EC 501: Problem Set 5, Solutions

1. (a) To test for returns to scale, suppose capital and labor inputs are fixed at K_0, L_0 and output is
$$q_0 = 5K_0^{\frac{3}{4}}L_0^{\frac{1}{4}}.$$
Now let's change capital and labor inputs by a factor λ to $\lambda K_0, \lambda L_0$ and see what happens to output. Output is now
$$q_1 = 5\left(\lambda K_0\right)^{\frac{3}{4}}\left(\lambda L_0\right)^{\frac{1}{4}}$$
$$= \lambda^{\frac{3}{4}+\frac{1}{4}} \cdot 5K_0^{\frac{3}{4}}L_0^{\frac{1}{4}} = \lambda q_0.$$
Therefore q has also gone up by the factor λ, indicating that the production function is characterized by constant returns to scale.

(b) Consider the marginal product of labor.
$$MP_L = \frac{\partial q}{\partial L} = \frac{5}{4}K^{\frac{3}{4}}L^{-\frac{3}{4}}.$$
Then
$$\frac{\partial MP_L}{\partial L} \equiv \frac{\partial^2 q}{\partial L^2} = -\frac{15}{16}K^{\frac{3}{4}}L^{-\frac{7}{4}} < 0.$$
Since this derivative is negative, the marginal product of labor is decreasing. It can similarly be shown that MP_K is also decreasing.

(c) In the short run, if K is fixed at $\bar{K}$, the labor input must be chosen to satisfy
$$5\bar{K}^{\frac{3}{4}}L^{\frac{1}{4}} = q.$$
Solving for L gives us the conditional demand function for labor:
$$L = \left(\frac{q}{5}\right)^4 \bar{K}^{-3}.$$
Since total cost is $C = wL + rK$, where w, r are the input prices, the cost function in this case is
$$C(q, \bar{K}, w, r) = w\left(\frac{q}{5}\right)^4 \bar{K}^{-3} + r\bar{K}.$$

(d) The long run cost function can be found by optimizing the short run cost function over $\bar{K}$. So start by minimizing short run cost by choice of K, treated now as a variable:
$$\frac{\partial C}{\partial K} = -3w\left(\frac{q}{5}\right)^4 \bar{K}^{-4} + r = 0.$$

Solving for K, we get

$$K = \left(\frac{q}{5}\right)\left(\frac{3w}{r}\right)^{\frac{1}{4}}.$$

Substitute this for $\bar{K}$ in the short run cost function:

$$C = w\left(\frac{q}{5}\right)^4\left(\frac{q}{5}\right)^{-3}\left(\frac{3w}{r}\right)^{-\frac{3}{4}}. + r\left(\frac{q}{5}\right)\left(\frac{3w}{r}\right)^{\frac{1}{4}}$$

$$= \left(\frac{q}{5}\right) w^{\frac{1}{4}} r^{\frac{3}{4}}\left[3^{-\frac{3}{4}} + 3^{\frac{1}{4}}\right].$$

This is the long run cost function and can be written as

$$C(q, w, r) = \frac{4}{5 \cdot 3^{\frac{3}{4}}} q w^{\frac{1}{4}} r^{\frac{3}{4}}.$$

2. (a) To test for returns to scale, suppose capital and labor inputs are fixed at K_0, L_0 and output is

$$q_0 = 3K_0 + 2L_0.$$

Now let's change capital and labor inputs by a factor λ to $\lambda K_0, \lambda L_0$ and see what happens to output. Output is now

$$q_1 = 3(\lambda K_0) + 2(\lambda L_0)$$

$$= \lambda \cdot (3K_0 + 2L_0) = \lambda q_0.$$

Therefore q has also gone up by the factor λ, indicating that the production function is characterized by constant returns to scale.

(b) In the short run, suppose K is fixed at $\bar{K}$. Then, since $q = 3K + 2L$ is the production function, the firm can produce $3\bar{K}$ units of output at zero marginal cost. If the firm wanted to produce more than $3\bar{K}$ units of output, it would have to use labor to produce the excess over $3\bar{K}$. In other words, it would set

$$L = \frac{q - 3\bar{K}}{2}$$

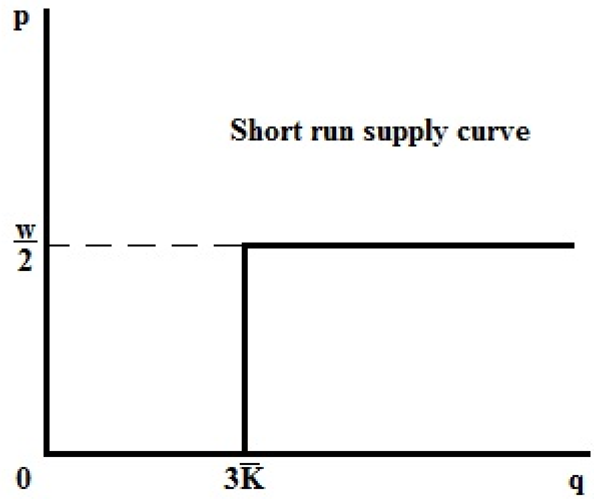
Since in this region each unit of labor produces 2 units of output, the marginal cost of production is

$$MC = \frac{w}{2}.$$

So, if $p < \frac{w}{2}$, the firm would supply only $3\bar{K}$ units of output, but if $p \geq \frac{w}{2}$, it would be willing to supply any amount. The supply curve is therefore

$$q^s = \begin{cases} 3\bar{K} & if\ p < \frac{w}{2} \\ \propto & if\ p \geq \frac{w}{2} \end{cases}$$

The supply curve can be illustrated by the dark line in the figure.



