Asset pricing in endowment and production economies: general ideas and loglinear benchmarks
Overview

A. Review and consolidation
B. Asset pricing in endowment economies
   1. Mehra-Prescott
   2. Campbell loglinear benchmark
C. Asset pricing in production economies
   1. General theory
   2. Jermann loglinear benchmark
A. Review and consolidation

• Last lecture’s complete markets model implies

\[ p(s_t) = \sum_{s_{t+1}} Q_1(s_{t+1} \mid s_t) [p(s_{t+1}) + d(s_{t+1})] = \sum_{j=0}^{\infty} \sum_{s_{t+j}} Q_j(s_{t+j} \mid s_t) d(s_{t+j}) \]

\[ = \sum_{s_{t+1}} \pi(s_{t+1} \mid s_t) \beta m(s_{t+1} \mid s_t) [p(s_{t+1}) + d(s_{t+1})] \]

\[ = \sum_{j=0}^{\infty} \sum_{s_{t+j}} \pi(s_{t+j} \mid s_t) \beta^j m(s_{t+j}, s_t) d(s_{t+j}) \]

\[ = E_t \{ \beta m(s_{t+1} \mid s_t) [p(s_{t+1}) + d(s_{t+1})] \} = E_t \{ \sum_{j=0}^{\infty} \sum_{s_{t+j}} \beta^j m(s_{t+j}, s_t) d(s_{t+j}) \} \]

where the latter expression renormalizes contingent claims prices
What is “m”?

• Normalized contingent claims price
• Stochastic discount factor
• Asset pricing kernel
• What could it be in complete markets?
  – Social planner’s endowment multiplier
  – Any asset holders marginal utility

\[
m(s_{t+j}, s_t) = \frac{\bar{\theta}(s_{t+j})}{\theta(s_t)} = \frac{u'(c^i_{t+j})}{u'(c^i_t)}
\]
Why combine macro and finance?

• Finance suggests diversifiable risk should not be priced
• Macroeconomics explains common factors across households, firms and industries
• Macroeconomics is thus important for understanding aggregate risks
• Financial markets are arguably important for macroeconomic developments
Holy Grail: Find “m”

- Practical (Markets): identify profits opportunities in asset markets (deviations from predicted asset prices)
- Practical (Markets): Create new assets
- Practical (Policy): Extract information about agent beliefs from asset prices
- Academic: Learn about asset price determination
- Academic: Additional discipline on modern macroeconomic models
Mechanics

• Evaluate pricing formulae and create practical model. That is, solve

\[ p(s_t) = E_t \{ \beta m(s_{t+1} \mid s_t)[p(s_{t+1}) + d(s_{t+1})]\} \]

\[ = E_t \{ \sum_{j=0}^{\infty} \sum_{s_{t+j}} \beta^j m(s_{t+j}, s_t)d(s_{t+j})\} \]
Digression on risk neutrality

- Notion of risk neutral probability measure as in LS (and in finance).
- Alternative risk neutral equation (units of prices and cash flows denominated in marginal utility)

\[ p(s_t) = E_t \{ \beta m(s_{t+1} \mid s_t) [p(s_{t+1}) + d(s_{t+1})] \} = E_t \{ \sum_{j=0}^{\infty} \sum_{s_{t+j}} \beta^j m(s_{t+j}, s_t) d(s_{t+j}) \} \]

\[ p(s_t) = \overline{E}_t \{ \beta [p(s_{t+1}) + d(s_{t+1})] \} = \overline{E}_t \{ \sum_{j=0}^{\infty} \sum_{s_{t+j}} \beta^j d(s_{t+j}) \} \]

where \( \overline{\pi}(s_{t+1} \mid s_t) = \pi(s_{t+1} \mid s_t) m(s_{t+1} \mid s_t) \)

\[ \theta(s_t) p(s_t) = E_t \{ \beta \theta(s_{t+1}) [p(s_{t+1}) + d(s_{t+1})] \} = E_t \{ \sum_{j=0}^{\infty} \sum_{s_{t+j}} \beta^j \theta(s_{t+j}, s_t) d(s_{t+j}) \} \]
B. Endowment Economies

