma_{H}^i(t, x, y) \theta_i(t, x, y) - \lambda J^i_{H}(t, x, y) y_i(\psi_i(t, x, y) - 1).
\end{align*}
\]

The diffusive volatilities and jumps components are given by

\[
\begin{align*}
\sigma^i_{H}(t, x, y) &= \sigma \text{ Cov}^C \left( \mathcal{H}_c(t, x, y) f_c(t, x, y), \partial_x \log f_c(t, x, y) \right), \\
\sigma_{H}^i(t, x, y) &= \sigma \text{ Cov}^C \left( \mathcal{H}_c(t, x, y) f_c(t, x, y), \partial_x \log f_c(t, x, y) \right) + \sigma \mathbb{E}^C \left[ \partial_x \left( \frac{\mathcal{H}_c}{f_c} \right)(t, x, y) \right], \\
J^i_{H}(t, x, y) &= \psi_j(t, x, y)^{-1} \text{ Cov}^C \left( \mathcal{H}_c(t, x, y) f_c(t, x, y), f_c(t, x, y)^{-1} \right), \\
J^i_{H}(t, x, y) &= -\mathbb{E}^C \left[ \frac{\mathcal{H}_c(t, x, y)}{f_c(t, x, y)} \right],
\end{align*}
\]

where \( \mathcal{H}_c \in \{B_c, S_c, -F^{0}_c\} \).

We decompose the risk premia of these securities into two parts in (15) as in Bai et al. (2015). The term in (16) is labeled the contagion component as it accounts for the \( ij \) — quantities of risk multiplied by the \( j \) — shocks market prices of risk, and thus, it measures the risk premium of security \( H_i \) associated to cash flows, and defaults risks of other firms. The term in (17) is labeled the cash-flow component, as it accounts for the \( ii \) — quantities of risk multiplied by the \( i \) — shocks market prices of risk and thus, it measures the risk premium of security \( H_i \) associated to its own cash flow and default risk.

A few more comments are worth pointing out. First, note that both terms depend on the vector \( (t, D(t), 1_d(t)) \), hence the properties of both components are determined by economy-wide effects. Second, the contagion component contains \( 2N - 2 \) terms when all firms are alive while the cash-flow component has only two terms. Despite this difference, it is not a trivial task to determine their relative sizes or signs. We explore a surprising limiting result in the next section and leave comparative statics for Section 6.
4. Large Number of Firms

The tractability of the market prices of risk and price levels expressions allow us to investigate their limiting behavior when the economy is populated by a large number of firms. In fact, a large body of the literature simply focus their analysis on a representative firm which prevents one from understanding the cross sectional effects of a large number of firms on equilibrium quantities (see Capponi and Larsson (2015)).

The next proposition presents the limiting behavior of the equilibrium quantities when the economy is populated by a large number of firms.

**Proposition 7.** As the number of firm in the economy becomes large, the following convergences hold $\mathbb{P}$-almost surely.

**Market prices of risk:**

$$
\lim_{N \to \infty} \psi_i(t) = f_b(t, D_i(t), 1_{\{\tau^*_i > t\}})^{-1},
$$

$$
\lim_{N \to \infty} \theta_i(t) = -\sigma \partial_x \log f_b(t, D_i(t), 1_{\{\tau^*_i > t\}}).
$$

**Quantities of risk:**

$$
\lim_{N \to \infty} \sigma_{ii}^H(t) = \sigma \partial_x \left( \frac{H_b}{f_b} \right) (t, D_i(t), 1_{\{\tau^*_i > t\}}),
$$

$$
\lim_{N \to \infty} J_{ii}^H(t) = -\left( \frac{H_b}{f_b} \right) (t, D_i(t), 1_{\{\tau^*_i > t\}}),
$$

$$
\lim_{N \to \infty} \sigma_{ij}^H(t) = \lim_{N \to \infty} J_{ij}^H(t) = 0.
$$

**Securities prices:**

$$
\lim_{N \to \infty} B_i(t) = \frac{B_b(t, D_i(t), 1_{\{\tau^*_i > t\}})}{f_b(t, D_i(t), 1_{\{\tau^*_i > t\}})},
\quad \lim_{N \to \infty} S_i(t) = \frac{S_b(t, D_i(t), 1_{\{\tau^*_i > t\}})}{f_b(t, D_i(t), 1_{\{\tau^*_i > t\}})},
$$

where $H \in \{B, S, \text{CDS}\}$ and $H_b \in \{B_b, S_b, -F^0_b\}$, respectively. The expected return associated with contagion effects is given by

$$
\lim_{N \to \infty} \mu^{\text{contagion}}_H(t) = 0.
$$

As the number of firms in the economy increases, market prices of risk not only converge to their well defined limits, but they converge to their value in bad state of the economy. However, the convergence is independent whether the true state of the economy is actually good or bad, that is, it is irrespective of the investors’ beliefs. Note further, that as shown in Proposition 2 (iii), $f_\epsilon$ is a decreasing function on $\mu_\epsilon$, which implies that the market price of jump risk takes its minimum value at the bad state.
Similarly, from Proposition 2 (iv), the market price of diffusion risk assume its minimum value at the bad state.

Critically, the results in Proposition 7 show that the contagion channel is eliminated when the economy is populated by a large number of firms. As the number of firms increases, the Contagion measure converges to the Dirac mass of the bad state. The main reason behind this result is that the probability of default of a firm in a bad state is always larger than the same probability of default in a good state of the economy because the expected cash-flow is lower and the reduced-form shocks arrive at a higher frequency. This idea is analytically represented by the inequality \( f_g(t, x, y)/f_b(t, x, y) \leq 1 \), shown in Proposition 2. For this reason, as the number of firms increases, the weighted average between bad and good states shifts toward the bad states and converges to it in the limit. Thus, a large number of firms not only brings the economy to the equilibrium in a bad state, but it also eliminates the contagion across firms.

An analysis of the price level of equity and debt is more involved, but we provide intuition. The higher the expected growth of cash-flow news process, the higher the likelihood that both debt and equity contract will be honored. Consequently, the higher the prices of such contracts. This implies that both prices reach their minimum value when the state of the economy is bad and the cash-flow drift is low. In this state, the risk of a structural default is perceived as higher which deplete prices and increase the likelihood of the firm’s default. Thus, as the number of firms growth, the price of debt and equity converge to their lowest level, which is the value they assume in the bad state of the economy. Our numerical exercise in Section 6 illustrates this point.

5. Application: Valuation of multiname credit derivatives

In recent decades, credit derivatives have become widely used financial instruments used by market participants to gain exposure or to diversify default risk. According to the Bank of International Settlements (BIS), the outstanding gross notional of credit derivatives by the end of 2016 was over $10 trillion dollars. Contagion is a major concern for the valuation of credit derivatives and therefore for financial stability. In this section we derive closed form expressions for prices and credit spreads for first to default (FtD) and credit default obligations (CDO) contracts. The sensitivity of the equilibrium quantities to the model parameters are explored in Section 6.

5.1. First to Default Basket

Contrary to a CDS that offers protection against a specific firm’s default, a first to default basket (FtD) protects the buyer against the default of any company pre-specified in the basket. Despite being considered the precursors of the CDOs, these contracts are
simpler to describe and price. In addition, its short-dated nature helped credit strategist
from large investment banks to market this product worldwide.\footnote{See Baird, Jane. “Investors return to simple structured credit tool.”, Reuters March 13 2008. Online.}

Formally, the price of the FtD at time $t$ can be expressed as

$$\text{FtD}(t, D(t), 1_d(t)) = \mathbb{E} \left[ \frac{M(T)}{M(t)} \mathbb{1}_{\{\min_i \tau_i \leq T\}} \bigg| \mathcal{F}_t \right].$$

The next proposition presents the FtD price along with its spread.

**Proposition 8.** The FtD price over the $N$ firms in our economy is

$$\text{FtD}(t, x, y) = 1 - \frac{1}{\mathbb{E}[F_{c}(t, x, y)]} \prod_{i=1}^{N} \mathbb{E}[\mathcal{F}_{i}(t, x_{i}, y_{i})]. \quad (19)$$

The FtD spread is

$$\mu_{FtD}^{i}(t, x, y) = \sum_{j=1}^{N} \sigma_{FtD}^{ij}(t, x, y) \theta_{j}(t, x, y) - \lambda \sum_{j=1}^{N} J_{FtD}^{ij}(t, x, y) \psi_{j}(t, x, y) - 1,$$

where $\sigma_{FtD}$ and $J_{FtD}$ are given in Appendix B.

The tractability of the model provides a closed form expression for the FtD price even for a very large number of firms. Contrary to the pricing methods investigated in Caldana et al. (2016), our closed form pricing expressions avoid costly Monte Carlo simulations that can be challenging in settings with many firms.

5.2. Collateralized Debt Obligation

A Collateralized Debt Obligation (CDO) is a contract that guarantees protection against losses due to defaults of firms pre-specified in the CDO contract. Instead of protecting against the default of every single firm, a CDO is divided into tranches, and provides protection against losses that represent a fraction of the total portfolio value outlined in the contract. A tranche with attachment points $A_{L}$ and $A_{U}$ does not suffer any losses if the total loss of the portfolio of bonds is less than $A_{L}$ percent of the initial value of the portfolio. If the losses are larger than $A_{L}$ percent of the initial value of the portfolio, then the tranche will suffer losses up to $A_{U}$ percent of the initial value of the portfolio.

