Lecture 8:
Mechanics of Real Business Cycles:


Questions posed by KPR

• What are the business cycle properties of the basic neoclassical model of capital accumulation, when it is augmented by productivity shocks?

• What are desirable strategies for computing linear approximation (or loglinear approximation) solutions to dynamic equilibrium models?

• Do these methods continue to work if there is a stochastic trend in productivity, i.e., there are permanent variations in the level of productivity?

• If government policy or private market failure leads competitive equilibrium to be suboptimal, are entirely new methods necessary?
Recipe: Part A

1. Start with model which includes technical progress and describes a growing economy (depends on t via deterministic trend growth)
2. Restrict preferences and technology so that steady state growth is feasible
3. As in growth models, find a transformed economy which is stationary (i.e., in which the equations do not depend on t except as subscript)
4. Write down a discrete time Lagrangian
5. Find the FOCs and TVC for this economy
6. Find the stationary point of these FOCS (which surely satisfies the TVC)
Recipe: Part B

1. Linear/loglinear approximation
2. Solve the resulting linear RE model in general: \( \{Y\} \) depends on \( \{X\} \)
3. Calculate the solution under a particular driving process, obtaining a solution in state space form
4. Use this solution to calculate
   a. Comparative dynamics (impulse responses)
   b. Population moments
   c. Stochastic simulations
5. Use economic analysis to interpret solution: e.g., permanent income theory including Fisher’s rule for consumption growth and real interest rate

SGZ Macro 2010, Lecture 8:
Mechanics of RBC models
Extensions of Recipe

• KPR88b: To suboptimal equilibria
• KPR88b: To permanent variations in productivity (stochastic trend productivity) motivated by unit root findings
• Later: to all sorts of real and monetary models, including New Keynesian models
• Now routine:
  – Write down FOCs and other conditions of equilibrium
  – Linearize around appropriately defined stationary point
  – Get dynamic outcomes using linear RE techniques
Recipe: Part A

• The original economy: production with deterministic technical progress expressible in labor augmenting form

\[ Y_t = A_t F(K_t, X_t N_t) \text{ and } X_{t+1} = \gamma X_t \]

\[ \Rightarrow y_t = A_t F(k_t, N_t) \text{ with } y_t = \frac{Y_t}{X_t} \text{ and } k_t = \frac{K_t}{X_t} \]

[form necessary for ss growth]
Consumption, Investment and Capital Accumulation

\[ C_t + I_t = Y_t \text{ and } K_{t+1} = (1 - \delta)K_t + I_t \]

\[ \Rightarrow c_t + i_t = y_t \text{ and } \gamma k_{t+1} = (1 - \delta)k_t + i_t \]

with \( c_t = \frac{C_t}{X_t} \) and \( i_t = \frac{I_t}{X_t} \)

\[ (1 - \delta) \frac{K_t}{X_t} + \frac{I_t}{X_t} = \frac{K_{t+1}}{X_{t+1}} = \frac{K_{t+1}}{X_{t+1}} \frac{X_{t+1}}{X_t} = \gamma k_{t+1} \]
Labor and leisure

- Per-capita hours of work and leisure must be constant in steady state
- This comes despite fact that wage rate will grow due to technical progress

\[ W_t = wX_t = wX_0 \gamma^t \]
Preference restrictions necessary for ss growth to be optimal

• Need: constant consumption growth in face of constant ss real interest rate (implied by above)
• Need: constant work level in face of real wage growth
• Utility (general): u(C,L)
Preference restrictions
(sometimes called KPR utility)

- Intertemporal MRS: must be constant elasticity (as in permanent income model)
- MRS between work and leisure, must display invariance to growing consumption and wage rate (offsetting income and substitution effects)

\[ u(c, L) = \frac{1}{1-\sigma} c^{1-\sigma} v(L) \] or \[ u(c, L) = \log(c) + v(L) \]
Original and modified utility

For first utility function above

\[ U = \sum_{t=0}^{\infty} \beta^t u(C_t, L_t) = (X_0)^{1-\sigma} \sum_{t=0}^{\infty} (\beta \gamma^{1-\sigma})^t u(c_t, L_t) \]

For both utility functions:

(i) \( X_0 \) affects welfare but not preferences (over \( c, L \)), so set \( X_0 = 1 \) for convenience and abstract from it

(ii) modification of discount factor
Solving optimization problem

• Via Lagrangian

\[ L = \sum_{t=0}^{\infty} (\beta^*)^t u(c_t, L_t) \]

\[ + \sum_{t=0}^{\infty} (\beta^*)^t \lambda_t [A_t f(k_t, N_t) + (1 - \delta)k_t - \gamma k_{t+1} - c_t] \]

\[ + \sum_{t=0}^{\infty} (\beta^*)^t \omega_t [1 - N_t - L_t] \]

Concepts: \((\beta^*)^t \lambda_t \) is shadow price of \( c, y \)

\((\beta^*)^t \omega_t \) is shadow price of \( N, L \)