1. Lucas, Mehra and Prescott
2. Discount bonds, coupon bonds, and strips
3. Mechanics of discount bond yields
4. The stochastic discount factor and bond pricing
5. A simple model of the term structure
6. Stripping other assets
1. Basic Endowment economy

• Lucas tree is complex asset. Ownership of tree today is claim to fruit tomorrow (d’) and future ownership (p’).
• Mehra-Prescott studied returns on this sort of tree, interpreting it as the stock market and contrasting the returns on stocks to those on bonds.
• Assumption of power utility (constant relative risk aversion)
• Markov chain on consumption growth rate.
Endowment

• Markov chain on growth rate

states: \( \gamma_1 < \gamma_2 < \ldots \gamma_J \)

\[ \text{prob}(g_{t+1} = \gamma_j \mid g_t = \gamma_h) = \pi_{hj} \]

• Asset pricing condition

Stock: \( p(g) = \beta E\{ (\frac{c'}{c})^{-\sigma} [p' + d'] \} \)

Bond: \( b(g) = \beta E\{ (\frac{c'}{c})^{-\sigma} \} \)
Asset returns

• Conceptually (conjectured solution)

Stock: \( p_t = f(g_t; g_{t-1})c_t; \quad d_t = c_t \)

\[
1 = \beta E\left\{ \left( \frac{c'}{c} \right)^{-\sigma} \left[ \frac{p' + d'}{p} \right] \right\}
\]

\[
= \beta E\{ (g')^{-\sigma} [g' f(g'; g) + g'] \} | g
\]

\[
R_s(g'; g) = g' f(g'; g) + g'
\]

Bond: \( 1 = \beta E\{(g')^{-\sigma} (R_b(g))\} | g \)
Computationally

• For each state $h$ (Markov-Bellman sense)

\[ 1 = \beta \sum_{j=1}^{J} \pi_{jh} (g_j)^{-\sigma} [R_s(\gamma_j; \gamma_h)] \]

\[ 1 = \beta \sum_{j=1}^{J} \pi_{jh} (g_j)^{-\sigma} [R_b(\gamma_h)] \]
MP finding

• For consumption process that matched aspects of US data (mean, standard deviation, serial correlation), there was a critical asset pricing anomaly: the spread in expected return between stocks and bonds was small for “reasonable” levels of risk aversion (less than 10) relative to observed data: .5%, say rather than 6% per annum.
B2. Stripping coupon bonds

- Pure discount bond of maturity $n$: face value payment of “$f$” dollars $n$ periods from now
- Coupon bond
  - Regular **coupon** payments of “$c$” dollars (say, every year, for simplicity) and one terminal payment **face value** payment of “$f$” dollars
- Strip market
  - Discount bonds created by selling coupons and face values separately
    - [http://www.riskglossary.com/link/treasury_strips.htm](http://www.riskglossary.com/link/treasury_strips.htm)
B3. Mechanics of discount bond yields

Discrete compounding: yield to maturity

\( p_{nt} \): n period discount bond price
\( i_{nt} \): n period discount bond yield

\[ p_{nt} = [(1 + i_{nt})]^{-n} \]

Continuous compounding: yield to maturity

\[ p_{nt} = \exp(-n \times i_{nt}) \]
B3. Stochastic discount factor and asset pricing

• (Gross) holding period yield on any asset

\[ h_{t+1} = \frac{P_{t+1} + d_{t+1}}{P_t} \]

• Stochastic discount factor \( m \)

\[
p_t = E_t[m_{t+1}(p_{t+1} + d_{t+1})] \\
= E_t[m_{t+1}d_{t+1}] + E_t[m_{t+2}m_{t+1}(p_{t+2} + d_{t+2})]
\]
Bond pricing with an SDF