Similar to the case of the FtD, we assume that the CDO is written over all the $N$
firms of our economy. Define the number of defaulted firms by time \( t \) by

\[
N_t = \sum_{i=1}^{N} \mathbb{1}_{\{\tau_i \leq t\}}.
\]

Formally, since all bonds have the same face value and covenant, the price of the a tranche of a CDO with attachment points \( A_L < A_U \), at time \( t \), is giving by

\[
\text{CDO}(t, D(t), I_d(t)) = \mathbb{E} \left[ \frac{M(T)}{M(t)} \frac{1}{N} (\min\{N_T, NA_U\} - \min\{N_T, NA_L\}) \mid \mathcal{F}_t \right].
\] (20)

**Proposition 9.** The price of a CDO tranche with attachment points \( A_L < A_U \), at time \( t = 0 \), is giving by

\[
\text{CDO}(0, D(0), I_d(0)) = \mathbb{E} \left[ A_U C_1^1(A_U) - A_L C_1^1(A_L) + \frac{1}{N} (C_2^2(A_U) - C_2^2(A_L)) \right],
\] (21)

where \( C_1^1 \) and \( C_2^2 \) are given by

\[
C_1^1(A) = \sum_{m=0}^{N} \frac{1}{\Lambda} \mathbb{1}_{\{m \geq NA\}} \left( \frac{e^{-\lambda T} F_{\epsilon}^0(0, D_0, 1)}{a_\epsilon(0, D_0, 0)} \right)^{N-m} \mathbb{P}(N_T = m \mid \epsilon),
\]

\[
C_2^2(A) = \sum_{m=0}^{N} \frac{1}{\Lambda} \mathbb{1}_{\{m < NA\}} m \left( \frac{e^{-\lambda T} F_{\epsilon}^0(0, D_0, 1)}{a_\epsilon(0, D_0, 0)} \right)^{N-m} \mathbb{P}(N_T = m \mid \epsilon),
\]

\[
N_T \mid \epsilon \sim \text{Bin}(N, q_\epsilon),
\]

\[
q_\epsilon = 1 - e^{-\lambda_T a_\epsilon(0, D_0, 0)}.
\]

The CDO price is provided for the beginning of the economy where all cash-flow levels are the same. Similar to the case of the FtD basket, the CDO contract is written over all firms in the economy. Consequently, its price depends on all cash-flow shocks by construction. Thus, the notion of contagion previously discussed for equity, bonds and CDS does not apply for FtD and CDO securities defined in our economy.

6. Comparative Statics

This section presents a numerical exercise to investigate the sensitivity of equilibrium quantities with respect to the model parameters. We analyze a symmetric equilibrium at time \( t = 0 \) when all the firms are alive and have same initial cash-flow level \( D_0 \). The parameters are chosen as in Bai et al. (2015) and set the prior probability of being in a good state as \( p = 0.75 \) and set \( \lambda_b = 1\% \) and \( \lambda_g = 0.2\% \) such that the average intensity stays at \( \bar{\lambda} = 0.4\% \). The time to maturity equals \( T = 5 \) and a risk aversion parameter \( \gamma = 2 \). To
model the cash-flow process, we follow the parametrization of Barberis et al. (2015) and set $\mu_b = 2\%$ and $\mu_g = 6\%$ such that the average cash-flow growth is $\bar{\mu} = p\mu_g + (1-p)\mu_b = 5\%$, while the volatility is $\sigma = 25\%$. All initial cash flows are set at $D_0 = 1.75$. The par value of debt is set at $P = 1.5$ and set covenant at $L = 1$. The economy is initially populated with $N = 10$ firms. Parameters are summarized in Table B.2.

Panel (a) of Figure B.1 shows the sensitivity of the market price of diffusion risk with respect to risk aversion parameter $\gamma$, the covenant $L$, the number of firms $N$ and the initial cash-flow $D_0$. The graph on the top left shows that the market price of diffusion risk is an increasing function of the risk aversion parameter. However, the relationship is not linear as risk aversion impacts in a nonlinear way $f_{\epsilon}(t,x_i,y_i)$; the market price of diffusion risk is a nonlinear transformation of $f_{\epsilon}(t,x_i,y_i)$. The top right graph shows that as the number of firms increases, the market price of risk decreases to a limit value, as predicted by Eq.(18) in Proposition 7, the market price of diffusion risk converges its value in the bad regime. The left bottom plot illustrates that the response of the market price of diffusion risk to variations on the covenant level $L$ vary significantly. Contrary to the previous cases, the probability of being in the good state of the economy $p$ has a significant effect on the level of the price of diffusion risk. In the bottom right graph, we show that, as the initial cash-flow $D_0$ increases, the level of the market price of diffusion risk decreases. The economic intuition is as follows: A higher level of initial cash-flow decreases the likelihood of the firm experiencing a structural default. Consequently, the price of risk associated with these shocks also decreases. Note that in all four plots, the market price of diffusion risk is always higher in the case when the probability of being in a good state is higher.

Panel (b) of Figure B.1 shows the sensitivity of the market price of jump risk with respect to the same variables. On the top row, the price of jump risk displays a similar behavior to the price of diffusion risk as risk aversion and the number of firms in the economy increase. The bottom left figure shows that, contrary to the price of diffusion risk, the market price of jump risk decreases monotonically with the covenant level $L$. The bottom right picture also shows that the effect on the price of jump risk is different from the one in the price of diffusion risk. As the initial cash-flow process increases and the likelihood of structural defaults decreases, firms increase their exposure to reduced-form default. Consequently, the price of risk associated with such shocks increases and converges to its upper bound.
Panels (a) and (b) of Figure B.2 show the sensitivity of bond and equity prices, respectively, with respect to risk aversion parameter $\gamma$, the par value of debt $P$, the number of firms $N$ and the initial cash-flow $D_0$. As illustrated in the top left figures, an increase in risk aversion reduces the price of both contracts. Also, similar to the analysis for the prices of risk, the plot in the top right illustrates the convergence of prices to its limits when the state of the economy is bad. The bottom left plot shows that the effect of the par value of debt is the opposite to debt and equity contracts, as the par value of debt increases, the debt price increases as the payoff at the terminal date also increases. At the same time, it becomes harder for cash-flows to end above the face value of debt, and the stock price decreases. The figures in the bottom right shows that bond and equity prices have a bell shape as a function of the initial cash-flow. When the cash-flow is very large, the likelihood of a structural default goes to zero, while the probability of debt repayment goes to one.

[Figure 2 about here.]

Figure B.3 shows the sensitivity of the bond and equity premium to the model parameters. Risk premium is decomposed into two components: the contagion and the cash-flow risk premium. The contagion risk premium is the compensation the representative agent requires to hold Firm $i$‘s security and for being exposed to shocks of the cash-flow of Firm $j$. The cash-flow risk premium is the compensation for being exposed to Firm $i$‘s shocks. Contrary to the findings of Bai et al. (2015), the contagion risk premium is considerably smaller than the risk premium generated by shocks affecting the firm’s cash-flow. The main reason for such difference is that, in our equilibrium model, the impact of the parameters on the diffusive volatility and jumps of the cash-flow shocks are substantially larger than the effects on the risk loadings of other cash-flow shocks. Thus, both the equity and the bond premium are mainly driven by the cash-flow risk premium. In addition, for most of the parameter configurations displayed in Figure B.3, the contagion risk premium has the opposite sign of the cash-flow risk premium, indicating the complementarity between assets.

[Figure 3 about here.]

Figure B.4 illustrates the behavior of the equilibrium leverage as we vary the risk aversion, the initial cash-flow, the face value and the covenant. The plot on the top left shows that risk aversion decreases the equity value faster than it decreases the debt value. Consequently, leverage increases as the risk aversion increases. The top right graph shows that when the initial cash-flow is very large or very close to the covenant, the leverage is
one, since on both cases the equity value goes to zero. The graph also show that there is an endogenous leverage level that is minimized for a specific $D_0$. Since the face value of debt does not impact the firm value, the left bottom graph displays the same pattern as the bottom left graph in B.2 that illustrates the effect of the face value of debt on the bond price level. Last, the bottom right figure shows that for large values of the covenant, the leverage converges to one since the equity converges to zero and debt to $L$. Nevertheless, the leverage is a non-monotonic function of the covenant.

Panel (a) and (b) in Figure B.5 shows the price level and the spread of the CDS, respectively. Similar to the case of bond and equity, the cash-flow premium command a large portion of the premium. The contagion premium is generally negative and considerably smaller than the contagion premium.

We investigate the sensitivity of the basket of derivatives to our parametrization. Figure B.6 shows the same analysis for the FtD. The price level plots show a quick convergence to the upper bound of the FtD, fixed at one in the example. Contrary to the previous contracts, the FtD contract is multiname, consequently, at the initial time, the contract is exposed equally to all cash-flow shocks by construction.