In the long run, we cannot use the Lagrange method because, with perfect substitutes as in this kind of production function, we will have corner solutions. Capital and labor are here perfect substitutes and so the firm in the long run will use only one or the other, depending on the factor prices. Now the typical isoquant has the slope

$$\frac{dK}{dL} = -\frac{2}{3}.$$

Therefore

$$\frac{w}{r} < \frac{2}{3} \quad \rightarrow \quad firm\ will\ use\ only\ L,$$
$$\frac{w}{r} > \frac{2}{3} \quad \rightarrow \quad firm\ will\ use\ only\ K.$$

If $\frac{w}{r} = \frac{1}{2}$, the firm could use any combination of K and L that lies on the isoquant. Therefore, we can write down the firm's cost function as

$$C = \begin{cases} \frac{wq}{2} & if\ \frac{w}{r} \leq \frac{2}{3} \\ \frac{rq}{3} & if\ \frac{w}{r} \geq \frac{2}{3} \end{cases}$$

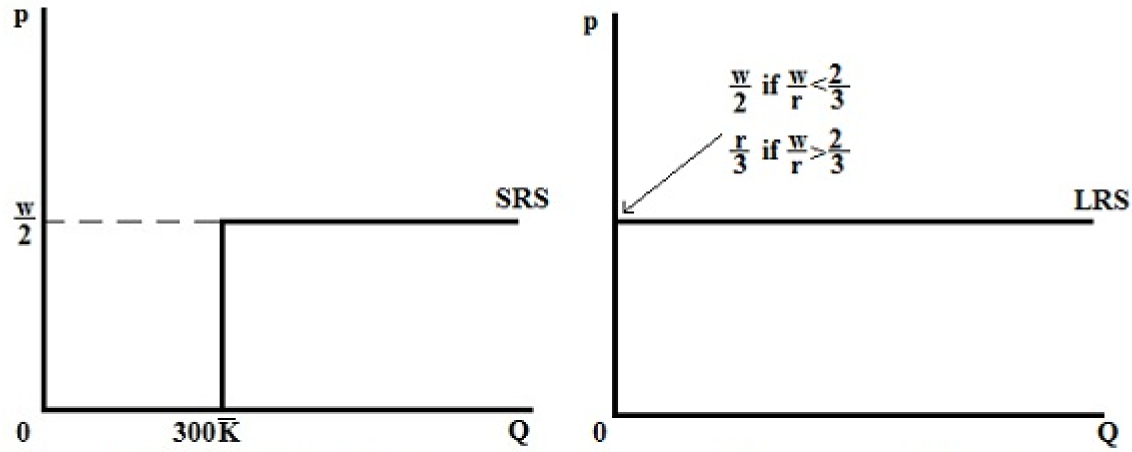
and therefore the marginal cost is

$$MC = \begin{cases} \frac{w}{2} & if\ \frac{w}{r} \leq \frac{2}{3} \\ \frac{r}{3} & if\ \frac{w}{r} \geq \frac{2}{3} \end{cases}$$

The long run MC curve is therefore a horizontal line at the height indicated by the previous equation. (c) The short run supply curve

would be the horizontal sum of all the individual supply curves. Since each firm supplies $3\bar{K}$ units of output if $p < \frac{w}{2}$, and there are 100 firms, the industry supply when $p < \frac{w}{2}$ will be $300\bar{K}$. At $p = \frac{w}{2}$, the supply curve will be perfectly elastic. The short run supply curve is shown in panel (a) of the Figure. In the long run, each firm's MC is flat and

so the industry supply curve will also be horizontal at the height of the individual firm MC curves. Panel (b) in the Figure illustrates.



(a) Short run supply curve

(b) Long run supply curve

3. (a) From the production function, we see that each unit of output requires $\frac{1}{5}$ units of capital and $\frac{1}{10}$ units of labor. Therefore, we can write total cost as

$$C(q, w, r) = \left(\frac{1}{5}r + \frac{1}{10}w\right) q.$$

Substituting the values $r = 1$ and $w = 3$, we get the total cost curve:

$$C(q) = \frac{1}{2}q.$$

To find the AC and MC curves:

$$AC = \frac{C(q)}{q} = \frac{1}{2}, \qquad and$$

$$MC = \frac{dC(q)}{dq} = \frac{1}{2}.$$

(b) If $\bar{K} = 10$ in the short run, the maximum feasible output is 50. The

short run cost function then is

$$C(q, \bar{K}, w, r) = r\bar{K} + \frac{1}{10}wq \qquad for\ q \leq 50$$

with output over 50 infeasible. Substituting the values, we can write the total cost curve as

$$C(q) = \begin{cases} 10 + \frac{3}{10}q & for\ q \leq 50 \\ \propto & for\ q > 50 \end{cases}$$

Then we can write down the AC and MC curves:

$$AC(q) = \begin{cases} \frac{10}{q} + \frac{3}{10} & for\ q \leq 50 \\ \propto & for\ q > 50 \end{cases}$$

$$MC(q) = \begin{cases} \frac{3}{10} & for\ q \leq 50 \\ \propto & for\ q > 50 \end{cases}$$