• Discount bond notation: bonds of maturity n at t are bonds of maturity n-1 at t+1

\[ p_{nt} = E_t[m_{t+1}p_{n-1,t+1}] \]
\[ = E_t[m_{t+n}m_{t+n-1}...m_{t+2}m_{t+1}] \]
\[ = E_t \exp([\sum_{j=1}^{n} \log(m_{t+j})]) \]
\[ i_{nt} = -\frac{1}{n} \log(p_{nt}) \]

• Expressions like these embolden Campbell and Jermann to study loglinear (lognormal) asset pricing
Key properties of lognormal random variable

• Suppose that $y$ is normal with mean $\mu$ and variance $\sigma^2$, then

$$E(\exp(y)) = \exp(\mu + \frac{1}{2}\sigma^2)$$
State space system

$$\log(m_{t+n}) = \kappa + \pi s_{t+n}$$

$$s_t = Ms_{t-1} + Ge_t$$

$$s_{t+n} = M^n s_t + \{Ge_{t+n} + MGe_{t+n-1} + ... M^{n-1}Ge_{t+1}\}$$
\[
\sum_{j=1}^{n} \log(m_{t+j})
= n\kappa + \pi\left[ s_{t+n} + s_{t+n-1} + s_{t+n-2} + \ldots s_{t+1} \right]
= n\kappa + \pi\left[ Ge_{t+n} + (I + M)s_{t+n-1} + s_{t+n-2} + \ldots s_{t+1} \right]
= n\kappa + \pi(G+\ldots M^n)s_t \quad \text{forecastable (known at } t) \\
+ \pi\left[ Ge_{t+n} + (I + M)Ge_{t+n-1} + (I + M + M^2)Ge_{t+n-2} \\
(I+M+\ldots M^{j-1})Ge_{t+n-j} + \ldots (I+M+\ldots M^{n-1})Ge_{t+1} \right]
q_n = q_{n-1} + \pi(I+M+\ldots M^{n-1})GE(e'G'(I+M+\ldots M^{n-1})'\pi')
Why useful?

• Simple formula for yield.
• Time-varying rate, but no change in risk premium

\[ p_{nt} = E_t \exp(\sum_{j=1}^{n} \log(m_{t+j,t+j-1})) \]

\[ i_{nt} = -\frac{1}{n} \log(p_{nt}) = -\frac{1}{n} [n\kappa + \pi(M+...+M^n)s_t + \frac{1}{2}q_n] \]

• More generally, can price single period cash flow period instruments in simple manner
Term structure

\[ p_{nt} = \exp(n\kappa + \pi(I+M+...M^n)s_t + \frac{1}{2}q_n) \]

\[ i_{nt} = -\frac{1}{n}\log(p_{nt}) = -(\kappa + \frac{q_n}{2n}) - \frac{1}{n}\pi(I+M+...M^n)s_t \]

\[ i_{1t} = -(\kappa + \frac{q_1}{2}) - \pi s_t \]

\[ E_t i_{1,t+j} = -(\kappa + \frac{q_1}{2}) - \pi M^j s_t \]

\[ i_{nt} = K_n + \frac{1}{n}E_t \left[ \sum_{j=0}^{n-1} i_{1,t+j} \right] \]
5. Stripping other assets

- Strip off and sell just the dividends at each date

\[ v_t[d_{t+n}] = E_t[m_{t+1} \ast v_{t+1}[d_{t+n}]] \]
C. Production economies

• Equations derived in Jermann (also IHW-part D)

• Interpretation:
  – “θ” is RC’s marginal value of having a little more of the consumption good. It depends on \( k_t, c_{t-1}, a_t \)
  – The social planner’s Lagrange multiplier on consumption goods in lecture 9, where these were endowments.

\[
c_t : \theta_t = u_1(c_t, c_{t-1}) + \beta E_t v_{c,t+1} \\
z_t : \theta_t = \lambda_t h'(z_t) \\
k_{t+1} : \lambda_t = \beta E_t v_{k,t+1} \\
p_t : c_t + i_t = a_t f(k_t, n) \\
\lambda_t : k_{t+1} = h(z_t)k_t + (1-\delta)k_t \\
ET : v_{c,t} = u_2(c_t, c_{t-1}) \\
ET : v_{kt} = h(z_t) + (1-\delta) \\
\]

\[
Y = [k_t, c_{t-1}, z_t, \theta_t, \lambda_t, v_{ct}, v_{kt}] 
\]

SGZ Macro 2010 WK1, lecture 10
What is “m”? 