Figure B.7 illustrates the sensitivity of the CDO price level. We note that the effects are of order magnitude smaller than the effects on the previous contracts. The reason is that as it can be observed, for small variations of the parameters, the CDO price converges to its upper limit $A_U - A_L = 0.3$. The spread is not provided since the explicit characterization is only provided for time $t = 0$ and the general formula for the CDO price on $t \in (0, T)$ cannot be obtained in closed-form.

7. Conclusion

Our paper shows that any model trying to explain the impact of contagion on credit spreads has to account for its effects on both quantities and prices of risk. Contrary to the existent models in the literature, we show that in multi-firm equilibrium setting, consistent with firm’s balance sheet aggregation, the contagion component of credit spreads becomes negligible as the number of firms increases. In contrast to the empirical findings of a
branch of the literature, we argue that in an exchange economy, where quantities of risk are specified endogenously, contagion through the equilibrium pricing kernel cannot be a sizable component of credit spreads if the number of firms is large. In addition, our limiting behavior analysis of equilibrium quantities provides a theoretical justification of the commonly adopted assumption on the conditionally diversifiability of firms’ default.

Our numerical exercise provides further support to the idea that the contagion component of equity and debt risk premium are quite small under an exchange economy with consistent aggregate firm’s balance sheet. Our findings are supported by some recent empirical studies that provide evidence of the importance of quantities of risk in understanding credit spreads. We conclude that to simply ignore the endogeneity of quantities of risk in equilibrium, as it is done in a multi-firm production economy setting in order to study contagion, is unlikely to provide models that fit both quantities and prices of risk and therefore explain both credit spreads and the volatility and jumps of a CDS simultaneously.
Appendix A. Default times

The understanding of the different types of stopping times involved in the modeling of the default time $\tau_i$ is crucial to determine the properties of such stopping time. A stopping time $\rho$ is said to be $\mathbb{F}^X$-predictable if there exists a sequence of $\mathbb{F}^X$ stopping times $(\rho_n)_{n \in \mathbb{N}}$ such that $\rho_n < \rho$ when $\rho \neq 0$ and $\rho_n \to \rho$ almost surely. Intuitively, a $\mathbb{F}^X$-predictable stopping time is announced by a sequence of $\mathbb{F}^X$-stopping times. Under mild assumptions on the filtration, the predictability of the stopping time $\rho$ is equivalent to having the form $\rho = \inf\{t \geq 0 : X_t = a\}$, for some continuous adapted process $(X_t)_{t \geq 0}$ and $a \in \mathbb{R}$, i.e., $\mathbb{F}^X$-predictable stopping times are given by passage times of a process $X$ generating the filtration. On the other hand, a stopping time is said to be $\mathbb{F}^X$-totally inaccessible if $\mathbb{P}(\rho = \rho_p < +\infty) = 0$ for any $\mathbb{F}^X$-predictable stopping time $\rho_p$. From the definition, it follows that a totally inaccessible stopping time cannot be predicted. The jump time of Poisson process is considered the classical example of a total inaccessible stopping time.

These definitions show that, while $\tau^*_i$ is a totally inaccessible stopping time and $\tau^s_i$ is a predictable stopping time. Default occurs at either type of stopping times. This raises the question of to what class of stopping time does $\tau_i$ belong to. In fact, we argue that $\tau_i = \tau^*_i \wedge \tau^s_i$ is neither totally inaccessible nor predictable. Firstly, take $\rho_p = \tau^s_i$ in the definition of totally inaccessible stopping time. It follows that

$$\mathbb{P}(\tau_i = \tau^s_i < +\infty) = \mathbb{P}(\tau^s_i < \tau^*_i < +\infty) > 0,$$

which proves that $\tau_i$ is not totally inaccessible. Moreover, if there was a sequence of stopping times $\rho_n$ satisfying $\rho_n < \tau_i$ with $\rho_n \to \tau_i$, then on the event $\{\tau^*_i < \tau^s_i\}$, that has positive probability, we would have $\rho_n < \tau^*_i$ and $\rho_n \to \tau^*_i$, which contradicts the fact that $\tau^*_i$ is totally inaccessible. Hence, $\tau_i$ is also not predictable. Nonetheless, when restricted to $\{\tau^*_i < \tau^s_i\}$ (resp. $\{\tau^*_i > \tau^s_i\}$), the stopping time $\tau_i$ is totally inaccessible (resp. predictable).

Appendix B. Proofs

Proof of Proposition 1. In equilibrium, the representative household holds all assets in positive supply. Consequently, the agent’s final wealth $W(T)$ consists of the sum of all equity and debt contracts in the economy and it is represented by

$$W(T) = L \sum_{i=1}^N 1_{\{\tau_i \leq T\}} + \left( \sum_{i=1}^N (D_i(T) - P)^+ + \min(P, D_i(T)) \right) 1_{\{\tau_i > T\}}$$

$$= LN + \left( \sum_{i=1}^N (D_i(T) - P)^+ + \min(P, D_i(T)) - L \right) 1_{\{\tau_i > T\}}$$

$$= LN + \sum_{i=1}^N (D_i(T) - L) 1_{\{\tau_i > T\}},$$

where the second equality follows from the identity $1_{\{\tau_i \leq T\}} = 1 - 1_{\{\tau_i > T\}}$. The pricing kernel $M(t)$ satisfies

$$\tilde{\Lambda}(t) = \mathbb{E} \left[ e^{-\gamma W(T)} \mid \mathcal{F}_t \right] = e^{-\gamma NL} \mathbb{E} \left[ e^{-\gamma \sum_{i=1}^N (D_i(T) - L) 1_{\{\tau_i > T\}}} \mid \mathcal{F}_t \right]. \quad (B.1)$$
By construction, $M$ is a true martingale with respect to the investor’s filtration $\mathcal{F}$. Using the shocks independence, we write $(D_i(T) - L)\mathbb{1}_{(\tau_i > T)}$ as

$$(D_i(T) - L)\mathbb{1}_{(\tau_i > T)} = (D_i(t) + \mu_i(T - t) + \sigma(Y_i(T) - Y_i(t)) - L)\mathbb{1}_{(\tau_i^* > T)} \mathbb{1}_{(\tau_i^* > T)},$$

and factor the right hand side of (B.1) as the following product of expectations:

\[
\tilde{\Lambda}(t)M(t) = e^{-\gamma NL} \prod_{i=1}^{N} \mathbb{E} \left[ e^{-\gamma(D_i(t) + \mu_i(T - t) + \sigma(Y_i(T) - Y_i(t)) - L)\mathbb{1}_{(\tau_i > T)}} \mid \mathcal{F}_t, \epsilon = g \right] \\
+ e^{-\gamma NL} (1 - p) \prod_{i=1}^{N} \mathbb{E} \left[ e^{-\gamma(D_i(t) + \mu_i(T - t) + \sigma(Y_i(T) - Y_i(t)) - L)\mathbb{1}_{(\tau_i > T)}} \mid \mathcal{F}_t, \epsilon = b \right] \\
= e^{-\gamma NL} \prod_{i=1}^{N} \mathbb{E} \left[ e^{-\gamma(D_i(t) - L + \sigma(\tilde{Y}_i^*(T) - \tilde{Y}_i^*(t)))\mathbb{1}_{(\tau_i > T)}} \mid \mathcal{F}_t, \epsilon = g \right] \\
+ e^{-\gamma NL} (1 - p) \prod_{i=1}^{N} \mathbb{E} \left[ e^{-\gamma(D_i(t) - L + \sigma(\tilde{Y}_i^*(T) - \tilde{Y}_i^*(t)))\mathbb{1}_{(\tau_i > T)}} \mid \mathcal{F}_t, \epsilon = b \right],
\]

where the expectations above are conditioned on the investor’s filtration and the state of the economy and the shifted Brownian motion $\tilde{Y}_i^*$, is defined as $\tilde{Y}_i^*(T) - \tilde{Y}_i^*(t) = Y_i(T) - Y_i(t) + \frac{\sigma}{\sqrt{2\gamma}}(T - t)$.

