EC 501: Problem Set 4, Solutions

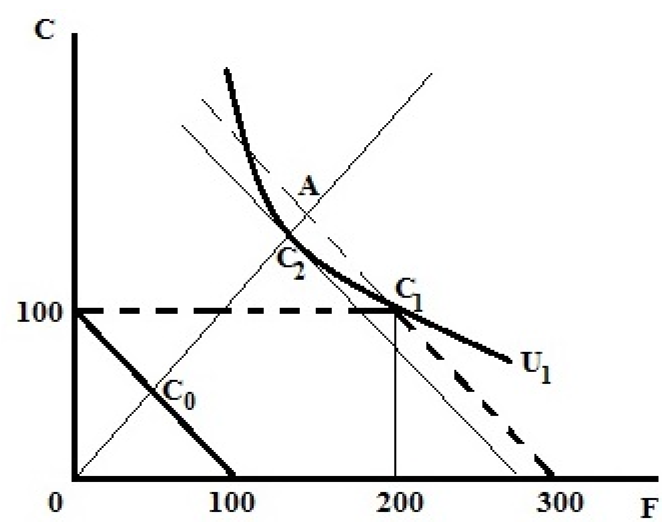
1. (a) With a Cobb-Douglas utility function, the household will maintain fixed budget shares of the goods, with the shares in this example being $\theta_f = \frac{1}{2}$ and $\theta_c = \frac{1}{2}$. Then, since $I = \$100, p_f = \$1, p_c = \$1$, the initial consumption bundle will be

$$F_0 = 50 \quad and \quad C_0 = 50.$$

(b) The initial consumption bundle is shown as the point C_0 in the graph (see next page). When the household is given 200 units of food, its constraint pushes out to the right by 200 units. The new constraint (the dashed line in the graph) is kinked: it has a flat portion up to the point C_1 and then a sloped portion parallel to the original constraint.

If the household had \$300 of income, they would want to consume at point A, but this point is outside the constraint and therefore is not available. Rather, they would consume the bundle C_1, which is at the kink in the constraint and where

$$F_1 = 200 \quad and \quad C_1 = 100.$$



(c) The household achieves the utility level U_1 in the graph. The cheapest way to achieve U_1 given the existing prices would be at the point C_2, where the proportion $\frac{F}{C} = 1$, as required in the optimal bundle. Now

$$U_1 = (200)^{\frac{1}{2}} \cdot (100)^{\frac{1}{2}} = 100\sqrt{2}.$$

This is the utility level that must be attained at C_2. Now the household's demand functions are

$$F = \frac{I}{2p_f} \quad and \quad C = \frac{I}{2p_c}$$

and, since $p_f = p_c = 1$, the indirect utility function of this household is

$$V(p, I) = \left(\frac{I}{2}\right)^{\frac{1}{2}} \cdot \left(\frac{I}{2}\right)^{\frac{1}{2}} = \frac{I}{2}.$$

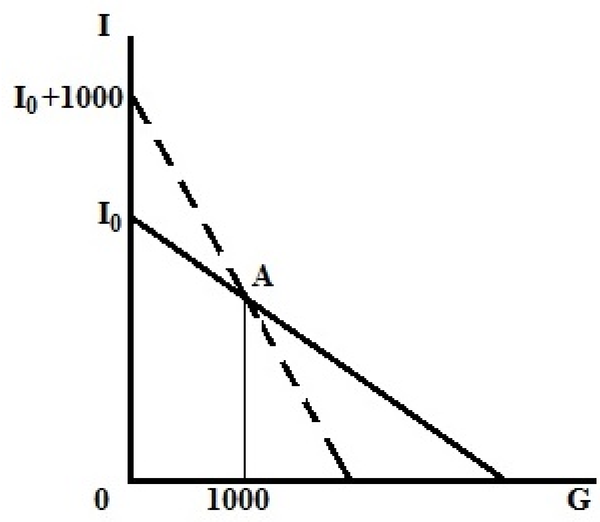
Therefore, the income needed to achieve U_1 must satisfy

$$\frac{I}{2} = 100\sqrt{2}$$

which can be solved to yield $I = 282.80$.

Thus, since it started with \$100, the household is better off by **\$182.80**. This is true whether we use the CV or the EV measure of welfare change, since prices have not changed in this situation.

2. Let's draw a graph with the quantity of gasoline G on the horizontal axis and income I on the vertical. Suppose the initial income is I_0, the budget constraint is the solid line in the graph and the chosen bundle A includes 1000 gallons of gasoline.