- According the analysis above,

\[ \theta(s_t) p(s_t) = \beta E_t \{ \theta(s_{t+1})[p(s_{t+1}) + d(s_{t+1})] \} \]

- For arbitrary p,d

- Note that the DP decision rules are

\[
\begin{align*}
\theta(s_t) \\
\begin{bmatrix}
k_{t+1} \\
c_t \\
a_{t+1}
\end{bmatrix}
= \\
\begin{bmatrix}
k_t[h(z(s_t)) + (1 - \delta)] \\
c(s_t) \\
f(a_t, e_t)
\end{bmatrix}
\end{align*}
\]
Jermann loglinear/lognormal approximation

- Approximate second block of equations with loglinear RE model with normal shocks
- Treat the first equation, the pricing equation, as exactly as possible
- What does “as exactly as possible” mean?
• Loglinearly approximate pricing kernel ($\theta$) and payouts ($d$)
• Strip payouts $t+j$ periods ahead.
• Use exact lognormal approximation to price strips
• Add the strips back up to get multiperiod payouts as a non-loglinear function of the state.
See you next October!

• We’ll study asset pricing in production economies as an example of use of higher order approximation methods.
• We’ll see how well Jermann’s approximation holds up.
6. Consumption and the SDF

\[ m_{t+1} = \frac{u_c(c_{t+1})}{u_c(c_t)} = \left( \frac{c_{t+1}}{c_t} \right)^{-\sigma} \]

\[ \beta = \exp(-\nu) \]

\[ \log(m_{t+1}) = -\sigma [\log(c_{t+1}) - \log(c_t)] \]
7. Valuing a stock

- A stock is just a portfolio of stripped dividends.

- Value the strips via loglinear asset pricing using a joint process for m and d.

- Add up the values
8. Equity premium puzzle once again

• Suppose that log consumption is a random walk with normal innovations
• Suppose we have constant elasticity marginal utility
• What is the return on a pure consumption discount bond?
Bond return

$$1 = R_b \beta E_t[m_{t+1}]$$

$$E_t[\beta \left( \frac{c_{t+1}}{c_t} \right)^{-\sigma}] = \beta \exp(-\sigma \gamma + \frac{1}{2} \sigma^2 E(e^2))$$

Fisher's rule under uncertainty

$$\log(R_b) = \nu + \sigma \gamma - \frac{1}{2} \sigma^2 E(e^2)$$
Suppose stock price is proportional to consumption

\[ R_{t+1} = \frac{(p_{t+1} + d_{t+1})}{p_t} = \left(\frac{c_{t+1}}{c_t}\right)k \]

\[ E(\log(R_{s,t+1})) = \log(k) + \gamma + \frac{1}{2}E(e^2) \]
Determining “k”

\[ 1 = E_t[m_{t+1} R_{s,t+1}] \]

\[ 1 = E_t[\beta \left( \frac{C_{t+1}}{c_t} \right)^{1-\sigma} k] = \beta k \exp((1-\sigma)\gamma + \frac{1}{2}(1-\sigma)^2 E(e^2)) \]

\[ \log(k) = \nu - (1-\sigma)\gamma - \frac{1}{2}(1-\sigma)^2 E(e^2) = \]

\[ = [\nu - \sigma \gamma - \frac{1}{2} \sigma^2 E(e^2)] - \gamma - \frac{1}{2} E(e^2) + \sigma E(e^2) \]
MP puzzle

\[ ER_s = R_b + \sigma E(e^2) + \frac{1}{2} E(e^2) \]