Given the realization of $\epsilon$, both filtrations, $\mathcal{F}$ and $\mathcal{G}$, have the same information. From the expression derived in Proposition 2 and denoting the left hand side of (B.2) by $e^{-\gamma NL} F(t, D(t), \mathbb{1}_d(t))$, it follows that

$$F(t, x, y) = pF_g(t, x, y) + (1 - p)F_b(t, x, y) = \mathbb{E}[F_{\epsilon}(t, x, y)],$$

where

$$F_{\epsilon}(t, x, y) = \prod_{i=1}^{N} f_{\epsilon}(t, x_i, y_i).$$

where $f_{\epsilon}(t, x_i, y_i)$ is defined in (6). Consequently, the expression for the state price density in (B.1) is completely characterized and reduces to

$$M(t) = \frac{F(t, D(t), \mathbb{1}_d(t))}{\Lambda},$$

where $\Lambda = F(0, D(0), \mathbb{1}_d(0)) = e^{\gamma NL} \tilde{\Lambda}$. This proves the existence and uniqueness of the equilibrium. \(\square\)

**Proof of Proposition 2.** From p.147 of Jeanblanc et al. (2009), the joint distribution between the shifted Brownian motion $\tilde{Y}_i^*(T)$ and its minimum $\tilde{Y}_i^* = \min_{s \in [0,T]} \tilde{Y}_i^*(s)$ is given by

$$p_{\epsilon}(u, v) = \frac{1}{\sqrt{2\pi T^3}} e^{\frac{-u^2}{2T}} e^{\frac{-2v^2}{2T}} e^{\frac{-2uv}{2T}},$$

if $v < u \land 0$.

\[
\text{if } v > u \land 0.
\]

(B.4)

Next, notice that $f_{\epsilon}(t, x, y)$ can be written as follows:

$$f_{\epsilon}(t, x, y) = \mathbb{E} \left[ e^{-\gamma(D_i(t) - L + \sigma(\tilde{Y}_i^*(T) - \tilde{Y}_i^*(t)))\mathbb{1}_{(\tau_i^* > T)}\mathbb{1}_{(\tau_i^* > T)}} \mid D_i(t) = x, \mathbb{1}_{(\tau_i^* > T)} = y, \epsilon \right].$$

24
Using the independence between the structural and the reduced-form defaults, the expression for $f_\epsilon(t,x,y)$ becomes

$$f_\epsilon(t,x,y) = 1 - ye^{-\lambda_\epsilon(T-t)} + ye^{-\lambda_\epsilon(T-t)} \mathbb{E} \left[ e^{-\gamma(x-L+\sigma(T_{\gamma_{T}}(T))\|T_{\gamma_{T}} - \epsilon\|_{\gamma_{T}})} \right].$$

Using the density described in (B.4), we rewrite the expectation as

$$\mathbb{E} \left[ e^{-\gamma(x-L+\sigma(T_{\gamma_{T}}(T))\|T_{\gamma_{T}} - \epsilon\|_{\gamma_{T}})} \right] = \int_{\frac{L-x}{\sigma}}^{+\infty} \int_{v}^{+\infty} e^{-\gamma(x-L+\sigma u)} p_\epsilon(u,v) dudv + \int_{-\infty}^{\frac{L-x}{\sigma}} \int_{v}^{+\infty} p_\epsilon(u,v) dudv.$$

Defining

$$a_\epsilon(t,x,\gamma) = \int_{\frac{L-x}{\sigma}}^{+\infty} \int_{v}^{+\infty} e^{-\gamma(x-L+\sigma u)} p_\epsilon(u,v) dudv,$$

and evaluating the integral, (7) and (8) follow. By construction, $a_\epsilon(t,x,\gamma) \geq 0$. From the relationship

$$\int_{-\infty}^{\frac{L-x}{\sigma}} \int_{v}^{+\infty} p_\epsilon(u,v) dudv = 1 - \int_{\frac{L-x}{\sigma}}^{+\infty} \int_{v}^{+\infty} p_\epsilon(u,v) dudv = 1 - a_\epsilon(t,x,0),$$

we write (6).

Properties of $f_\epsilon$:

(i) The result follows immediately from the inequality

$$0 < e^{-\gamma(D_\epsilon(T)-L)1_{\{\gamma_{T} > T\}}1_{\{\gamma_{T} > T\}}} \leq 1,$$

since $D_\epsilon(T) \geq L$.

(ii) To verify that $\partial_x f_\epsilon \leq 0$, we use the Leibniz integral rule. Given the relationship

$$f_\epsilon(t,x,y) = 1 - ye^{-\lambda_\epsilon(T-t)} + ye^{-\lambda_\epsilon(T-t)} \left( \int_{\frac{L-x}{\sigma}}^{+\infty} \int_{v}^{+\infty} e^{-\gamma(x-L+\sigma u)} p_\epsilon(u,v) dudv + \int_{-\infty}^{\frac{L-x}{\sigma}} \int_{v}^{+\infty} p_\epsilon(u,v) dudv \right),$$

it follows that

$$\partial_x f_\epsilon(t,x,y) = ye^{-\lambda_\epsilon(T-t)} \left( \frac{1}{\sigma} \int_{\frac{L-x}{\sigma}}^{+\infty} e^{-\gamma(x-L+\sigma u)} p_\epsilon \left( u, \frac{L-x}{\sigma} \right) du \right)$$

$$- \int_{\frac{L-x}{\sigma}}^{+\infty} \int_{v}^{+\infty} e^{-\gamma(x-L+\sigma u)} p_\epsilon(u,v) dudv \frac{1}{\sigma} \int_{\frac{L-x}{\sigma}}^{+\infty} p_\epsilon \left( u, \frac{L-x}{\sigma} \right) du$$

$$= ye^{-\lambda_\epsilon(T-t)} \left( \frac{1}{\sigma} \int_{\frac{L-x}{\sigma}}^{+\infty} e^{-\gamma(x-L+\sigma u)} p_\epsilon \left( u, \frac{L-x}{\sigma} \right) du \right)$$

$$- \int_{\frac{L-x}{\sigma}}^{+\infty} \int_{v}^{+\infty} e^{-\gamma(x-L+\sigma u)} p_\epsilon(u,v) dudv.$$
Therefore, for all $u \geq (L - x)/\sigma$, we have $e^{-\gamma(x-L+\sigma u)} \leq 1$, which implies $\partial_u f_e \leq 0$.

(iii) Consider $f_e$ as a function of $\lambda_e$ and $\mu_e$ and denote

$$f(t, x, y, \mu_e, \lambda_e) = f_e(t, x, y), \quad \text{and} \quad a(t, x, \gamma, \mu_e) = a_e(t, x, \gamma)$$

so that $f(t, x, y, \mu_e, \lambda_e) = 1 + ye^{-(T-t)}(a(t, x, \gamma, \mu_e) - a(t, x, \mu_e))$. Notice now that, since $0 \leq f \leq 1$, we must have $a(t, x, \gamma, \mu_e) - a(t, x, \mu_e) \leq 0$. Then

$$\partial_\lambda f(t, x, y, \mu_e, \lambda_e) = -(T-t)e^{-(T-t)}(a(t, x, \gamma, \mu_e) - a(t, x, \mu_e)) > 0,$$

which implies that

$$f(t, x, y, \mu_e, \lambda_b) > f(t, x, y, \mu_e, \lambda_0) = 1 + ye^{-(T-t)}(a(t, x, \gamma, \mu_e) - a(t, x, 0, \mu_e)).$$

Hence, once we prove that $\kappa(\mu) = a(t, x, \gamma, \mu) - a(t, x, 0, \mu)$ is a decreasing function of $\mu$, the result is proved. In order to simplify notation, we drop the arguments $T - t$ and $\mu$ from $d_+$. For a strictly positive $\gamma$ and for $\gamma = 0$, respectively, we have

$$\partial_\mu a(t, x, \gamma, \mu) = -\gamma(T-t)a(t, x, \gamma, \mu) + e^{-\gamma(x-L)}e^{-(T-t)}\frac{\sqrt{T-t}}{\sigma} \left( \phi(d_+(x-L, \gamma)) - e^{\frac{2(L-x)(\mu-\gamma^2)}{\sigma^2}} \phi\left(d_+(L-x, \gamma)\right) \right)$$

$$- \frac{2(L-x)}{\sigma^2} e^{-\gamma(x-L)}e^{-(T-t)}\frac{2(L-x)}{\sigma^2} e^{\frac{2L-x}{\sigma^2}} \Phi(d_+(L-x, \gamma), d_+(L-x, 0)) \right)$$

By defining

$$C(\gamma, \mu) = e^{-\gamma(x-L)}e^{-(T-t)}\frac{\sqrt{T-t}}{\sigma} \phi\left(d_+(L-x, \gamma)\right) + \frac{2(L-x)}{\sigma^2} \Phi(d_+(L-x, \gamma), d_+(L-x, 0))$$

$$\kappa_1(\gamma, \mu) = -C(\gamma, \mu) \left( \frac{\sqrt{T-t}}{\sigma} \phi(d_+(L-x, \gamma)) - \frac{2(L-x)}{\sigma^2} \phi\left(d_+(L-x, \gamma)\right) \right)$$

$$\kappa_2(\gamma, \mu) = e^{-\gamma(x-L)}e^{-(T-t)}\frac{\sqrt{T-t}}{\sigma} \phi\left(d_+(L-x, \gamma)\right),$$

it follows that

$$\partial_\mu \kappa(\mu) = \partial_\mu a(t, x, \gamma, \mu) - \partial_\mu a(t, x, 0, \mu)$$

$$= -\gamma(T-t)a(t, x, \gamma, \mu) + \kappa_1(\gamma, \mu) - \kappa_1(0, \mu) + \kappa_2(\gamma, \mu) - \kappa_2(0, \mu).$$