Now draw the new budget constraint after the p_G rises by \$1 and income rises by \$1000. This is the dotted line in the graph. It must be steeper than the original budget constraint (since p_G has gone up), and it must pass through A, since the cost of A has risen by exactly the amount by which I has gone up. Therefore, you cannot be worse off (since A is still available) and are probably better off if there is any possibility of substitution. You are exactly as well off as before only in the extreme case of zero substitutability.

3. Let's look to see if Jones can still buy the bundle q_1 that he consumed in year 1. It would cost in year 2:

$$C(p_2, q_1) = (20 * 6) + (30 * 30) = 1020.$$

Since this is less than his income of \$1050, Jones must be better off now.

4. In this problem, bread and cheese are perfect complements for Jane. A representative utility function would be

$$U(B, C) = min\left[\frac{B}{2}, \frac{C}{2}\right].$$

With this utility function, U would simply be the number of sandwiches Jane is able to make.

(a) To find the elasticities of demand, we need to find Jane's demand function for cheese. We know she needs 2 slices of cheese and 2 slices of bread for each sandwich and therefore the number of sandwiches she can make is

$$Q_{sandwiches} = \frac{I}{2p_b + 2p_c}.$$

Therefore the demand function for cheese is

$$C = \frac{2I}{2p_b + 2p_c}. \tag{1}$$

Then the elasticity of demand for cheese is

$$\epsilon_c = \frac{\partial C}{\partial p_c} \cdot \frac{p_c}{C}. \tag{2}$$

Now

$$\frac{\partial C}{\partial p_c} = \frac{-4I}{(2p_b + 2p_c)^2}.$$

Substituting this and equation (1) in (2) and simplifying, we get a formula for the elasticity of demand for cheese:

$$\epsilon_c = \frac{-2p_c}{2p_b + 2p_c}. \tag{3}$$

Similarly, we can find the cross-price elasticity of demand for cheese with respect to the price of bread, defined as

$$\epsilon_{cb} = \frac{\partial C}{\partial p_b} \cdot \frac{p_b}{C}.$$

By following the same procedure as before, we can find the cross-price elasticity to be

$$\epsilon_{cb} = \frac{-2p_b}{2p_b + 2p_c}. \tag{4}$$

Finally, for the income elasticity of demand, defined as

$$\eta_c = \frac{\partial C}{\partial I} \cdot \frac{I}{C},$$

note that

$$\frac{\partial C}{\partial I} = \frac{2}{2p_b + 2p_c}$$

and therefore $\eta_c = 1$.

(b) If $p_b = p_c$, let them each equal p. Then, from (3), we can find the elasticity of demand for cheese to be

$$\epsilon_c = \frac{-2p}{4p} = -\frac{1}{2}.$$

From (4), we can find the cross-price elasticity of demand to be

$$\epsilon_{cb} = \frac{-2p}{4p} = -\frac{1}{2}.$$

We can now check that

$$\epsilon_c + \epsilon_{cb} + \eta_c = -\frac{1}{2} - \frac{1}{2} + 1 = 0,$$

as expected.

(c) If $p_c = 2p_b$, we can write down the elasticities on the basis of the formulae:

$$\epsilon_c = \frac{-4p_b}{6p_b} = -\frac{2}{3},$$

$$\epsilon_{cb} = \frac{-2p_b}{6p_b} = -\frac{1}{3}$$

and of course $\eta_c = 1$.

5. In this example, p_x, I are unchanged. p_y has gone up from \$10 to \$12, so $\Delta p_y = 2$. X has risen from 500 to 600, so $\Delta X = 100$. So a simple estimate of the cross-price elasticity of demand for X is

$$\epsilon_{xy} = \frac{\Delta X}{\Delta p_y} \cdot \frac{p_y^0}{X_0} = \frac{100}{2} \cdot \frac{10}{500} = 1.$$

The arc elasticity of demand would use the mid-point values of X and p_y:

$$\epsilon_{xy}^{arc} = \frac{\Delta X}{\Delta p_y} \cdot \frac{\frac{p_x^0 + p_x^1}{2}}{\frac{X_0 + X_1}{2}} = \frac{100}{2} \cdot \frac{11}{550} = 1.$$