SGZ Macro 2010, Lecture 8:
Mechanics of RBC models
FOCs + TC

\[ c_t : (\beta^*)^t[D_1u(c_t, L_t) - \lambda_t] = 0 \]
\[ L_t : (\beta^*)^t[D_2u(c_t, L_t) - \omega_t] = 0 \]
\[ N_t : (\beta^*)^t[-\omega_t + \lambda_t A_t D_2 f(k_t, N_t)] = 0 \]
\[ k_{t+1} : (\beta^*)^t[-\lambda_t + \beta^* \lambda_{t+1} (A_{t+1} D_1 f(k_{t+1}, N_{t+1}) + (1 - \delta))] = 0 \]
\[ (\beta^*)^t \lambda_t : [A_t f(k_t, N_t) + (1 - \delta)k_t - \gamma k_{t+1} - c_t] = 0 \]
\[ (\beta^*)^t \omega_t : [1 - N_t - L_t] = 0 \]

\[ TVC : \lim_{t \to \infty} (\beta^*)^t \lambda_t k_{t+1} = 0 \]
Stationary point

- Constrained by

\[
\begin{align*}
    c : [D_1 u(c, L) - \lambda] &= 0 \\
    L : [D_2 u(c, L) - \omega] &= 0 \\
    N : -\omega + \lambda AD_2 f(k, N) &= 0 \\
    k : [-\lambda + \beta^* \lambda (AD_1 f(k, N) + (1 - \delta))] &= 0 \\
    \lambda : [Af(k, N) + (1 - \delta)k - \gamma k - c] &= 0
\end{align*}
\]
Recipe: Part B

- Linearization/loglinearization

\[
\begin{align*}
\xi_{cc} \hat{c}_t + \xi_{cL} \hat{L}_t - \hat{\lambda}_t &= 0 \\
\xi_{Lc} \hat{c}_t + \xi_{LL} \hat{L}_t - \hat{\omega}_t &= 0 \\
-\hat{\omega}_t + \hat{A}_t + \xi_{nk} \hat{k}_t + \xi_{nk} \hat{k}_t &= 0 \\
N * \hat{N}_t + L * \hat{L}_t &= 0 \\
-\hat{\lambda}_t + \hat{\lambda}_{t+1} + \eta_A \hat{A}_{t+1} + \eta_k \hat{k}_{t+1} + \eta_N \hat{N}_{t+1} &= 0 \\
-\left[ \hat{A}_t + s_N \hat{N}_t + s_k \hat{k}_{t+1} \right] + \left[ s_c \hat{c}_t + s_i \phi \hat{k}_{t+1} - s_i (\phi - 1) \hat{k}_t \right] &= 0
\end{align*}
\]
Solution of Linear Model

- Stage 1: Relate endogenous variables to sequences of exogenous variables (as in Blanchard-Kahn)

\[ \hat{k}_{t+1} = \mu_1 \hat{k}_t + \psi_1 \hat{A}_t + \psi_2 \sum_{j=0}^{\infty} \mu_2^{-j} E_t \hat{A}_{t+j+1} \]

- Stage 2: Assuming driving process and evaluate discounted sums

\[ \hat{A}_t = \rho \hat{A}_{t-1} + e_t \]

\[ \hat{k}_{t+1} = \mu_1 \hat{k}_t + \left[ \psi_1 + \psi_2 \frac{\rho}{1 - (\rho / \mu_2)} \right] \hat{A}_t = \mu_1 \hat{k}_t + \pi_{kA} \hat{A}_t \]
\[ Y_t = \Pi s_t \quad s_t = Ms_{t-1} + Ge_t \]

- Linear difference system (Bellman)

- State space model (Hamilton)

- RE solution (from BK or KW)

- Used for calculations of responses to one-time shocks (impulse responses); stochastic simulations; calculations of moments…
\[
\begin{bmatrix}
    y_t \\
    c_t \\
    i_t \\
    n_t \\
    ... \\
\end{bmatrix}
= \begin{bmatrix}
    \pi_{yk} & \pi_{ya} \\
\end{bmatrix}
\begin{bmatrix}
    k_t \\
    a_t \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
    k_t \\
    a_t \\
\end{bmatrix}
= \begin{bmatrix}
    \mu & \pi_k \alpha \\
    0 & \rho \\
\end{bmatrix}
\begin{bmatrix}
    k_{t-1} \\
    a_{t-1} \\
\end{bmatrix}
+ \begin{bmatrix}
    0 \\
    1 \\
\end{bmatrix} e_t
\]

SGZ Macro 2010, Lecture 8: Mechanics of RBC models
Substantive conclusions from KPR

- Productivity shocks (A) must be persistent to generate business cycle phenomena: these give rise to important wealth effects on consumption.

- Although labor is invariant to trend growth in productivity (X), it varies sharply in response to productivity shocks (A).

- Large productivity variations are necessary to produce substantial volatility in output [not stressed in paper, but implicit in graphs].
Do we need large productivity shocks?

• KR in *Handbook of Macroeconomics* review developments from 1988 to 1999 when there was:
  – A huge number of RBC studies
  – A rising concern that productivity shocks measured via Solow residual were “too big” and “too important” to RBC modeler conclusions
KR strategy

• Modified basic 1988 model to introduce several key features that had previously been studied separately
  – Labor choice on extensive margin (to work or not) rather than intensive margin (how many hours to work)
  – Varying utilization of capital
Implications

- Model economy:
  - Solow residual as poor proxy for actual productivity shock (utilization not observed)
  - High response to actual productivity shocks (so the implied shocks could be smaller)
  - Alternative method of extracting shocks