From the computation of the derivatives of $\kappa_1$ with respect to $\gamma$,

$$\partial_\gamma \kappa(\gamma) = (x - L + \mu(T-t)) + \gamma\sigma^2(T-t) + d_+(x-L, \gamma)\sigma(T-t)) \kappa_2(\gamma, \mu)$$

$$= (-d_+(x-L, \gamma)\sigma(T-t) + d_+(x-L, \gamma)\sigma(T-t)) \kappa_2(\gamma, \mu) = 0,$$
it follows that $\kappa_2(\gamma, \mu) = \kappa_2(0, \mu)$. In addition, notice that
\[
\partial_{\gamma}\kappa_1(\gamma, \mu) = \left(\frac{-(x-L+\mu(T-t)) + \gamma\sigma^2(T-t) - 2(L-x)}{d_+(L-x, \gamma)\sigma(T-t)}\right) \kappa_1(\gamma, \mu)
- C(\gamma, \mu)\phi(d_+(L-x, \gamma)) \left( (T-t)d_+(L-x, \gamma) - \frac{2(L-x)\sqrt{T-t}}{\sigma} \right)
= \frac{d_+(L-x, \gamma)\sigma\sqrt{T-t}}{\sigma} C(\gamma, \mu) \frac{\sqrt{T-t}}{\sigma} \phi(d_+(L-x, \gamma))
+ \frac{d_+(L-x, \gamma)\sigma\sqrt{T-t}}{\sigma} C(\gamma, \mu) \frac{2(L-x)}{\sigma^2} \Phi(d_+(L-x, \gamma))
- C(\gamma, \mu)\phi(d_+(L-x, \gamma)) \left( (T-t)d_+(L-x, \gamma) - \frac{2(L-x)\sqrt{T-t}}{\sigma} \right)
= C(\gamma, \mu) \frac{2(L-x)\sqrt{T-t}}{\sigma} \left( \phi(d_+(L-x, \gamma)) + d_+(L-x, \gamma)\Phi(d_+(L-x, \gamma)) \right).
\]
From the well-known fact that $\phi(z) + z\Phi(z) > 0$, for any $z \in \mathbb{R}$, we conclude that $\partial_{\gamma}\kappa_1(\gamma, \mu) < 0$. Thus, $\kappa_1(\gamma, \mu) < \kappa_1(0, \mu)$, we conclude that $\partial_{\gamma}\kappa(\gamma, \mu) < 0$. Thus,
\[
f_b(t, x, y) > 1 + ye^{-\lambda_b(T-t)}(a(t, x, \gamma, \mu_b) - a(t, x, 0, \mu_b))
\geq 1 + ye^{-\lambda_b(T-t)}(a(t, x, \gamma, \mu_b) - a(t, x, 0, \mu_b)) = f_g(t, x, y).
\]
(iv): Define
\[
h(t, x, y) = \log f_b(t, x, y) - \log f_g(t, x, y).
\]
Then $h(t, L, y) = 0$ and $h(t, x, y) > 0$, for all $x > L$, by part (iv). Hence, there must exist $\Delta L > 0$, such that $\partial_{\gamma}h(t, x, y) > 0$, for $x \in (L, L + \Delta L)$.

Proof of Proposition 3. To obtain the market prices of diffusion and jump risk, we apply Itô’s lemma on (B.3) and match the coefficients with their counterparts on the stochastic differential equation for the state price density
\[
\frac{dM(t)}{M(t-)} = -\sum_{i=1}^N \theta_i(t_-)dZ_i(t) + \sum_{i=1}^N (\psi_i(t_-) - 1) dm_i(t), \quad (B.5)
\]
where $(Z, m)$ are given in (1) and (2). The process $M(t)$ is a true martingale under $\mathcal{F}$ given the
assumption of \( r = 0 \). An application of Itô’s lemma on (B.3) gives

\[
\Lambda \, dM(t) = p \, dF_g(t, D(t), \mathbb{1}_d(t_\lambda)) + (1 - p) \, dF_b(t, D(t), \mathbb{1}_d(t_\lambda))
\]

\[
= \sum_{i=1}^{N} \left( p \sigma \frac{\partial f_g}{\partial x}(t, D_i(t), \mathbb{1}_{\{\tau_i^+ > t\}}) \prod_{j \neq i} f_g(t, D_j(t), \mathbb{1}_{\{\tau_j^+ > t\}}) \\
+ (1 - p) \sigma \frac{\partial f_b}{\partial x}(t, D_i(t), \mathbb{1}_{\{\tau_i^+ > t\}}) \prod_{j \neq i} f_b(t, D_j(t), \mathbb{1}_{\{\tau_j^+ > t\}}) \right) dZ_i(t)
\]

\[
\quad + \sum_{i=1}^{N} \left( p \prod_{j \neq i} f_g(t, D_j(t), \mathbb{1}_{\{\tau_j^+ > t\}})(1 - f_g(t, D_i(t), \mathbb{1}_{\{\tau_i^+ > t\}})) \\
+ (1 - p) \prod_{j \neq i} f_b(t, D_j(t), \mathbb{1}_{\{\tau_j^+ > t\}})(1 - f_b(t, D_i(t), \mathbb{1}_{\{\tau_i^+ > t\}})) \right) dm_i(t).
\]

Dividing both sides of (B.6) by (B.3) and matching the coefficients with the ones in (B.5), we obtain

\[
\psi_i(t, x, y) = \frac{p \prod_{j \neq i} f_g(t, x_j, y_j) + (1 - p) \prod_{j \neq i} f_b(t, x_j, y_j)}{p \prod_{i=1}^{N} f_g(t, x_i, y_i) + (1 - p) \prod_{i=1}^{N} f_b(t, x_i, y_i)} = E \left[ \frac{C_i(t, x, y)}{f_i(t, x_i, y_i)} \right],
\]

\[
\theta_i(t, x, y) = -\frac{p \sigma \partial_x f_g(t, x_i, y_i) \prod_{j \neq i} f_g(t, x_j, y_j) + (1 - p) \sigma \partial_x f_b(t, x_i, y_i) \prod_{j \neq i} f_b(t, x_j, y_j)}{p \prod_{i=1}^{N} f_g(t, x_i, y_i) + (1 - p) \prod_{i=1}^{N} f_b(t, x_i, y_i)}
\]

\[
= -\sigma E [C_i(t, x, y) \partial_x \log f_i(t, x_i, y_i)].
\]

Results in (9) and (10) follow from an application of the Contagion measure.

\[
\square
\]

\textbf{Proof of Proposition 4.} The risky debt price is derived using the pricing kernel derived in the previous section. According to Duffie and Lando (2001), the fair price of Firm i’s risky debt can be expressed as

\[
M(t)B_i(t, D(t), \mathbb{1}_d(t)) = E \left[ M(T) \left( P \mathbb{1}_{\{\tau_i^+ > T\}} + D_i(T) \mathbb{1}_{\{\tau_i^+ > T\}} \mathbb{1}_{\{L \leq D_i(T) \leq P\}} + L \mathbb{1}_{\{\tau_i^+ \leq T\}} \right) \mid \mathcal{F}_t \right]
\]

\[
= LM(t) + E \left[ M(T) \left( P \mathbb{1}_{\{\tau_i^+ > T\}} + D_i(T) \mathbb{1}_{\{\tau_i^+ > T\}} \mathbb{1}_{\{L \leq D_i(T) \leq P\}} - L \mathbb{1}_{\{\tau_i^+ \leq T\}} \right) \mid \mathcal{F}_t \right]
\]

\[
= LM(t) + \frac{P}{\Lambda} \prod_{j \neq i} f_g(t, D_j(t), \mathbb{1}_{\{\tau_j^+ > t\}})(P F_g^0 - F_g^0) + F_g^0(t, D_i(t)) \mathbb{1}_{\{\tau_i^+ > t\}}
\]

\[
+ \frac{1 - P}{\Lambda} \prod_{j \neq i} f_b(t, D_j(t), \mathbb{1}_{\{\tau_j^+ > t\}})(P F_b^0 - F_b^0) + F_b^0(t, D_i(t)) \mathbb{1}_{\{\tau_i^+ > t\}}
\]

\[
+ \frac{1}{\Lambda} \prod_{j \neq i} f_r(t, D_j(t)) \mathbb{1}_{\{\tau_j^+ > t\}} + \mathbb{1}_{\{\tau_i^+ > t\}} (P F_r^0 - F_r^0) + F_r^0(t, D_i(t)) \mathbb{1}_{\{\tau_i^+ > t\}}
\]

where

\[
F_{g,k}^0 = E \left[ (\hat{Y}_{1 i}^r(t) - \hat{Y}_{1 i}^r(t))^{k} e^{-\gamma(D_i(t) - L)\mathbb{1}_{\{\tau_i^+ > T\}} \mathbb{1}_{\{L \leq D_i(t) \leq P\}}} \mathcal{F}_t, \epsilon \right],
\]

28
Moreover, since
\[ \mathcal{F}_{\epsilon}^k = \mathbb{E}\left[ (\hat{Y}^i_t(T) - \hat{Y}^i_t(t))^k e^{-\gamma(D_t(T) - L)\mathbb{1}_{\{\tau^i > T\}} \mathbb{1}_{\{\tau_t > T\}} \mid \mathcal{F}_t, \epsilon]\right]. \]

we only need to evaluate the expectations $\mathcal{F}_{\epsilon}^0$ and $\mathcal{F}_{\epsilon}^{P,0}$. Hence,
\[ \mathcal{F}_{\epsilon}^{P,0} = ye^{-\gamma(x - L) - \lambda_s(T - t)} \mathbb{E}\left[ e^{-\gamma\sigma(Y^i_t(T) - Y^i_t(t))\mathbb{1}_{\{\tau^i > T\}} \mathbb{1}_{\{\tau^i > T\}} \mid \mathcal{F}_t, \epsilon}\right] \]