6. (a) A simple way to think about this is to treat Joe's income as $p_a \cdot A_0$ and then find the demand functions in the usual way. Since the utility function is Cobb-Douglas, with equal coefficients for the two goods, we can write down the demand functions as

$$A = \frac{p_a \cdot A_0}{2p_a} \qquad \rightarrow \qquad A = \frac{A_0}{2}$$

$$B = \frac{p_a \cdot A_0}{2p_b}.$$

(b) The demand functions are of Cobb-Douglas form and can be written

$$A = \frac{1}{2}A_0 p_a^0 p_b^0 \qquad \textit{and} \qquad B = \frac{1}{2}A_0 p_a^1 p_b^{-1}.$$

Since we know that elasticities in Cobb-Douglas functions are given by the exponents on the respective variables, we can write down the elasticities by reading them from the demand functions:

$$\epsilon_a = 0, \epsilon_{ab} = 0, \qquad \textit{and} \qquad \epsilon_b = -1, \epsilon_{ba} = 1.$$

7. (a) Suppose Bill works for L hours per day. Then his income will be $I = wL$ and this will equal his expenditure on goods (whose average price is 1.5):

$$wL = 1.5y \qquad \rightarrow \qquad L = \frac{1.5}{w}y.$$

Since Bill's time constraint is L+T=*24*, we can write his constraint as

$$\frac{1.5}{w}y + T = 24.$$

Bill wants to maximize $U(y,T) = y^{\frac{1}{2}}T$ subject to this constraint. The Lagrangian for this problem is

$$\mathcal{L} = y^{\frac{1}{2}}T + \lambda\left[24 - \frac{1.5}{w}y - T\right].$$

The first-order conditions, aside from the constraint, are:

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{1}{2}y^{-\frac{1}{2}}T - \lambda \cdot \frac{1.5}{w} = 0 \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial T} = y^{\frac{1}{2}} - \lambda = 0. \tag{6}$$

Rearranging and then dividing (5) by (6), we get

$$\frac{1}{2} \cdot \frac{T}{y} = \frac{1.5}{w},$$

which can be written as

$$\frac{1.5}{w}y = \frac{1}{2}T.$$

Substituting this in the constraint, we get

$$\frac{1}{2}T + T = 24 \qquad or \qquad T = 16.$$

Thus Bill would have 16 non-working hours and would work for 8 hours per day.

(b) Since we found that Bill would work for 8 hours per day without having had to specify the wage rate, it must be the case that he would not change his labor supply in response to a 10% increase in his wage rate; he would continue to work 8 hours per day regardless of the wage rate.

(c) Suppose Bill is offered an overtime wage rate of v and then works x overtime hours. Remember he receives a wage of $5 per hour for the first 8 hours he works. So his total income now will be

$$I = 40 + vx.$$

Since he spends this income on goods y whose average price is $1.50, we have

$$40 + vx = 1.5y,$$

which can be written as

$$x = \frac{1.5y - 40}{v}.$$

Since we know that $8 + x + T = 24$, we can write Bill's constraint as

$$\frac{1.5y - 40}{v} + T = 16,$$

which can be rearranged as

$$\frac{1.5}{v}y + T = 16 + \frac{40}{v}. \tag{7}$$

This is now Bill's constraint and he wants to maximize his utility subject to this constraint.
The Lagrangian for this problem is

$$\mathcal{L} = y^{\frac{1}{2}}T + \lambda\left[16 + \frac{40}{v} - \frac{1.5}{v}y - T\right].$$

The first-order conditions, aside from the constraint, are:

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{1}{2}y^{-\frac{1}{2}}T - \lambda \cdot \frac{1.5}{v} = 0 \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial T} = y^{\frac{1}{2}} - \lambda = 0. \tag{9}$$

Rearranging and then dividing (8) by (9), we get

$$\frac{1}{2} \cdot \frac{T}{y} = \frac{1.5}{v},$$

which can be written as

$$\frac{1.5}{v} y = \frac{1}{2} T.$$

Substituting this in the constraint (7), we get

$$\frac{1}{2}T + T = 16 + \frac{40}{v} \qquad \textit{or} \qquad \frac{3}{2}T = 16 + \frac{40}{v}.$$

This can be rearranged as

$$v = \frac{40}{\frac{3}{2}T - 16}. \tag{10}$$

The equation (10) shows what the overtime wage rate v would have to be in order to induce Bill to work a total of *(24-T)* hours. Since his boss wants him to work a total of 10 hours, he wants to induce $T = 14$ and so we can substitute this value in (10) in order to find the required overtime wage rate:

$$v = \frac{40}{\frac{3}{2}(14) - 16} = 8.$$

Therefore Bill's boss must offer him an overtime wage rate of \$8 per hour in order to induce him to work a total of 10 hours (of which 2 hours is overtime).