Thus, the expression for the bond price of Firm $i$ is given in (12). Similarly, we characterize the stock price of Firm $i$ using the no-arbitrage condition
\[
M(t)S_i(t, D(t), \mathbb{1}_d(t)) = \mathbb{E}t \left[ M(T)(D_t(T) - P)\mathbb{1}_{\{\tau^i > T\}} \mathbb{1}_{\{\tau_t > T\}} \mid \mathcal{F}_t, \epsilon = g \right]
\]
\[
= p \left( (D_t(t) - P)\mathbb{E}t \left[ M(T)\mathbb{1}_{\{\tau^i > T\}} \mathbb{1}_{\{D_t(T) > P\}} \mid \mathcal{F}_t, \epsilon = g \right] \right) + \sigma \mathbb{E}t \left[ M(T)(\hat{Y}^i_t(T) - \hat{Y}^i_t(t))\mathbb{1}_{\{\tau^i > T\}} \mathbb{1}_{\{D_t(T) > P\}} \mid \mathcal{F}_t, \epsilon = g \right] + (1 - p) \left( (D_t(t) - P)\mathbb{E}t \left[ M(T)\mathbb{1}_{\{\tau^i > T\}} \mathbb{1}_{\{D_t(T) > P\}} \mid \mathcal{F}_t, \epsilon = b \right] \right) + \sigma \mathbb{E}t \left[ M(T)(\hat{Y}^i_t(T) - \hat{Y}^i_t(t))\mathbb{1}_{\{\tau^i > T\}} \mathbb{1}_{\{D_t(T) > P\}} \mid \mathcal{F}_t, \epsilon = b \right]
\]
\[
= \frac{p}{\Lambda} \prod_{j \neq i} f_0(t, D_j(t), \mathbb{1}_{\{\tau^j > t\}}) ((D_t(t) - P)(\mathcal{F}_{g}^0 - \mathcal{F}_{g}^{P,0})(t, D_t(t), \mathbb{1}_{\{\tau^i > t\}}) + \sigma(\mathcal{F}_{g}^1 - \mathcal{F}_{g}^{P,1})(t, D_t(t), \mathbb{1}_{\{\tau^i > t\}})
\]
\[
+ \frac{1 - p}{\Lambda} \prod_{j \neq i} f_0(t, D_j(t), \mathbb{1}_{\{\tau^j > t\}}) ((D_t(t) - P)(\mathcal{F}_{b}^0 - \mathcal{F}_{b}^{P,0})(t, D_t(t), \mathbb{1}_{\{\tau^i > t\}}) + \sigma(\mathcal{F}_{b}^1 - \mathcal{F}_{b}^{P,1})(t, D_t(t), \mathbb{1}_{\{\tau^i > t\}})).
\]

Thus, the stock price expression reduces to (11).

**Proof of Proposition 5.** The CDS price is derived following the same arguments presented in the proof of Proposition 4. From the definition of the CDS, the fair price is
\[
\text{CDS}_i(t, D(t), \mathbb{1}_d(t)) = \mathbb{E}t \left[ \frac{M(T)}{M(t)} \mathbb{1}_{\{\tau_t \leq T\}} \mid \mathcal{F}_t \right]
\]
which implies

\[
= \mathbb{E} \left[ \frac{M(T)}{M(t)} (1 - \mathbb{1}_{\{T > t\}}) \mid \mathcal{F}_t \right],
= 1 - \mathbb{E} \left[ C_e(t, D(t), \mathbb{1}_{\{t \leq T\}}) \frac{J^0_e(t, D_{ij}(t), \mathbb{1}_{\{T > t\}})}{f_e(t, D_{ij}(t), \mathbb{1}_{\{T > t\}})} \right].
\]

and thus (14) follows.

\[\square\]

**Proof of Proposition 6.** Prices are generically represented by \( \mathbb{E}[C_e(t, x, y)G_e(t, x_i, y_i)] \), so that diffusive volatilities take the form

\[\sigma^{ij}(t, x, y) = \sigma \mathbb{E} \left[ \frac{\partial C_e}{\partial x_j}(t, x, y)G_e(t, x_i, y_i) \right],\]

\[\sigma^{ii}(t, x, y) = \sigma \left( \mathbb{E} \left[ \frac{\partial C_e}{\partial x_i}(t, x, y)G_e(t, x_i, y_i) \right] + \mathbb{E}[C_e(t, x, y)\partial_xG_e(t, x_i, y_i)] \right).\]

Noticing that

\[\frac{\partial f_e}{\partial x_j}(t, x, y) = \left( \prod_{k \neq j} f_e(t, x_k, y_k) \right) \frac{\partial f_e}{\partial x_j}(t, x, y) = f_e(t, x, y)\partial_x \log f_e(t, x, y),\]

and that

\[\frac{\partial C_e}{\partial x_j}(t, x, y) = \frac{1}{\mathbb{E}[f_e(t, x, y)]} \frac{\partial f_e}{\partial x_j}(t, x, y) - \frac{\mathbb{E}[C_e(t, x, y)\partial_x \log f_e(t, x, y)]}{\mathbb{E}[f_e(t, x, y)]} \mathbb{E} \left[ \frac{\partial f_e}{\partial x_j}(t, x, y) \right] = C_e(t, x, y)\partial_x \log f_e(t, x, y) - C_e(t, x, y)\mathbb{E}[C_e(t, x, y)\partial_x \log f_e(t, x, y)],\]

we can write the expression for diffusive volatilities as

\[\sigma^{ij}(t, x, y) = \sigma \mathbb{E} \left[ C_e(t, x, y)\partial_x \log f_e(t, x, y) \right] G_e(t, x_i, y_i)\]

\[\quad - \mathbb{E}[C_e(t, x, y)G_e(t, x_i, y_i)] \mathbb{E}[C_e(t, x, y)\partial_x \log f_e(t, x, y)],\]

\[\sigma^{ii}(t, x, y) = \sigma \left( \mathbb{E} \left[ C_e(t, x, y)\partial_x \log f_e(t, x, y) \right] G_e(t, x_i, y_i)\right)\]

\[\quad - \mathbb{E}[C_e(t, x, y)G_e(t, x_i, y_i)] \mathbb{E}[C_e(t, x, y)\partial_x \log f_e(t, x, y)]\]

\[\quad + \mathbb{E}[C_e(t, x, y)\partial_x G_e(t, x_i, y_i)].\]

To obtain the jump expressions, notice that when \( y_j = 1 \),

\[J^{ij}(t, x, y) = \mathbb{E} \left[ C_e(t, x, y - y_j e_j)G_e(t, x_i, y_i - y_i \mathbb{1}_{\{i = j\}}) \right] - \mathbb{E}[C_e(t, x, y)G_e(t, x_i, y_i)],\]

where \( e_j \) is the vector of zeros whose \( j^{th} \) entry equals one. Thus, it follows that

\[C_e(t, x, y - y_j e_j) = \frac{f_e(t, x, y - y_j e_j)}{\mathbb{E}[f_e(t, x, y - y_j e_j)]} = \frac{f_e(t, x, y)}{\mathbb{E}[f_e(t, x, y)]} \frac{C_e(t, x, y)}{f_e(t, x, y)},\]

which implies

\[J^{ij}(t, x, y) = \frac{1}{\psi_j(t, x, y)} \mathbb{E} \left[ C_e(t, x, y - y_j e_j)G_e(t, x_i, y_i - y_i \mathbb{1}_{\{i = j\}}) \right] - \mathbb{E}[C_e(t, x, y)G_e(t, x_i, y_i)].\]
To derive the risk premium of a security $H_i \in \{B_i, S_i, CDS_i\}$, we apply Itô’s lemma on the left hand side of the following $\mathcal{F}$-martingale

$$
M(t)H_i(t, D(t), \mathbb{1}_d(t)) = \mathbb{E} [M(T)H_i(T, D(T), \mathbb{1}_d(T)) \mid \mathcal{F}_t].
$$

and use no-arbitrage condition to obtain the expected return representation

$$
\mu^H(t, x, y) = \sum_{j=1}^{N} \sigma^H_{ij}(t, x, y) \theta_j(t, x, y) - \lambda \sum_{j=1}^{N} y_j J^H_{ij}(t, x, y)(\psi_j(t, x, y) - 1)
$$

The representation in (15) follows from identifying the terms $ii$ from $ij$ in the above equation. The formula for the spread follows from setting $G_i = (B_i, S_i, -\mathcal{F}^0_t) f^{-1}$ for $H_i \in \{B_i, S_i, CDS_i\}$ and the market prices of risk in (9) and (10).

Proof of Proposition 7. Let

$$
X_i(t) = \frac{f_g(t, D_i(t), \mathbb{1}_{(\tau^i > t)})}{f_b(t, D_i(t), \mathbb{1}_{(\tau^i > t)})},
$$

so that

$$
C_g(t, D(t), \mathbb{1}_d(t)) = \frac{1}{p + (1-p) \prod_{i=1}^{N} \frac{1}{X_i(t)}}, \quad C_b(t, D(t), \mathbb{1}_d(t)) = \frac{1}{(1-p) + p \prod_{i=1}^{N} X_i(t)}.
$$

Since $0 \leq f_g(t, x, y) < f_b(t, x, y)$, it follows that $X_i(t) \in (0, 1)$. Since the sequence of random variables $(X_i(t))_{i \in \mathbb{N}}$ is i.i.d., the law of large numbers implies that

$$
\lim_{N \to \infty} \left( \prod_{i=1}^{N} X_i(t) \right)^{1/N} = e^{\mathbb{E}[\log X_i(t)]} \in (0, 1), \text{ } \mathbb{P}\text{-a.s.}
$$

Thus, for large enough $N$, there exists $c \in (0, 1)$ such that

$$
\prod_{i=1}^{N} X_i(t) \leq c^N, \text{ } \mathbb{P}\text{-a.s.}
$$

As a result,

$$
\lim_{N \to \infty} \prod_{i=1}^{N} X_i(t) = 0 \text{ and } \lim_{N \to \infty} \prod_{i=1}^{N} \frac{1}{X_i(t)} = \infty, \text{ } \mathbb{P}\text{-a.s.} \quad \text{(B.9)}
$$

Hence,

$$
\lim_{N \to \infty} C_g(t, D(t), \mathbb{1}_d(t)) = 0 \text{ and } \lim_{N \to \infty} C_b(t, D(t), \mathbb{1}_d(t)) = \frac{1}{1-p}, \text{ } \mathbb{P}\text{-a.s.}
$$

Next, we prove that the cross diffusive volatilities and jumps presented in (B.7) and (B.8) converge to zero. Consider $G_i = \mathcal{H}_i / f_i$, with $\mathcal{H}_i \in \{B_i, S_i, -\mathcal{F}^0_t\}$ for a claim $H_i \in \{B_i, S_i, CDS_i\}$, and denote the probability of the good state with respect to the contagion measure $\mathcal{C}$ by $p^N$, where we make the dependence on $N$ explicit and omit the dependence on $(t, D(t), \mathbb{1}_d(t))$ for the sake of a simpler notation. We also will use the a superscript $j$ instead of the argument $(t, D_j(t), \mathbb{1}_{(\tau^j > t)})$.
The convergence in (B.9) guarantees that

$$\lim_{N \to \infty} p_N^c = 0 \text{ and } p_N^c \leq \frac{P}{P + (1-P)c - N}, \text{ P-a.s.}$$

Moreover,

$$Cov^c(H_c, G_t) = p_N^c H_c G_t + (1 - p_N^c) H_c G_t - (p_N^c H_c + (1 - p_N^c) H_c) (p_N^c G_t + (1 - p_N^c) G_t)$$

$$= (p_N^c - (p_N^c)^2) H_c G_t + (1 - p_N^c - (1 - p_N^c)^2) H_c G_t$$

$$= p_N^c (1 - p_N^c) H_c G_t - p_N^c (1 - p_N^c) H_c G_t$$

Thus, the quantities of risk become

$$\sigma^c(t) = p_N^c (1 - p_N^c) (\partial_x \log f_y^j - \partial_y \log f_y^j)(H^c_{ij} \frac{f_y^j}{f_y^j} - \frac{H^c_{ij}}{f_y^j})$$

$$J^c_H(t) = p_N^c (1 - p_N^c) \left( \frac{1}{f_y^j} - \frac{1}{f_y^j} \right) \left( H^c_{ij} \frac{f_y^j}{f_y^j} - \frac{H^c_{ij}}{f_y^j} \right).$$

Hence

$$\lim_{N \to \infty} \sigma^c_H(t) = 0 \text{ and } \lim_{N \to \infty} J^c_H(t) = 0, \text{ P-a.s.}$$

The convergence for $\sigma^c_H(t)$ and $J^c_H(t)$ follows a similar argument.

Next, we evaluate the asymptotic behavior of the contagion risk premium. In this case, the convergence analysis is more subtle because it involves an infinite summation of quantities that are converging to zero. We outline the argument for the diffusive volatility component of $\mu^c_{t, \text{contagion}}(t)$. The jump component follows the exact same argument.

Since

$$\theta_j(t) = -\sigma (\partial_x \log f_y^j + p_N^c (\partial_x \log f_y^j - \partial_y \log f_y^j)),$$

the cross diffusive reward of the contagion risk premium is

$$\sum_{j \neq i} \sigma^c_H(t) \theta_j(t) = -\sigma (p_N^c)^2 (1 - p_N^c) \left( \frac{H^c_{ij}}{f_y^j} - \frac{H^c_{ij}}{f_y^j} \right) \sum_{j \neq i} (\partial_x \log f_y^j - \partial_y \log f_y^j)^2$$

$$\quad - \sigma p_N^c (1 - p_N^c) \left( \frac{H^c_{ij}}{f_y^j} - \frac{H^c_{ij}}{f_y^j} \right) \sum_{j \neq i} (\partial_x \log f_y^j - \partial_y \log f_y^j) \partial_x \log f_y^j.$$

Since the random variables $(\partial_x \log f_y^j - \partial_y \log f_y^j)^2$ and $(\partial_x \log f_y^j - \partial_y \log f_y^j) \partial_x \log f_y^j$ are i.i.d. with finite mean, the law of large numbers implies that both averages

$$\frac{1}{N} \sum_{j \neq i} (\partial_x \log f_y^j - \partial_y \log f_y^j)^2 \text{ and } \frac{1}{N} \sum_{j \neq i} (\partial_x \log f_y^j - \partial_y \log f_y^j) \partial_x \log f_y^j,$$

converge P-a.s.

Since

$$\lim_{N \to \infty} N p_N^c = 0,$$

32
this concludes the proof.

Proof of Proposition 8. Using that the shocks are independent, we have

\[
M(t) \ FtD(t, D(t), \1_d(t)) = \mathbb{E} \left[ M(T) \mathbb{1}_{\{\min, \tau \leq T\}} \mid \mathcal{F}_t \right]
\]
\[
= \mathbb{E} \left[ M(T) \left( 1 - \prod_{i=1}^{N} \mathbb{1}_{(\tau_i > T)} \right) \mid \mathcal{F}_t \right],
\]
\[
= M(t) - \mathbb{E} \left[ M(T) \prod_{i=1}^{N} \mathbb{1}_{(\tau_i > T)} \mid \mathcal{F}_t \right]
\]
\[
= M(t) - \frac{1}{A} \mathbb{E} \left[ \prod_{i=1}^{N} e^{-\gamma(D_i(T) - L)\mathbb{1}_{(\tau_i > T)}} \mathbb{1}_{(\tau_i > T)} \mid \mathcal{F}_t \right]
\]
\[
= M(t) - \frac{1}{A} \mathbb{E} \prod_{i=1}^{N} \left[ e^{-\gamma(D_i(T) - L)\mathbb{1}_{(\tau_i > T)}} \mathbb{1}_{(\tau_i > T)} \mid \mathcal{F}_t \right]
\]
\[
= M(t) - \frac{1}{A} \mathbb{E} \prod_{i=1}^{N} \mathbb{E} \left[ J^0_{\epsilon}(t, D_i(t), \mathbb{1}_{(\tau_i > T)}) \right].
\]

Hence, (19) follows from a simplifying terms above. The diffusive volatility and jumps can be recovered from an application of Itô’s formula on the previous expression:

\[
d FtD(t, D(t), \1_d(t)) = \left[ \ldots \right] dt + \sum_{i=1}^{N} \sigma^i_{FLD}(t, D(t, \1_d(t)) dZ_i(t) + \sum_{i=1}^{N} J^i_{FLD}(t, D(t, \1_d(t)) dm_i(t),
\]

where

\[
\sigma^i_{FLD}(t, x, y) = - \frac{1}{\mathbb{E}[F_i(t, x, y)]} \prod_{j \neq i} \mathbb{E}[J^0_{\epsilon}(t, x_j, y_j)]
\]
\[
\left( \sigma \mathbb{E} \left[ \partial_x J^0_{\epsilon}(t, x_i, y_i) \right] + \theta_i(t, x, y) \mathbb{E}[J^0_{\epsilon}(t, x_i, y_i)] \right),
\]
\[
= \frac{1}{\mathbb{E}[F_i(t, x, y)]} \prod_{i=1}^{N} \mathbb{E}[J^0_{\epsilon}(t, x_i, y_i)].
\]

From no-arbitrage condition,

\[
\mu_{FLD}(t, x, y) = \sum_{i=1}^{N} \sigma^i_{FLD}(t, x, y) \theta_i(t, x, y) - \lambda \sum_{i=1}^{N} J^i_{FLD}(t, x, y) y_i \psi_i(t, x, y) - 1),
\]

which concludes the characterization.