8. (a) Area B represents the opportunity cost of the time that the worker is giving up when he or she works. Since the wage is w_0 per hour, the worker enjoys a "surplus" on any units of work where the wage is higher than the opportunity cost. The area A is the aggregation of this surplus and is a measure of the net gain the worker gets from working.

   (b) If $w = 10$, we can substitute this value in Andrew's supply curve to find that he would work

$$L = -1 + \frac{10}{2} = 4 \quad \textit{hours per day.}$$

   If we wish to induce Andrew to work for 8 hours per day, we would need to offer him a wage rate w where

$$8 = -1 + \frac{w}{2}$$

which can be solved to yield

$$w = 18.$$

At this wage rate, since he would work 8 hours per day, his total earnings would be $18 * 8 = \$144$ per day.

(c) If Andrew does not accept the job, he has two other options: stay at home (*i.e.*, do no work), or work in the market at the wage of \$10 per hour. Our strategy then to answer this question is to compare the surplus from the job over not working with the surplus from working at the market wage, and seeing which option gives him the highest surplus.

If Andrew worked at the market wage of \$10 per hour, we know from part (b) that he would want to work for 4 hours per day. The shaded area in Figure 1 then represents his surplus in this situation. We can calculate his surplus in this situation to be

$$Surplus_{market} = \frac{1}{2} \cdot 4 \cdot 8 = 16.$$

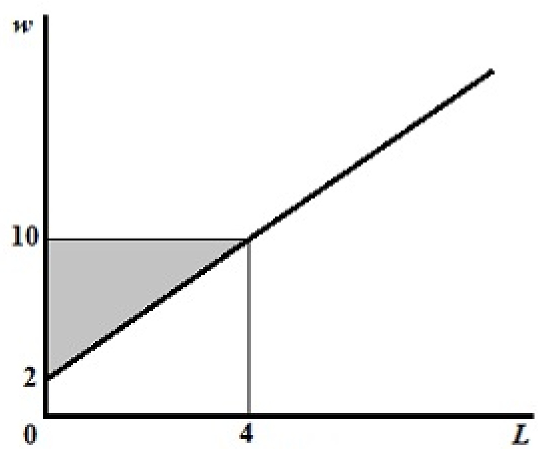


Fig 1: Surplus with \$10 market work

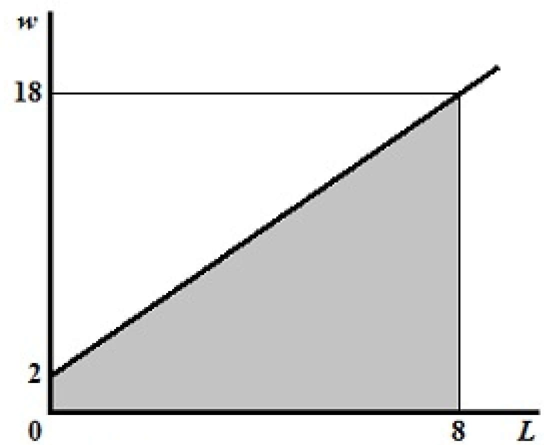


Fig 2: Opportunity Cost under job

Alternatively, if Andrew accepted the job paying \$90 per day, he would have to work for 8 hours per day. The opportunity cost of this time is illustrated by the shaded area in Figure 2 and his surplus from accepting the job would be \$90 minus this opportunity cost:

$$Surplus_{job} = 90 - \left[(2 * 8) + \left(\frac{1}{2} \cdot 8 \cdot 16 \right) \right] = 10.$$

Thus Andrew's surplus is higher when he simply works in the market at \$10 per hour; he will therefore reject the job and follow this option instead.

9. This statement is false, because there is some ambiguity, in certain situations, about the effect of a rise in the interest rate on consumption.

   For a consumer who is a borrower, a rise in the interest rate leads unambiguously to a fall in present consumption, since both income and substitution effects work in this direction, but the effect on future consumption is ambiguous, since the two effects work in opposite directions.

   For a consumer who is a lender, a rise in the interest rate leads unambiguously to a rise in future consumption, since both income and substitution effects work in this direction, but the effect on present consumption is ambiguous, since the two effects work in opposite directions.

   For a detailed discussion, see pp. 101-104 in the text (Chapter 4).