Proof of Proposition 9. In order to compute the expectation in (20), we evaluate the following expectations:

\[
C^1_{\epsilon}(A) = \mathbb{E} \left[ M(T) \mathbb{1}_{\{N_T \geq N_A\}} \mid \epsilon \right] \quad \text{and} \quad C^2_{\epsilon}(A) = \mathbb{E} \left[ M(T) N_T \mathbb{1}_{\{N_T < N_A\}} \mid \epsilon \right].
\]

where \( A = \{A_U, A_L\} \). Note that

\[
C^1_{\epsilon}(A) = \sum_{m=0}^{N} \mathbb{E} \left[ M(T) \mathbb{1}_{\{N_T \geq N_A\}} \mid N_T = m \right] \mathbb{P}(N_T = m \mid \epsilon)
\]

33
\[
\sum_{m=0}^{N} \frac{1}{\Lambda} \mathbb{1}_{\{m \geq NA\}} \mathbb{E} \left[ \prod_{i=1}^{N} e^{-\gamma(D_i(T)-L)} \mathbb{1}_{\{\tau_i > T\}} \mid \mathcal{N}_T = m, \epsilon \right] \mathbb{P}(\mathcal{N}_T = m \mid \epsilon).
\]

Moreover, for a given state of the economy at time 0, all the companies have the same distribution and are independent. Thus,

\[
\mathbb{E} \left[ \prod_{i=1}^{N} e^{-\gamma(D_i(T)-L)} \mathbb{1}_{\{\tau_i > T\}} \mid \mathcal{N}_T = m, \epsilon \right] = \mathbb{E} \left[ \prod_{i=m+1}^{N} e^{-\gamma(D_i(T)-L)} \mid \tau_i^* > T, \epsilon \right] = \mathbb{E} \left[ \prod_{i=m+1}^{N} e^{-\gamma(D_i(T)-L)} \mid \tau_i^* > T, \epsilon \right] \mathbb{P}(\tau_i^* > T \mid \epsilon) = \left( \frac{e^{\lambda T F_0^0(0, D_0, 1)}}{a_e(0, D_0, 0)} \right)^{N-m}.
\]

Then

\[
C_1(A) = \sum_{m=0}^{N} \frac{1}{\Lambda} \mathbb{1}_{\{m \geq NA\}} \left( \frac{e^{\lambda T F_0^0(0, D_0, 1)}}{a_e(0, D_0, 0)} \right)^{N-m} \mathbb{P}(\mathcal{N}_T = m \mid \epsilon),
\]

where the last expectation is with respect to the random variable \(\mathcal{N}_T\). Additionally,

\[
C_2(A) = \sum_{m=0}^{N} \frac{1}{\Lambda} \mathbb{1}_{\{m < NA\}} m \left( \frac{e^{\lambda T B_0^0(0, D_0, 1)}}{a_e(0, D_0, 0)} \right)^{N-m} \mathbb{P}(\mathcal{N}_T = m \mid \epsilon).
\]

If we define \(q_e = \mathbb{P}(\tau_i \leq T \mid \epsilon)\), it follows that \(\mathcal{N}_T \mid \epsilon \sim Bin(N, q_e)\). Additionally, \(q_e\) can be computed explicitly. Using the notation of the proof of Proposition 1, it follows that

\[
1 - q_e = \mathbb{P}(\tau_i > T \mid \epsilon) = \mathbb{P}(\tau_i^* > T, \tau_i^* > T \mid \epsilon) = \mathbb{P}(\tau_i^* > T \mid \epsilon) \mathbb{P}(\tau_i^* > T \mid \epsilon) = e^{-\lambda T} \int_0^\infty \int_{L-D_0 \sigma}^{+\infty} p_e(u, v) dudv.
\]

The expression in (21) follows from collecting terms. \(\square\)
References


(a) Market Price of Diffusion Risk.

(b) Market Price of Jump Risk.

Figure B.1: Market Prices of Risk. These figures show the market price of diffusion and jump risk on Panels (a) and (b), respectively. There are three different scenarios of being in a good regime: $p = 25\%$ (dotted line), $p = 50\%$ (solid line) and $p = 75\%$ (dashed line). The top left figure shows the sensitivity of the market price of risk with respect to the risk aversion parameter $\gamma \in (0, 5]$. The top right graph shows how the prices of risk respond to an increase of the number of firms $N$ from 1 to 50. The bottom left plot shows how the prices of risk respond to an increase in $L$, which we vary from 50% to 95% of $D_0$. The bottom right panel graphs the effect of the probability of being in a good state $p$ on the market prices of risk.
Figure B.2: Securities Prices. These figures show the price of debt and equity on Panels (a) and (b), respectively. We investigate the effect of the risk aversion, the initial cash-flow level, the par value of debt and the number of firms in the economy. The top left figure shows the sensitivity of the security price with respect to the risk aversion parameter $\gamma \in [0, 5]$. The top right graph illustrates how the security price responds to a change of the initial cash-flow level $D_0$ from $L$ to $2P$. The bottom left plot shows how the security price behaves as a function of the par value $P$, which we vary from $L$ to $2D_0$. The bottom right panel graphs the effect of the number of firms in the economy on the security price, with $N \in \{1, 2, \ldots, 50\}$. 

(a) Bond Price.

(b) Stock Price.
Figure B.3: Risk Premium. Panels (a) and (b) show the effect on the bond and equity risk premium, respectively, as we vary the model’s parameters. The top left figure shows the sensitivity of the risk premium with respect to the risk aversion parameter $\gamma \in (0, 5]$. The top right graph illustrates how the risk premium responds to a change of the covenant $L$ from $0.5D_0$ to $0.95D_0$. The bottom left plot shows how the risk premium behaves as a function of the par value $P$ when we increase it from $L$ to $2D_0$. The bottom right panel graphs the effect of the number of firms in the economy on the risk premium, with $N \in \{1, 2, \ldots, 50\}$.
Figure B.4: Leverage. The firm leverage consists of the ratio between debt and firm value. The figure on the top left illustrates how the firm value changes when the risk aversion \( \gamma \in (0, 5] \). The graph on the top right shows how the leverage behaves as the covenant \( L \in (0.01D_0, 0.99D_0) \). The second row shows how the firm’s leverage responds to changes in the par value of debt (bottom left) and the covenant level (bottom right).
Figure B.5: Credit Default Swap. The graphs in Panel (a) illustrate the effects of the model’s parameters on the price level of the CDS. The figure on the top left illustrates how the CDS is affected by the risk aversion parameter. The graph on the bottom right shows the effect on the CDS when we vary the face value of debt from $L$ to $3D_0$. On the top right, the chart illustrates the impact of the initial value of the cash-flow $D_0$ on the value of the CDS. The figure on the bottom left shows the impact of the number of firms on price. Panel (b) quantify the impact of the same parameters on the CDS spread.
Figure B.6: First to Default Basket. The graphs in Panel (a) illustrate the effects of the model’s parameters on the price level of the FtD. The figure on the top left illustrates how the FtD is affected by the risk aversion parameter. The graph on the bottom right shows the effect on the FtD when we vary the face value of debt from $L$ to $3D_0$. On the top right, the chart illustrates the impact of the initial value of the cash-flow $D_0$ on the value of the FtD. The figure on the bottom left shows the impact of the number of firms on price. Panel (b) quantify the impact of the same parameters on the FtD spread.
Figure B.7:  

CDO Price. The figure on the top left illustrates how the CDO is affected by the risk aversion parameter $\gamma \in (0, 5]$. The graph on the top right shows the effect on the CDO when we vary the initial cash-flow from $D_0 \in (1, 5.25]$. On the bottom left, the chart illustrates the impact of the covenant $L$ on the value of the CDO. The figure on the bottom right shows the impact of the number of firms on the CDO price.
\[ S_i(T) + B_i(T) = V_i(T) \]

<table>
<thead>
<tr>
<th></th>
<th>No Default</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_i(T) )</td>
<td>( \max{D_i(T) - P, 0} )</td>
<td>( \max{L - P, 0} ) = 0</td>
</tr>
<tr>
<td>( P_i(T) )</td>
<td>( \min{D_i(T), P} )</td>
<td>( \min{L, P} ) = ( L )</td>
</tr>
</tbody>
</table>

\( D_i(T) = L \)

**Table B.1:** Firm \( i \)'s payoff at time \( T \) in the case of no default and default.
### Table B.2: Table of Parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_g$</td>
<td>0.2%</td>
</tr>
<tr>
<td>$\lambda_b$</td>
<td>1%</td>
</tr>
<tr>
<td>$p$</td>
<td>75%</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>2</td>
</tr>
<tr>
<td>$\mu_g$</td>
<td>6%</td>
</tr>
<tr>
<td>$\mu_b$</td>
<td>2%</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>25%</td>
</tr>
<tr>
<td>$L$</td>
<td>1</td>
</tr>
<tr>
<td>$P$</td>
<td>1.5</td>
</tr>
<tr>
<td>$D_0$</td>
<td>1.75</td>
</tr>
<tr>
<td>$N$</td>
<td>10</td>
</tr>
<tr>
<td>$t$</td>
<td>0</td>
</tr>
<tr>
<td>$T$</td>
<td>5</td>
</tr>
</tbody>
</table>